"Mapping class groups and quantum topology"

IRMA, University of Strasbourg, 25–29 June 2012.

mini-course

"Johnson-Morita theory and the Goldman-Turaev Lie bialgebra"

Part II : "The Goldman-Turaev Lie bialgebra" (a geometric approach)

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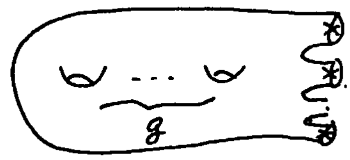
Contents:

S : connected compact oriented surface with $\partial S \neq \phi$.

$\Longrightarrow$ Classification of surfaces $\quad S \cong \Sigma_{g,r} :=$  $\; \} r$

$r \geqq 1$

$\mathcal{M}(S) = \left\{ \varphi : S \to S : \text{orientation preserving diffeomorphism}, \ \varphi|_{\partial S} = id_{\partial S} \right\} \Big/ \begin{array}{l}\text{isotopy} \\ \text{fixing } \partial S \\ \text{pointwise}\end{array}$

mapping class group

$E \subset \partial S$ finite subset s.t. $\text{inclusion}_* : \pi_0(E) \xrightarrow{\cong} \pi_0(\partial S)$

$\mathcal{I}(S) = Ker(\mathcal{M}(S) \to Aut(H_1(S, E ; \mathbb{Z})))$

the "smallest" Torelli group in the sense of Putman

(In the case $r = 1$, $\mathcal{I}(S) = \mathcal{I}_{g,1}$.)

Abstract of Part II

(Recall : $\mathbb{Q}\widehat{\pi_1(\Sigma_{g,1})}$: $\mathcal{M}_{g,1}$-module $\Rightarrow$ Johnson homomorphism)

$\mathbb{Q}\hat{\pi}'(S)$: Goldman - Turaev Lie bialgebra

$*_0, *_1 \in \partial S$, $I = [0,1] = \{t \in \mathbb{R} \,;\, 0 \le t \le 1\} \subset \mathbb{R}$ unit interval

$\Pi S(*_0, *_1) := [(I, 0, 1), (S, *_0, *_1)]$ homotopy classes of paths from $*_0$ to $*_1$

$\mathbb{Q}\Pi S(*_0, *_1)$: $\mathbb{Q}\hat{\pi}'(S)$-bimodule

$\xrightarrow[\text{completion}]{}$ $\widehat{\mathbb{Q}\Pi S}(*_0, *_1)$: $\widehat{\mathbb{Q}\hat{\pi}}(S)$-bimodule

$\Longrightarrow$ $\tau : \mathcal{I}(S) \hookrightarrow \widehat{\mathbb{Q}\hat{\pi}}(S)$ embedding

Putman's generators of $\mathcal{I}(S)$

Dehn twist formula by Kuno-K.

a geometric generalization of the Johnson homomorphism

(In the case $S = \Sigma_{g,1}$
$$\begin{tikzcd} \mathcal{I}_{g,1} \arrow[r, "\tau"] \arrow[dr, "\text{Massuyeau's } \tau^\theta"'] & \widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{g,1}) \\ & \hat{\mathfrak{h}}_{g,1} \arrow[u] \end{tikzcd}$$
)

$\Longrightarrow$ Image $\tau \subset$ Ker(Turaev cobracket)

$\forall$ diffeomorphism preserves the self-intersection of any curve on S

$\Longrightarrow$ a topological interpretation of the Morita traces

a theorem of Massuyeau - Turaev

§ II-1. Goldman bracket

S (connected) oriented surface.

$\hat{\pi} = \hat{\pi}(S) := [S^1, S] = \pi_1(S)/\text{conj}.$

$|\ | : \pi_1(S) \to \hat{\pi}(S)$ quotient map, forgetting the basepoint

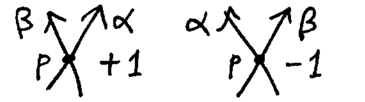
$\mathbb{Z}\hat{\pi}$: the free $\mathbb{Z}$-module over the set $\hat{\pi}$

$\alpha, \beta : S^1 \to S$ maps in general position ($\Rightarrow \#(\alpha \cap \beta) \lneqq +\infty$)

(i.e., $\alpha \sqcup \beta : S^1 \sqcup S^1 \to S$ is an immersion with at worst transverse double points

(Abuse of Notation : We use the same α for the map α itself and for its homotopy class $[\alpha] \in \hat{\pi}$.

Goldman bracket $\quad [\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p : \alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$

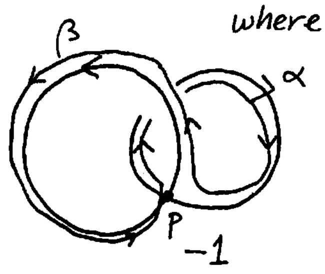


where $\varepsilon(p : \alpha, \beta) \in \{\pm 1\}$ local intersection number

α_p (resp. β_p) $\in \pi_1(S, p)$ based loop along α (resp. β) based at p.

$\underrightarrow{\alpha_p \beta_p} \in \pi_1(S, p) \overset{|\ |}{\longmapsto} |\alpha_p \beta_p| \in \hat{\pi}$

first traverse α_p, then β_p



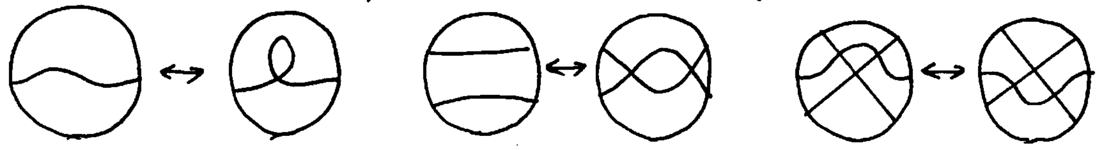
Theorem (Goldman, Invent. math., 85, 263–302 (1986))

$[\ ,\ ] : \mathbb{Z}\hat{\pi} \times \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}$ is well-defined.

$(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra $\square$

well-defined $\Leftarrow$ invariance of $[\ ,\ ]$ under the following 3 local moves

birth-death of a monogon, birth-death of a bigon, jumping over a double point



Lie algebra
- (skew) $[\alpha,\beta] = -[\beta,\alpha]$ $(\because\ \varepsilon(p;\alpha,\beta) = -\varepsilon(p;\beta,\alpha))$
- (Jacobi) $[\alpha,[\beta,\gamma]] + [\beta,[\gamma,\alpha]] + [\gamma,[\alpha,\beta]] = 0$ ($\because$ straight-forward computation)

Remark · Goldman extracted this Lie algebra structure from Wolpert's formula on the Teichmüller space, and his own study of the moduli space of flat bundles on S

- $1 \in \hat{\pi}$ constant loop, $1 \in \mathrm{Center}(\mathbb{Z}\hat{\pi})$ (i.e. $\forall \alpha \in \hat{\pi}$ $[1,\alpha] = 0$)
  $\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}1$ quotient Lie algebra.

Action of a free loop on a path

$\mathrm{Int}\, S := S \setminus \partial S$ interior

$*_0, *_1 \in S$.

