Holomorphic foliations with singularities (Some integrability results)

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Contents

Chapter 1

Introduction

This text has been written with the aim of providing a quick introduction to the framework of holomorphic foliations with singularities. Our methodology is based on the proofs of a couple of important results. The first is the theorem of Mattei–Moussu on the existence of holomorphic first integrals for germs of holomorphic foliations. The second is the linearization theorem of Camacho–Lins Neto–Sad for foliations in the complex projective plane. With these results, we address both local and global aspects of this interesting subject. This text is strongly influenced by the author's personal interests, and it is intended neither to exhaust the subject nor to provide a complete introduction to this beautiful field. I also recommend various other texts, for instance those by D. Cerveau and J.-F. Mattei, Y. Ilyashenko, and F. Loray. I hope these notes will be helpful to those interested in this field of mathematics.

The theory of foliations is one of those subjects in mathematics that brings together several distinct domains, such as topology, dynamical systems, and geometry, among others. Its origins go back to the classical works of C. Ehresmann, W. Shih, and G. Reeb. It provides a natural and valuable approach to the qualitative study of dynamics and ordinary differential equations on manifolds.

Although its origins are in the classical framework of real functions and manifolds, the notion of foliation is also very useful in the holomorphic world. Indeed, it has ancient roots in the study of complex differential equations. From these early problems, the introduction of singularities as objects of study is a natural step. We mention, in particular, the works of P. Painlevé and Malmquist. With Painlevé, the study of complex rational differential equations of the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$$

received some of its first specific methods and results. After Painlevé, many authors contributed to the initial development of the theory, among them E. Picard, G. Darboux, H. Poincaré, H. Dulac, Briot, and Bouquet.

Complex differential equations appear naturally in mathematics and in the natural sciences. Among the classical motivations one finds problems coming from electrical circuits and electromagnetic phenomena. Another important motivation is the search for, and study of, new classes of transcendental functions, such as Liouvillian functions.

With the advent of the geometric theory of foliations and the modern results of Cartan, Oka, Nishino, Suzuki, and others in the theory of analytic functions of several complex variables, together with developments in algebraic and analytic geometry, this field of research became very active again. To this day it remains an active branch of modern mathematics. It also has important connections with algebra and algebraic geometry. For Suzuki's contributions related to holomorphic flows, see [23, 24, 25].

Chapter 2

Foliations with rational first integral

2.1 Motivation

It is a well-known fact that differential equations define "new" functions. For example, ordinary differential equations with polynomial coefficients may define non-algebraic (transcendental) functions, such as exponential and logarithmic functions. On the other hand, it is generally believed that *an algebraic differential equation admitting sufficiently many algebraic solutions must have all of its solutions algebraic.*

In what follows, we shall discuss this and state some results reinforcing it. We shall address the global point of view. Let us introduce some notions:

Definition 2.2. A *holomorphic foliation with singularities of codimension one* on a complex manifold M^m is given by a pair $\mathcal{F} = (\mathcal{F}_0, S)$, where

- (i) $S \subset M$ is an analytic subset of dimension $\leq m-2$ (this means that for each point $x \in S$ we have $\dim_x S_j^x \leq m-2$ for every local branch of S at x).

  (figure)