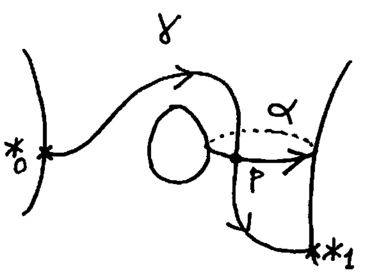
$\Pi S(*_0,*_1) := [(I,0,1),(S,*_0,*_1)] = \{\gamma : I \to S : \text{conti map}, \ell(0) = *_0, \ell(1) = *_1\}/\underset{\text{rel }\partial}{\simeq}$

$\mathbb{Z}\Pi S(*_0,*_1)$: the free $\mathbb{Z}$-module over the set $\Pi S(*_0,*_1)$

$S^* := S \setminus (\{*_0,*_1\} \cap \mathrm{Int}\, S)$

$\alpha \in \hat{\pi}(S^*)$, $\gamma \in \Pi S(*_0,*_1)$ in general position

Action of α on γ



$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p;\alpha,\gamma)\, \gamma_{*_0 p}\, \alpha_p\, \gamma_{p *_1} \in \mathbb{Z}\Pi S(*_0,*_1)$$

$\gamma_{*_0,p} \in \Pi S(*_0,p)$ path from $*_0$ to p along γ

$\gamma_{p,*_1} \in \Pi S(p,*_1)$ from p to $*_1$

Theorem (Kuno–K., arXiv: 1008.5017, 1109.6479)

$\sigma : \mathbb{Z}\hat{\pi}(S^*) \times \mathbb{Z}\Pi S(*_0,*_1) \to \mathbb{Z}\Pi S(*_0,*_1)$ well-defined

$\mathbb{Z}\Pi S(*_0,*_1)$: left $\mathbb{Z}\hat{\pi}(S^*)$-module via σ

$\forall \alpha \in \hat{\pi}$, $\sigma(\alpha)$ is a "derivation" i.e., $\forall *_0, \forall *_1, \forall *_2 \in S$ $\forall \gamma_1 \in \Pi S(*_0,*_1), \forall \gamma_2 \in \Pi S(*_1,*_2)$

$$\sigma(\alpha)(\gamma_1\gamma_2) = \sigma(\alpha)(\gamma_1)\gamma_2 + \gamma_1\sigma(\alpha)(\gamma_2)$$

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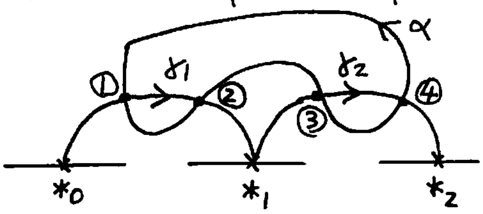
<u>well-defined</u> $\Leftarrow$ invariance of $\sigma(\alpha)(\gamma)$ under the 3 local moves

<u>left $\mathbb{Z}\hat{\pi}$-module</u> $\alpha, \beta \in \hat{\pi}$, $\gamma \in \Pi S(*_0, *_1)$

$$\sigma([\alpha,\beta])(\gamma) = \sigma(\alpha)\sigma(\beta)(\gamma) - \sigma(\beta)\sigma(\alpha)(\gamma) \quad (\because \text{ straight-forward computation})$$

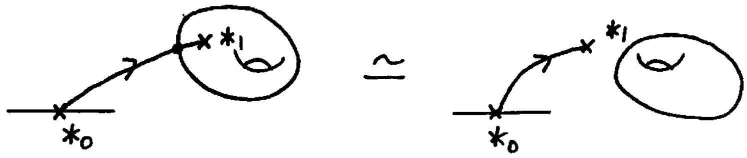
<u>"derivation"</u>



$① + ② = \sigma(\alpha)(\gamma_1)\gamma_2$

$③ + ④ = \gamma_1 \sigma(\alpha)(\gamma_2)$

<u>Remark</u> • If we consider $\hat{\pi}(S)$ instead of $\hat{\pi}(S^*)$, then σ is <u>not</u> well-defined



• Massuyeau-Turaev gives an interpretation of σ as the "derived form" of the homotopy intersection form on S

<u>Small category</u> $\mathbb{Q}\mathcal{C}(S,E)$, $E \subset S$ subset, "groupoid ring"

<u>object</u> $* \in E$ (i.e., $Ob(\mathbb{Q}\mathcal{C}(S,E)) = E$)

<u>morphism</u> $\mathbb{Q}\mathcal{C}(S,E)(*_0, *_1) := \mathbb{Q}\Pi S(*_0, *_1)$, $*_0, *_1 \in E$

morphism from $*_0$ to $*_1$

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<u>derivations of $\mathbb{Q}\mathcal{C}(S,E)$</u>

$$\mathrm{Der}(\mathbb{Q}\mathcal{C}(S,E)) := \left\{ D = \{D^{(*_0,*_1)}\}_{*_0,*_1 \in E} \;;\; \begin{array}{l} D^{(*_0,*_1)} \in \mathrm{End}(\mathbb{Q}\Pi S(*_0,*_1)) \\ D^{(*_0,*_2)}(u_1 u_2) \\ = (D^{(*_0,*_1)} u_1) u_2 + u_1 (D^{(*_1,*_2)} u_2) \\ \forall *_0, *_1, *_2 \in E \\ \forall u_1 \in \mathbb{Q}\Pi S(*_0,*_1),\ \forall u_2 \in \mathbb{Q}\Pi S(*_1,*_2) \end{array} \right\}$$

Lie algebra.

$S^* := S \setminus (E \cap \mathrm{Int}\, S)$

$\sigma : \mathbb{Q}\hat{\pi}(S^*) \to \mathrm{Der}(\mathbb{Q}\mathcal{C}(S,E))$ well-defined Lie algebra homomorphism

<u>Remark</u> • $\sigma(1) = 0$ ($1 \in \hat{\pi}(S^*)$ constant loop)

$\sigma : \mathbb{Q}\hat{\pi}'(S^*) \to \mathrm{Der}(\mathbb{Q}\mathcal{C}(S,E))$

• $\partial S \neq \phi$, $\pi_0(E \cap \partial S) \xrightarrow[\text{incl}_*]{} \pi_0(\partial S)$ surjective ($\Leftrightarrow \pi_0(\partial S, E \cap \partial S) = *$)

$\Rightarrow \mathrm{Ker}\,\sigma = \mathbb{Q}1.$

§ II-2. Completion and Dehn twists

(Recall: G: group, $IG := \mathrm{Ker}(\varepsilon : \mathbb{Q}G \to \mathbb{Q})$ augmentation ideal
$\sum_{x\in G} a_x x \mapsto \sum a_x$

$\widehat{\mathbb{Q}G} := \varprojlim \mathbb{Q}G/(IG)^n$ the completed group ring)

$n \geq 0$

$F_n \mathbb{Q}\Pi S(*_0, *_1) := \gamma_0 (I\pi_1(S, q))^n \gamma_1 \subset \mathbb{Q}\Pi S(*_0, *_1)$

where $q \in S$, $\gamma_0 \in \Pi S(*_0, q)$, $\gamma_1 \in \Pi S(q, *_1)$

the RHS does not depend on q, γ_0 and γ_1 ($\Leftarrow$ $(IG)^n \subset \mathbb{Q}G$ two-sided ideal)

$\widehat{\mathbb{Q}\Pi S}(*_0, *_1) := \varprojlim_{n\to\infty} \mathbb{Q}\Pi S(*_0, *_1)/F_n \mathbb{Q}\Pi S(*_0, *_1)$

(Small Category $\widehat{\mathbb{Q}\mathcal{C}}(S, E)$ "completed groupoid ring"

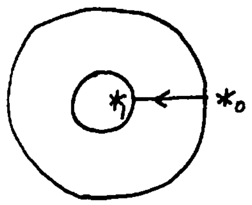
object $\mathrm{Ob}\, \widehat{\mathbb{Q}\mathcal{C}}(S, E) = E$

morphism $\widehat{\mathbb{Q}\mathcal{C}}(S, E)(*_0, *_1) := \widehat{\mathbb{Q}\Pi S}(*_0, *_1)$, $*_0, *_1 \in E$)

$\mathcal{M}(S, E) \overset{\text{def}}{=} \{\varphi : S \to S : \text{ori. pres. diffeo};\ f|_{(\partial S)\cup E} = id_{(\partial S)\cup E}\} /$ isotopy fixing $(\partial S)\cup E$ pointwise

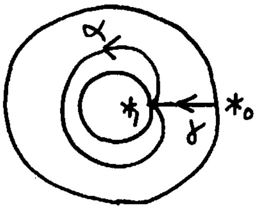
$\curvearrowright \widehat{\mathbb{Q}\mathcal{C}}(S, E)$ natural action

Dehn twist on an annulus $\quad S =$



, $E := \{*_0, *_1\} \subset \partial S$

annulus

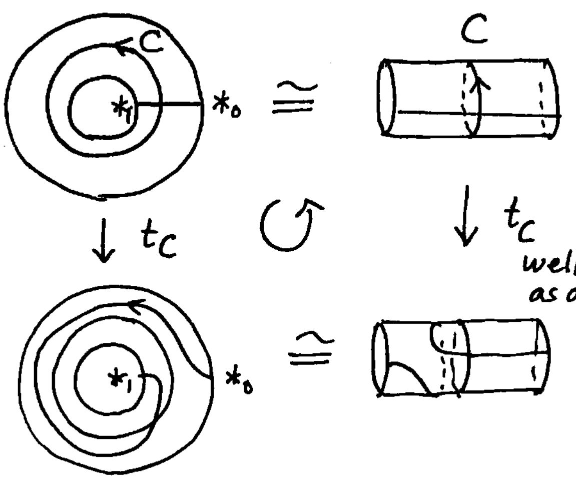


$\alpha \in \pi_1(S, *_1)$

$\Pi S(*_0, *_1) = \{\gamma \alpha^n ; n \in \mathbb{Z}\}$

$\widehat{\mathbb{Q}\Pi S}(*_0, *_1) = \{\gamma u ; u \in \widehat{\mathbb{Q}\langle\alpha\rangle}\}$

(right handed) Dehn twist $\quad t_C \in \mathcal{M}(S, E)$, $C = |\alpha| \in \hat{\pi}(S)$



well-defined as an element of $\mathcal{M}(S, E)$

$$\boxed{\begin{array}{l} t_C(\gamma) = \gamma\alpha \\ t_C(\alpha) = \alpha \end{array}}$$

$\gamma\alpha = \gamma e^{\log\alpha} \in \widehat{\mathbb{Q}\Pi S}(*_0,*_1)$, $\log\alpha \in \widehat{\mathbb{Q}\langle\alpha\rangle}$

$\log(t_C) \in \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{E}}(S,E))$ "derivation"

$$(*1)\begin{cases} \log(t_C)(\gamma) \stackrel{\text{def}}{=} \gamma\log\alpha \\ \log(t_C)(\alpha) \stackrel{\text{def}}{=} 0 \end{cases}$$

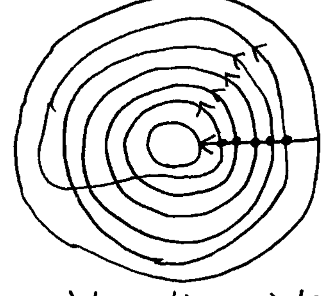
$$\Longrightarrow \quad e^{\log(t_C)} \left(\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!}(\log(t_C))^n\right) = t_C \text{ on } \widehat{\mathbb{Q}\mathcal{E}}(S,E)$$

(∵) $(\log(t_C))^n(\alpha) = 0$, $(\log(t_C))^n(\gamma) = \gamma(\log\alpha)^n$ if $n \geq 1$)

On the other hand

$\sigma(C^n)(\gamma) = n\gamma\alpha^n$, $\sigma(C^n)(\alpha) = 0$ if $n \geq 1$

(∵) $n=5$



- n intersection points
- contribution of each pt $= +\gamma\alpha^n$

$f(x)$: polynomial in x

$$(*2)\begin{cases} \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \\ \sigma(f(C))(\alpha) = 0 \end{cases}$$

Compare (*1) and (*2)

$$\alpha f'(\alpha) = \log\alpha$$

$$f(x) = \int_1 \frac{1}{x}\log x \, dx = \frac{1}{2}(\log x)^2$$

Hence $\log t_C = \frac{1}{2}(\log C)^2 \notin \mathbb{Q}\hat{\pi}$

$\in \widehat{\mathbb{Q}\hat{\pi}}$ a completion.

$n \geq 1$

$\mathbb{Q}\hat{\pi}(S)(n) := |\mathbb{Q}1 + (I\pi_1(S,g))^n|$

where $g \in S$, $1 \in \pi_1(S,g)$ constant loop.

the RHS does not depend on g

Lemma (1) $*_0, *_1 \in S$, $S^* = S \setminus (\{*_0,*_1\} \cap \mathrm{Int}\, S)$, $\forall n\ \forall m \geq 1$

$$\sigma(\mathbb{Q}\hat{\pi}(S^*)(n))(F_m\mathbb{Q}\Pi S(*_0,*_1)) \subset F_{m+n-2}\mathbb{Q}\Pi S(*_0,*_1)$$

(2) $\forall n, \forall m \geq 1$

$$[\mathbb{Q}\hat{\pi}(S)(m), \mathbb{Q}\hat{\pi}(S)(n)] \subset \mathbb{Q}\hat{\pi}(S)(m+n-2)$$

(∵ straight-forward computation)

Lem (2)

$\widehat{\mathbb{Q}\hat{\pi}}(S) \stackrel{\text{def}}{=} \varprojlim \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(n)$ the completed Goldman Lie algebra

$\widehat{\mathbb{Q}\hat{\pi}}(S)(n) := \mathrm{Ker}(\widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(n))$ Lie subalgebra

$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S^*) \to \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{E}}(S,E)) := \{$continuous derivations of $\widehat{\mathbb{Q}\mathcal{E}}(S,E)\}$

well-defined Lie algebra homomorphism ($\Leftarrow$ Lem (1))

$\frac{1}{2}(\log C)^2 \in \widehat{\mathbb{Q}\hat{\pi}}$ (annulus)

$\Downarrow$ van Kampen theorem for the fundamental groupoids

Theorem (Kuno–K., arXiv: 1008.5017, 1109.6479)

S : connected oriented surface, $E \subset S$ subset, $S^* := S \setminus (E \cap \operatorname{Int} S)$

$C \subset S^* \setminus \partial S$ simple closed curve

$L(C) \overset{\text{def}}{=} \frac{1}{2}(\log C)^2 \in \widehat{\mathbb{Q}\hat{\pi}}(S^*)$

$\Longrightarrow t_C = \exp(\sigma(L(C))) \in \operatorname{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S, E))$ ——

Remarks · The original version (1008.5017) covers only the case $S = \Sigma_{g,1}$, $E = \{*\} \subset \partial S$, and involves a symplectic expansion of $\pi_1(\Sigma_{g,1})$.

· Massuyeau–Turaev (arXiv: 1109.5248) gives another generalization of the original version;

Theorem (Massuyeau–Turaev) S : (connected) oriented surface, $* \in S$

$C \subset S \setminus \{*\}$ simple closed curves.

$\Longrightarrow$ (1) $t_C = \exp(\sigma(L(C))) \in \operatorname{Aut}(\widehat{\mathbb{Q}\pi_1(S,*)})$ if $* \in \partial S$

(2) $t_C = \exp(\sigma(L(C))) \in \operatorname{Out}(\widehat{\mathbb{Q}\pi_1(S)})$ if $* \in \operatorname{Int} S$ ——

· Independently, Kuno–K. (1109.6479) gives the generalization stated above.

Generalized Dehn twists

Even if C is not simple, we can define

$$t_C := e^{\sigma(L(C))} \in \operatorname{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S, E)),$$

which Kuno named the generalized Dehn twist along C

(i) Kuno (arXiv: 1104.2107)

$S = \Sigma_{g,1}$, $E = \{*\} \subset \partial S$, $C =$  (non-trivial) figure-eight

$\Longrightarrow t_C \notin$ Image of $\mathcal{M}_{g,1}$

(ii) Massuyeau–Turaev (1109.5248) defined the notion of "twists" in a general algebraic framework.

(iii) Kuno–K. (arXiv: 1112.3841), $\partial S \neq \emptyset$

C : not simple, π_1(regular neighbourhood of C) $\to \pi_1(S)$ injective

$\Longrightarrow t_C \notin$ Image of the mapping class group $\mathcal{M}(S)$ ——

Infinitesimal Dehn-Nielsen Theorem

easier? half.

Theorem (Kuno-K. 1109.6479v2)

S : compact with $\partial S \neq \phi$

$\pi_0(\partial S, E \cap \partial S) = *$ (i.e., $\pi_0(E \cap \partial S) \to \pi_0(\partial S)$ surjective)

$\Longrightarrow \sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \longrightarrow \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}}(S, E))$ injective

harder? half

Conjecture

S : compact with $\partial S \neq \phi$, $\pi_0(\partial S, E \cap \partial S) = *$

$\overset{?}{\Longrightarrow}$ $\mathrm{Image}\,\sigma = \left\{ D \in \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}}(S,E)) \; ; \; \begin{matrix} (D : \text{continuous, and}) \\ D(\forall \text{boundary loop}) = 0 \end{matrix} \right\}$

Remarks · If Conjecture is true, we need not Dehn-twist formula for our re-constuction of the Johnson homomorphisms

· Conjecture is true for $S = \Sigma_{g,1}$, $E = \{*\} \subset \partial S$.
($\Leftarrow$ symplectic expansion)

§ II-3. (geometric) Johnson homomorphism

S : connected compact oriented surface with $\partial S \neq \phi$.
i.e., $S \cong \Sigma_{g,r}$ with $r \geq 1$.

$E \subset \partial S$ finite subset s.t. $\pi_0(E) \overset{\cong}{\twoheadrightarrow} \pi_0(\partial S)$ $(\Rightarrow S^* = S)$

Coproduct on $\widehat{\mathbb{Q}\mathcal{C}}(S, E)$

$\Delta : \widehat{\mathbb{Q}\Pi S}(*_0, *_1) \longrightarrow \widehat{\mathbb{Q}\Pi S}(*_0, *_1) \hat{\otimes} \widehat{\mathbb{Q}\Pi S}(*_0, *_1)$ coproduct

$\gamma \in \Pi S(*_0, *_1) \longmapsto \gamma \hat{\otimes} \gamma$

$L^+(S, E) := \{ u \in \widehat{\mathbb{Q}\hat{\pi}}(S)(3) ; \Delta \sigma(u) = (\sigma(u) \hat{\otimes} \sigma(u)) \Delta \}$

$\subset \widehat{\mathbb{Q}\hat{\pi}}(S)$ Lie subalgebra $\quad (L^+(\Sigma_{g,1}, \{*\}) \cong \hat{\mathfrak{h}}^+_{g,1})$

$\exp \circ \sigma : L^+(S,E) \to \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$, $u \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!} \sigma(u)^n$ $\quad$ (converges $\Leftarrow L^+(S,E) \subset \widehat{\mathbb{Q}\hat{\pi}}(3)$.

injective ($\because$ σ : injective, $L^+(S,E) \subset \widehat{\mathbb{Q}\hat{\pi}}(3)$)

$\mathrm{Image}(\exp \circ \sigma) \subset \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$ subgroup

($\because$ $L^+(S,E) \subset \widehat{\mathbb{Q}\hat{\pi}}(3)$, Baker-Campbell-Hausdorff formula)

$\leadsto$ We regard $L^+(S,E)$ as a group via $\exp \circ \sigma$

17.

Lemma $C, C_1, C_2 \subset S$ closed curves

(1) $[C] = 0 \in H_1(S;\mathbb{Z}) \Rightarrow L(C) \in L^+(S,E)$

(2) $\pm[C_1] = \pm[C_2] \in H_1(S;\mathbb{Z}) \Rightarrow L(C_1) - L(C_2) \in L^+(S,E)$ _

(straight-forward computation)

$\mathcal{I}(S) = \mathrm{Ker}(\mathcal{M}(S) \to \mathrm{Aut}(H_1(S,E;\mathbb{Z})))$ the "smallest" Torelli group in the sense of Putman

Theorem (Putman, Geom. Top. 11 (2007), 829-865)

If genus$(S) \geq 1$, then $\mathcal{I}(S)$ is generated by the union

$\{t_C \,;\, C \subset S$ simple closed curve, $[C] = 0 \in H_1(S;\mathbb{Z})\}$

$\cup \{t_{C_1} t_{C_2}^{-1} \,;\, C_1, C_2 \subset S$: disjoint simple closed curves $\pm[C_1] = \pm[C_2] \in H_1(S;\mathbb{Z})\}$

In the case $g=0$, a similar result holds under some modification.

As a corollary,

$$\begin{tikzcd} L^+(S,E) \arrow[r, hook, "\exp\circ\sigma"] & \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E)) \\ & \mathcal{I}(S) \arrow[ul, dashed, "\exists! \tau"] \arrow[u, hook] \end{tikzcd}$$

injective (essentially due to Dehn-Nielsen)

$\tau : \mathcal{I}(S) \hookrightarrow L^+(S,E)$ injective group homomorphism

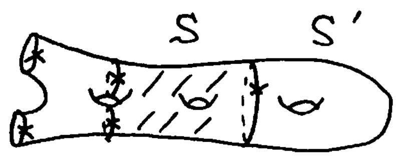
the (geometric) Johnson homomorphism

18.

Remark In the case $S = \Sigma_{g,1}$, $E = \{*\} \subset \partial S$,

$$\begin{tikzcd} \mathcal{I}_{g,1} \arrow[r, hook, "\tau"] \arrow[dr, "\text{Massuyeau's } \tau\theta"'] & L^+(S,E) \arrow[d, "\lambda_\theta", "\cong"'] \\ & \widehat{\mathfrak{h}}_{g,1}^+ \end{tikzcd}$$

<u>Naturality of</u> $\tau : \mathcal{I}(S) \hookrightarrow L^+(S,E)$.



(S', E') similar to (S,E)

$\iota : S \hookrightarrow S'$ embedding of surfaces

$\iota : \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}\hat{\pi}(S')$ inclusion homomorphism

$\iota : \mathcal{M}(S) \to \mathcal{M}(S')$ extending diffeo's by $\mathrm{id}_{S' \setminus S}$

$\Longrightarrow$ The diagram

$$\begin{tikzcd} \mathcal{I}(S) \arrow[r, "\iota"] \arrow[d, "\tau"'] & \mathcal{I}(S') \arrow[d, "\tau"] \\ L^+(S,E) \arrow[r, "\iota"'] & L^+(S',E') \end{tikzcd}$$

is well-defined and commutes.

<u>Remark</u> In the case $r \geqq 2$, Putman defines some other variants of the Torelli groups, and Church (arXiv:1108.4511) defines the 1^{st} Johnson homomorphism for each of these variants. I don't know how Church's Johnson homomorphism and ours are related to each other. Anyway, to get the geometric Johnson homomorphism for other Torelli groups, we need change the filtrations $\{\mathbb{Q}\hat{\pi}(m)\}_{n\geq 1}$ and $\{F_n \mathbb{Q}\Pi S(*_0, *_1)\}_{n\geq 1}$.

§ II-4. Turavev cobracket

S : connected oriented surface.

$\hat{\pi} = \hat{\pi}(S) = [S^1, S]$, $1 \in \hat{\pi}$ constant loop

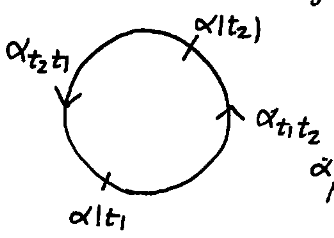
$\mathbb{Q}\hat{\pi}' = \mathbb{Q}\hat{\pi}'(S) := \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1$ Lie algebra ($\because$ $\mathbb{Q}1 \subset \mathrm{Center}(\mathbb{Q}\hat{\pi})$)

$\sigma : \mathbb{Q}\hat{\pi}' \to \mathrm{Der}\,\mathbb{Q}\mathcal{C}(S, E)$ well-defined ($\because$ $\sigma(1) = 0$)

$|\ |' : \mathbb{Q}\pi_1(S) \to \mathbb{Q}\hat{\pi}/\mathbb{Q}1 = \mathbb{Q}\hat{\pi}'$ quotient map.

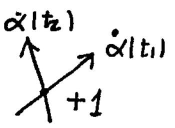
<u>Turaev cobracket</u>

$\alpha \in \hat{\pi}$ in general position



$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1;\ t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$ double points

$$\delta(\alpha) := \sum_{(t_1,t_2)\in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Q}\hat{\pi}' \otimes \mathbb{Q}\hat{\pi}'$$



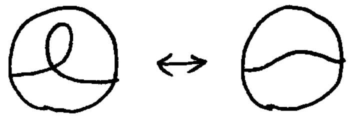
where $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number

Theorem (Turaev, Ann. Sci. ENS 24 635–704. (1991))

$\delta : \mathbb{Q}\hat{\pi}' \to \mathbb{Q}\hat{\pi}' \otimes \mathbb{Q}\hat{\pi}'$ well-defined

$(\mathbb{Q}\hat{\pi}, [\,,\,], \delta)$: Lie bialgebra ··· Chas involutive

Remark We must take $\mathbb{Q}\hat{\pi}' = \mathbb{Q}\hat{\pi} / \mathbb{Q}1$ to kill the ambiguity coming from "monogons"



Involutive Lie bialgebra /$\mathbb{Q}$

Definition $(\mathfrak{g}, [\,,\,], \delta)$: involutive Lie bialgebra /$\mathbb{Q}$

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $(\mathfrak{g}, [\,,\,])$: Lie algebra /$\mathbb{Q}$

i.e., 1-1) (skew) $\forall X, \forall Y \in \mathfrak{g}$. $[X, Y] = -[Y, X]$

1-2) (Jacobi) $\forall X, \forall Y, \forall Z \in \mathfrak{g}$. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

2) $(\mathfrak{g}, \delta)$: Lie coalgebra/$\mathbb{Q}$

i.e., 2-1) (coskew)

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\delta"] \arrow[dr, dashed, "\exists"'] & \mathfrak{g}\otimes\mathfrak{g} \\ & \Lambda^2\mathfrak{g} \arrow[u, hook] \end{tikzcd}$$

2-2) (coJacobi) $N(\delta\otimes 1)\delta = 0 : \mathfrak{g} \xrightarrow{\delta} \mathfrak{g}^{\otimes 2} \xrightarrow{\delta\otimes 1} \mathfrak{g}^{\otimes 3} \xrightarrow{N} \mathfrak{g}^{\otimes 3}$

where $N(XYZ) = XYZ + YZX + ZXY$. $(X, Y, Z \in \mathfrak{g})$ (to be continued

3) (combatibility) $\forall X, \forall Y \in \mathfrak{g}$

$$\delta[X,Y] = \mathrm{ad}(X)(\delta Y) - \mathrm{ad}(Y)(\delta X)$$

(where $\mathrm{ad}(X)(Y\otimes Z) = [X,Y]\otimes Z + Y \otimes [X,Z]$, $(X, Y, Z \in \mathfrak{g})$)

4) (involutivity) $[\,,\,] \circ \delta = 0 : \mathfrak{g} \to \mathfrak{g}$ —

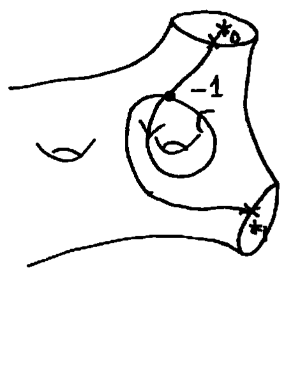
(1)~3) : original definition by Drinfel'd.)

$\mathrm{Ker}\,\delta < \mathfrak{g}$ Lie subalgebra $\Leftarrow$ 3)

($\because$) $X, Y \in \mathrm{Ker}\,\delta \Rightarrow \delta[X,Y] = \mathrm{ad}(X)(0) - \mathrm{ad}(Y)(0) = 0$ //)

(right) comodule structure on $\mathbb{Q}\widehat{\Pi S}(*_0, *_1)$

$*_0 \neq *_1 \in \partial S$, $\gamma \in \Pi S(*_0, *_1)$ in general position

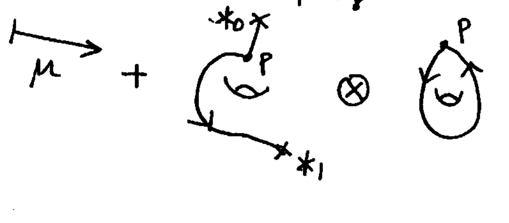


$\Gamma_\gamma := \{\text{double points of } \gamma\} \subset S$

$p \in \Gamma_\gamma$, $\gamma^{-1}(p) = \{t_1^p, t_2^p\} \subset I$. $t_1^p < t_2^p$

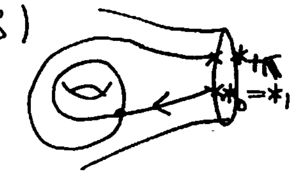
$$\mu(\gamma) := -\sum_{p\in\Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \left(\gamma_{0 t_1^p}\, \gamma_{t_2^p 1}\right) \otimes |\gamma_{t_1^p t_2^p}|'$$

$\in \mathbb{Q}\Pi S(*_0, *_1) \otimes \mathbb{Q}\hat{\pi}'(S)$



In the case $*_0 = *_1$, we move $*_1$ slightly in the positive direction,

then we get $\mu_+ : \mathbb{Q}\pi_1(S, *_0) \to \mathbb{Q}\pi_1(S, *_0) \otimes \mathbb{Q}\hat{\pi}'(S)$



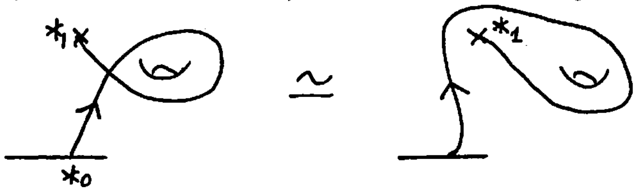
Remark These definitions are inspired by Turaev (Math. USSR Sbornik 35 (1979) 229-250) —

Theorem (Kuno-K., arXiv:1112.3841)

$\mu\,(\mu_+) : \mathbb{Q}\Pi S(*_0, *_1) \longrightarrow \mathbb{Q}\Pi S(*_0, *_1) \otimes \mathbb{Q}\hat{\pi}'(S)$ well-defined

$(\mathbb{Q}\Pi S(*_0, *_1), \sigma, \mu\,(\mu_+))$: involutive $\mathbb{Q}\hat{\pi}'(S)$-bimodule —

Remark The condition $*_0, *_1 \in \partial S$ is essential for the well-definedness of μ



δ and $\mu(\mu_+)$ are compatible with the filtrations $\{\mathbb{Q}\hat{\pi}(m)\}$ and $\{F_n \mathbb{Q}\Pi S\}$

$\Longrightarrow$ $\widehat{\mathbb{Q}\hat{\pi}}(S)$: (complete) involutive Lie bialgebra

$\widehat{\mathbb{Q}\Pi S}(*_0, *_1)$: (complete) involutive $\widehat{\mathbb{Q}\hat{\pi}}(S)$-bimodule.

${}^\forall$mapping class $\in \mathcal{M}(S)$ preserves $\mu\,(\mu_+)$

$\Longrightarrow$ ${}^\forall u \in \tau(\mathfrak{g}(S))$ ${}^\forall n \in \mathbb{Z}$ $e^{n\sigma(u)}$ preserves μ

i.e., ${}^\forall v \in \widehat{\mathbb{Q}\Pi S}(*_0, *_1)$ $(*_0, *_1 \in E)$

$$\mu(e^{n\sigma(u)} v) = (e^{n\sigma(u)} \hat{\otimes} e^{n\sigma(u)})\mu(v) \quad ({}^\forall n \in \mathbb{Z})$$

$\underset{\text{linear term in } n}{\Longrightarrow}$ $\mu(\sigma(u)v) = (\sigma(u) \otimes 1 + 1 \otimes \sigma(u))\mu(v)$ $(=: \sigma(u)\mu(v))$

"Compatibility" for $\mathbb{Q}\hat{\pi}$-bimodule $\widehat{\mathbb{Q}\Pi S}(*_0, *_1)$

$\sigma(u)\mu(v) - \mu(\sigma(u)v) = (\bar{\sigma} \otimes 1_{\widehat{\mathbb{Q}\hat{\pi}}})(v \otimes \delta u)$

(where $\bar{\sigma} : \widehat{\mathbb{Q}\Pi S} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\Pi S}$, $v \otimes u \mapsto -\sigma(u)(v)$)

$\Longrightarrow$ $(\bar{\sigma} \otimes 1_{\widehat{\mathbb{Q}\hat{\pi}}})(v \otimes \delta u) = 0$ $({}^\forall v \in \widehat{\mathbb{Q}\Pi S})$

$\Longrightarrow$ $\delta u = 0$ ($\because$ $\sigma : \widehat{\mathbb{Q}\hat{\pi}} \to \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}})$ injective)

Theorem (Kuno-K; 1112.3841)

$(\delta|_{L^+}) \circ \tau = 0 : \mathfrak{g}(S) \xrightarrow{\tau} L^+(S, E) \xrightarrow{\delta|_{L^+}} \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$

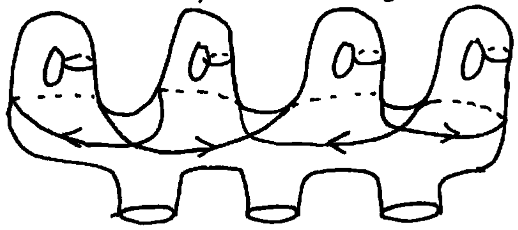
i.e., $\overline{\tau(\mathfrak{g}(S))}^{\text{Zariski closure}} \subset \mathrm{Ker}(\delta|_{L^+})$: Lie subalgebra

geometric obstruction of the surjectivity of $\tau : \mathfrak{g}(S) \to L^+(S, E)$

Proposition $\delta|_{L^+(S,E)} \neq 0$ if genus$(S) \geqq 2$

(pf)

C



$L(C) \in L^+(S,E)$

C has a self-intersection if $g \geqq 2$

$\delta L(C) \neq 0 \in \mathbb{Q}\hat{\pi}(S) \hat{\otimes} \mathbb{Q}\hat{\pi}(S)$ //

Conjecture

$$\overline{\tau(\mathcal{I}(S))}^{\text{Zariski closure}} \overset{?}{=} \operatorname{Ker}(\delta|_{L^+(S,E)})$$

evidences

(positive) Turaev's characterization of based simple loops

(negative) Chas' counter-example for Turaev's conjecture on free simple loops

§ II-5. Tensorial computation of the Turaev cobracket

$g \geqq 1$, $r=1$, $S = \Sigma = \Sigma_{g,1} =$  $* \in \partial S$, $E = \{*\}$

$\pi = \pi_1(\Sigma, *)$, $H = H_1(\Sigma; \mathbb{Q})$

$\hat{T} = \hat{T}(H) = \prod_{m=0}^{\infty} H^{\otimes m}$ completed tensor algebra.

$\theta : \pi \to \hat{T}$ symplectic expansion.

$N : \hat{T} \to \hat{T}$: cyclic symmetrizer (cyclicizer, norm map)

$N|_{H^{\otimes 0}} := 0$

$N(X_1 X_2 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$ $(X_j \in H)$, $m \geqq 1$

$\cdot : H \times H \to \mathbb{Q}$, $(X, Y) \mapsto X \cdot Y$, intersection number

Identify $H \overset{\text{P.d.}}{=} H^*$, $X \mapsto (Y \mapsto Y \cdot X)$ Poincaré duality

$\operatorname{Der}_\omega(\hat{T}) := \{ D : \text{continuous derivation of } \hat{T} ;\ D\omega = 0 \}$ ω: sympl. form

$$\begin{array}{ccccccc} \operatorname{Der}_\omega(\hat{T}) & \cong & N(\hat{T}) & \subset & H \otimes \hat{T} & \overset{\text{P.d.}}{=} & H^* \otimes \hat{T} \\ \Psi & & & & & & \Psi \\ D & & \longmapsto & & & & D|_H \end{array}$$

$\mathfrak{a}_g^- := N(\hat{T}) = \operatorname{Der}_\omega(\hat{T})$ (identified)

Theorem (Kuno-K, 1008.5017) $\quad \forall\theta$: symplectic expansion.

(1) $-\lambda_\theta := -N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} \mathfrak{a}_g^- = N(\hat{T})$, $|x| \mapsto -N\theta(x)$. Lie algebra isom. $\qquad (\mathfrak{a}_g^- /\!/ \mathrm{Der}_\omega(\hat{T}))$

(2)

$$\begin{tikzcd} \widehat{\mathbb{Q}\hat{\pi}} \otimes \widehat{\mathbb{Q}\pi} \arrow[r, "\sigma"] \arrow[d, "-N\theta\otimes\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{\mathbb{Q}\pi} \arrow[d, "\theta", "\cong"'] \\ \mathrm{Der}_\omega(\hat{T}) \hat{\otimes} \hat{T} \arrow[r, "\text{derivation}"'] & \hat{T} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \widehat{\mathbb{Q}\pi} \arrow[r, "|\ |'"] \arrow[d, "\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{\mathbb{Q}\hat{\pi}} \arrow[d, "-\lambda_\theta", "\cong"'] \\ \hat{T} \arrow[r, "-N"'] & N(\hat{T}) = \mathfrak{a}_g^- \end{tikzcd}$$

(3) $\lambda_\theta(L^+(S,E)) = \widehat{\mathfrak{h}_{g,1}^+} \subset \mathfrak{a}_g^-$ $\quad \square$

$\delta^\theta := ((-\lambda_\theta)\hat{\otimes}(-\lambda_\theta)) \circ \delta \circ (-\lambda_\theta)^{-1} : \mathfrak{a}_g^- \to \mathfrak{a}_g^- \hat{\otimes} \mathfrak{a}_g^-$ $\quad$ Turaev cobracket

$\tau^\theta : \mathcal{I}_{g,1} \to \widehat{\mathfrak{h}_{g,1}^+} \subset \mathfrak{a}_g^-$ $\quad$ Massuyeau's total Johnson map (stated in Part I)

$\delta^\theta \circ \tau^\theta = 0 : \mathcal{I}_{g,1} \to \mathfrak{a}_g^- \hat{\otimes} \mathfrak{a}_g^-$ $\quad (\Leftarrow \text{Thm})$

<u>Question 1</u> Explicit description of δ^θ ?

<u>not</u> known !! $\qquad \delta^\theta$ depends on θ $\left(\because \begin{array}{l} \{\text{sympl. exp.}\} \cong \widehat{\mathfrak{h}_{g,1}^+} \\ \delta|_{L^+} \neq 0 \end{array} \right)$

<u>Recall</u> original Johnson homomorphism

$\tau : gr(\mathcal{I}_{g,1}) \to gr(\widehat{\mathfrak{h}_{g,1}^+}) = \mathfrak{h}_{g,1}^+$

<u>Question 2</u> Explicit description of $gr(\delta^\theta)$?

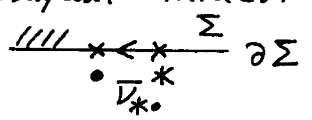
$\Uparrow$

Massuyeau - Turaev's tensorial description of the homotopy intersection form on Σ.

<u>The homotopy intersection form</u>

(originated by Papakyriakopoulos and Turaev, independently
modified by Massuyeau - Turaev.

$*, \bullet \in \partial\Sigma$

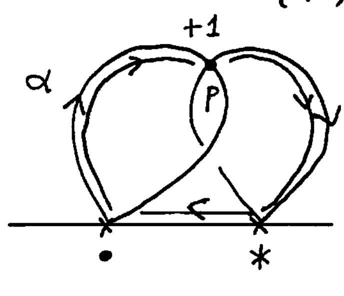


Identify $\pi_1(\Sigma, \bullet) = \pi_1(\Sigma, *) = \pi$ by $\alpha \mapsto \overline{\nu_{*\bullet}} \alpha (\overline{\nu_{*\bullet}})^{-1}$

$\eta : \pi_1(\Sigma, \bullet) \times \pi_1(\Sigma, *) \to \mathbb{Z}\pi_1(\Sigma, *)$

$\qquad \alpha \in \pi_1(\Sigma,\bullet), \ \beta \in \pi_1(\Sigma,*)$ in general position

$\eta(\alpha, \beta) := \sum_{p \in \alpha\cap\beta} \varepsilon(p; \alpha, \beta)\, \overline{\nu_{*\bullet}}\, \alpha_{\bullet p}\, \beta_{p*} \in \mathbb{Z}\pi_1(\Sigma, *)$



<u>Theorem</u> $\left(\begin{array}{l}\text{Papakyriakopoulos, Ann. of Math. Studies } \underline{84}\ (1975)\ 261\text{–}292 \\ \text{Turaev, Math. USSR Sbornik } \underline{35}\ (1979)\ 229\text{–}250\end{array}\right)$

(1) $\eta : \pi_1(\Sigma, \bullet) \times \pi_1(\Sigma, *) \to \mathbb{Z}\pi_1(\Sigma, *)$ well-defined

(2) $\eta(\alpha_1\alpha_2, \beta) = \eta(\alpha_1, \beta) + \alpha_1 \eta(\alpha_2, \beta)$
$\quad \eta(\alpha, \beta_1\beta_2) = \eta(\alpha, \beta_1)\beta_2 + \eta(\alpha, \beta_2)$ $\quad \left(\begin{array}{l}\alpha, \alpha_1, \alpha_2 \in \pi_1(\Sigma, \bullet) \\ \beta, \beta_1, \beta_2 \in \pi_1(\Sigma, *)\end{array}\right)$

<u>Theorem</u> (Massuyeau–Turaev, arXiv: 1109.5248)

θ : symplectic expansion

$$\Longrightarrow \begin{array}{ccc} \mathbb{Q}\pi \otimes \mathbb{Q}\pi & \xrightarrow{\eta} & \mathbb{Q}\pi \\ \theta\otimes\theta \downarrow & \circlearrowleft & \downarrow \theta \\ \hat{T} \otimes \hat{T} & \xrightarrow[\rho]{} & \hat{T} \end{array}$$

where

$$\rho(a, b) := (a - \varepsilon(a)) \overset{\bullet}{\rightsquigarrow} (b - \varepsilon(b)) + (a - \varepsilon(a))\, s(\omega)\, (b - \varepsilon(b)) \quad (a, b \in \hat{T})$$

$\varepsilon : \hat{T} \to \mathbb{Q}$ augmentation

$X_1 \cdots X_n \overset{\bullet}{\rightsquigarrow} Y_1 \cdots Y_m := (X_n \cdot Y_1)\, X_1 \cdots X_{n-1} Y_2 \cdots Y_m \quad (X_i, Y_j \in H)$

$\omega = \sum A_i B_i - B_i A_i \in H^{\otimes 2} \subset \hat{T}$

symplectic form

$$s(\omega) = \frac{1}{e^{-\omega} - 1} + \frac{1}{\omega} = -\frac{1}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \omega^{2k-1}$$

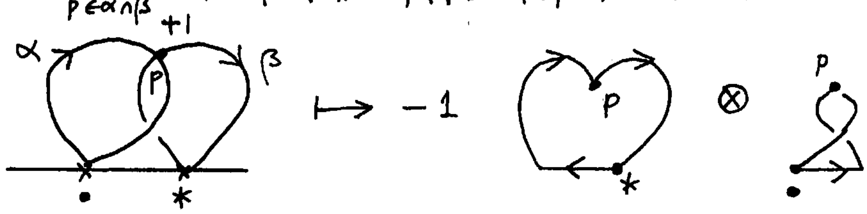
B_{2k} : Bernoulli number

<u>Computation</u>

(i) $\left(\begin{array}{l} \Delta : \mathbb{Q}\pi \to \mathbb{Q}\pi \otimes \mathbb{Q}\pi, \text{ coproduct}, \ x \in \pi \mapsto x \otimes x \\ \iota : \mathbb{Q}\pi \to \mathbb{Q}\pi, \text{ antipode}, \ x \in \pi \mapsto x^{-1} \end{array}\right)$

$\alpha, \beta \in \pi$ in general position

$$- (1 \otimes \beta)\big((1 \otimes \iota) \Delta \eta(\alpha, \beta)\big)(1 \otimes \alpha)$$
$$= - \sum_{p \in \alpha \cap \beta} \varepsilon(p ; \alpha, \beta)(1 \otimes \beta)\big(\overline{\nu_{*\bullet}}\, \alpha_{\bullet p} \beta_{p *} \otimes (\beta_{p*})^{-1} (\alpha_{\bullet p})^{-1} (\overline{\nu_{*\bullet}})^{-1}\big)\big(1 \otimes \overline{\nu_{*\bullet}}\, \alpha (\overline{\nu_{*\bullet}})^{-1}\big)$$
$$= - \sum_{p \in \alpha \cap \beta} \varepsilon(p ; \alpha, \beta)\big(\overline{\nu_{*\bullet}}\, \alpha_{\bullet p} \beta_{p *} \otimes \beta_{* p} \alpha_{p \bullet} (\nu_{*\bullet})^{-1}\big) =: \kappa_+(\alpha, \beta)$$



(ii) $\mu_+(\alpha\beta) = \mu_+(\alpha)(\beta \otimes 1) + (\alpha \otimes 1)\mu_+(\beta) + (1 \otimes \iota \iota')\kappa_+(\alpha, \beta)$

(i.e., μ "cobounds" κ)

(iii) $u, v \in \mathbb{Q}\pi$

$$\kappa_+(u, v) := - \sum (1 \otimes v'')\big((1 \otimes \iota)(\Delta \eta(u', v'))\big)(1 \otimes u'')$$

where $\Delta u = \sum u' \otimes u''$, $\Delta v = \sum v' \otimes v''$ $\quad \left(\text{cf. } \begin{array}{l} \Delta\alpha = \alpha \otimes \alpha \\ \Delta\beta = \beta \otimes \beta \end{array}\right)$

(iv) θ: symplectic expansion

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi}\,\hat{\otimes}\,\widehat{\mathbb{Q}\pi} & \xrightarrow{\kappa_+} & \widehat{\mathbb{Q}\pi}\,\hat{\otimes}\,\widehat{\mathbb{Q}\pi} \\ {\scriptstyle \theta\hat{\otimes}\theta}\downarrow{\scriptstyle \wr\!\!\mid} & \circlearrowleft & {\scriptstyle \wr\!\!\mid}\downarrow{\scriptstyle \theta\hat{\otimes}\theta} \\ \widehat{T}\,\hat{\otimes}\,\widehat{T} & \dashrightarrow & \widehat{T}\,\hat{\otimes}\,\widehat{T} \end{array}$$

$\exists!\,\kappa^\theta$: independent of θ (Massuyeau–Turaev)

$\forall X, \forall Y \in H$, $\Delta X = X\hat{\otimes}1 + 1\hat{\otimes}X$, $\Delta Y = Y\hat{\otimes}1 + 1\hat{\otimes}Y$

$$\begin{aligned} \kappa^\theta(X,Y) &= -(1\hat{\otimes}1)\big((1\hat{\otimes}L)\Delta\rho(X,Y)\big)(1\hat{\otimes}1) \\ &= -(1\hat{\otimes}L)\,\Delta\big((X\cdot Y)1 + X s(\omega) Y\big) \\ &= -(X\cdot Y)(1\hat{\otimes}1) - (1\hat{\otimes}L)\,\Delta(X s(\omega) Y) \end{aligned}$$

(v) μ "cobounds" κ $\Rightarrow$ $\forall X_1, \dots, \forall X_n \in H$

$$\begin{aligned} \mu^\theta(X_1\cdots X_n) &= (1\hat{\otimes}(-N)) \sum_{1\le i<j\le n} (X_1\cdots X_{i-1}\hat{\otimes}1)\,\kappa^\theta(X_i, X_j)\,(X_{j+1}\cdots X_n\hat{\otimes}X_{i+1}\cdots X_{j-1}) \\ &\quad + \sum_{i=1}^{n} (X_1\cdots X_{i-1}\hat{\otimes}1)\,\mu^\theta(X_i)\,(X_{i+1}\cdots X_n\hat{\otimes}1) \\ &= \sum_{1\le i<j\le n} (X_i\cdot X_j)\, X_1\cdots X_{i-1}X_{j+1}\cdots X_n \hat{\otimes} N(X_{i+1}\cdots X_{j-1}) \quad (\text{degree} = n-2) \\ &\quad + (1\hat{\otimes}N) \sum_{1\le i<j\le n} (X_1\cdots X_{i-1}\hat{\otimes}1)\big((1\hat{\otimes}L)\Delta(X_i s(\omega) X_j)\big)(X_{j+1}\cdots X_n\hat{\otimes}X_{i+1}\cdots X_{j-1}) \\ &\quad + \sum_{i=1}^{n} (X_1\cdots X_{i-1}\hat{\otimes}1)\,\mu^\theta(X_i)\,(X_{i+1}\cdots X_n\hat{\otimes}1) \end{aligned}$$

degree $\geq n$ $\left(\because\ \mu^\theta(X_i) \in \underset{\text{degree} \geq 1}{\underline{\widehat{T}\otimes\mathfrak{a}_g^-}} \right)$

(vi)

$$\begin{array}{ccc} \widehat{\mathbb{Q}\pi} & \xrightarrow{\mu_+} & \widehat{\mathbb{Q}\pi}\,\hat{\otimes}\,\widehat{\mathbb{Q}\hat{\pi}} \\ {\scriptstyle \theta}\downarrow{\scriptstyle \wr\!\!\mid} & \circlearrowleft & {\scriptstyle \wr\!\!\mid}\downarrow{\scriptstyle -\theta\otimes N\theta} \\ \widehat{T} & \xrightarrow{\mu^\theta} & \widehat{T}\,\hat{\otimes}\,\mathfrak{a}_g^- \\ {\scriptstyle -N}\downarrow & \circlearrowleft & \downarrow{\scriptstyle (1-T)\circ((-N)\hat{\otimes}1)} \\ \mathfrak{a}_g^- & \xrightarrow{\delta^\theta} & \mathfrak{a}_g^-\,\hat{\otimes}\,\mathfrak{a}_g^- \end{array}$$

$\left(\begin{array}{l} T : \mathfrak{a}_g^-\hat{\otimes}\mathfrak{a}_g^- \circlearrowleft \\ \quad u\hat{\otimes}v \mapsto v\hat{\otimes}u \\ \quad \text{switch map} \end{array} \right)$

$\Longrightarrow$ $\forall X_1, \dots, \forall X_n \in H$

$$\delta^\theta(N(X_1\cdots X_n)) = \underbrace{\delta^{alg}(N(X_1\cdots X_n))}_{\text{degree} = n-2} + \underbrace{\text{higher terms}}_{\text{degree} \geq n}$$

where

$$\delta^{alg}(N(X_1\cdots X_n)) := \sum_{i<j} (X_i\cdot X_j)\big\{ N(X_{i+1}\cdots X_{j-1})\hat{\otimes}N(X_{j+1}\cdots X_n X_1\cdots X_{i-1}) - N(X_{j+1}\cdots X_n X_1\cdots X_{i-1})\hat{\otimes}N(X_{i+1}\cdots X_{j-1}) \big\}$$

(a cobracket on $\mathfrak{a}_g^-$)

$\left(\begin{array}{l} \text{Schedler} \\ \text{Int. Math. Res. Note} \\ \text{2005 No.12,} \\ \text{725–760} \end{array} \right)$

<u>Theorem</u> (Massuyeau-Turaev, Kuno-K., independently)

$$\mathrm{gr}(\delta^\theta) = \delta^{alg} \quad \text{on} \quad \mathrm{gr}(\mathfrak{a}_g^-) = \bigoplus_{n=1}^{\infty} N(H^{\otimes n})$$

Laurent expansion of the Turaev cobracket δ^θ

$$\delta^\theta = \delta^{alg} + \delta^\theta_{(0)} + \delta^\theta_{(1)} + \cdots + \delta^\theta_{(k)} + \cdots$$

where $\delta^\theta_{(k)} : (\mathfrak{a}_g^-)_{(m)} \to (\mathfrak{a}_g^- \hat{\otimes} \mathfrak{a}_g^-)_{(m+k)} \quad (\forall m)$

$$gr(\tau)(\mathcal{I}_{g,1}) \subset \operatorname{Ker} \delta^{alg} \cap \bigcap_{k=0}^{\infty} \operatorname{Ker} \delta^\theta_{(k)}$$

[recent computation
(Massuyeau-Turaev, Kuno-K.) $\forall \theta$, $\delta^\theta_{(0)} = 0$ (Kuno-K.) $\exists \theta$, $\delta^\theta_{(1)} = 0$]

Morita traces

$$\mathfrak{a}_g^- = \prod_{m=0}^{\infty} N(H^{\otimes m}) \xrightarrow[\text{inclusion}]{\iota} \hat{T} = \prod_{m=0}^{\infty} H^{\otimes m} \xrightarrow[\text{projection}]{p_1} H$$

$$\mathcal{S} : \mathfrak{a}_g^- \hat{\otimes} \mathfrak{a}_g^- \xrightarrow[(p_1 \circ \iota) \otimes \iota]{} H \otimes \hat{T} \hookrightarrow \hat{T} \xrightarrow[\text{projection}]{} \widehat{Sym}(H) := \prod_{m=0}^{\infty} Sym^m(H)$$

$$\mathcal{S} \circ \delta^{alg} \circ \tau = 0 : gr(\mathcal{I}_{g,1}) \to \widehat{Sym}(H) \qquad (\Leftarrow \text{Thm})$$

[Theorem (Kuno-K.)

$$\mathcal{S} \circ \delta^{alg}|_{N(H^{\otimes(m+2)})} = (-m) \times \text{the Morita trace } \overset{Tr_{m+1}}{} : N(H^{\otimes(m+2)}) \to Sym^m(H)$$
]

i.e. All the Morita traces are derived from the geometric fact:

Any diffeomorphism preserves the self-intersection of any curve on the surface.