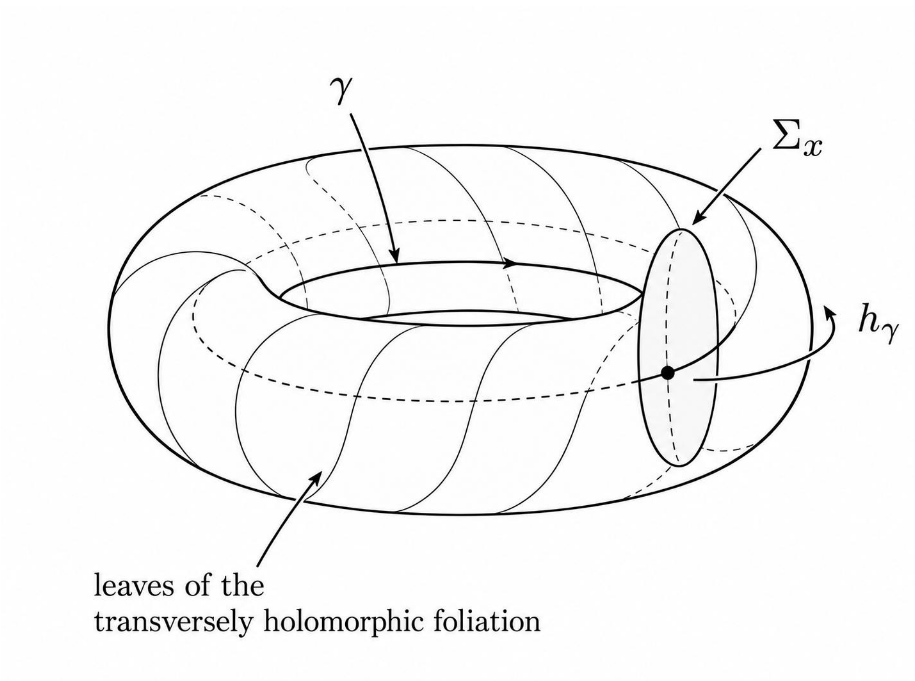
- (ii) A holomorphic foliation $\mathcal{F}_0$ of codimension one (in the usual sense, without singularities) in the open subset $M \setminus S$. (figure)

This definition can also be stated as follows.

Definition 2.3. $\mathcal{F}$ is given by a collection $\{(\omega_j, U_j)\}_{j\in J}$ where $\bigcup_{j\in J} U_j = M$ is an open cover, and on each U_j we have a holomorphic one-form ω_j satisfying the integrability condition $\omega_j \wedge d\omega_j = 0$. On each nonempty intersection $U_i \cap U_j$, we require the compatibility condition $\omega_i = g_{ij}\omega_j$ for some holomorphic function $g_{ij} \colon U_i \cap U_j \to \mathbb{C}^*$.

We may assume that $\mathrm{sing}(\omega_j) = \{p \in U_j : \omega_j(p) = 0\}$ has codimension at least two for every $j \in J$. The analytic subset $\mathrm{sing}(\mathcal{F}) = \bigcup_{j\in J} \mathrm{sing}(\omega_j) \subset M$ is the singular set of $\mathcal{F}$.

Example 2.4. We consider an ordinary differential equation $A(x,y)dx + B(x,y)dy = 0$ with polynomial coefficients $A, B \in \mathbb{C}[x,y]$ in the complex plane $\mathbb{C}^2$. The one-form $\omega = Adx + Bdy$ induces two other one-forms in the complex projective plane $\mathbb{C}P^2 = \mathbb{C}^2{}_{(x,y)} \cup \mathbb{C}^2_{(u,v)} \cup \mathbb{C}^2_{(r,s)}$ as follows:



$$u = \frac{1}{x}, v = \frac{y}{x}$$

$$\omega_1(u,v) = u^m \omega\left(\frac{1}{u}, \frac{v}{u}\right),$$
$$\omega_2(r,s) = r^m \omega\left(\frac{s}{r}, \frac{1}{r}\right)$$

for some $m \in \mathbb{N}$.

Then the collection $\left\{\left(\omega, \mathbb{C}^2_{(x,y)}\right), \left(\omega_1, \mathbb{C}^2_{(u,v)}\right), \left(\omega_2, \mathbb{C}^2_{(r,s)}\right)\right\}$ defines a foliation $\mathcal{F}$ on $\mathbb{CP}^2$ with a finite singular set in $\mathbb{C}P^2$.

(figure)

This example is also general in the following sense:

Proposition 2.5. *Every holomorphic foliation with singularities of codimension one on the complex projective space $\mathbb{C}P^m$, $m \geqslant 2$, is given, in any affine chart with coordinates $(x_1, \ldots, x_m) \in \mathbb{C}^m$, by a polynomial one-form*

$$\omega(x_1, \ldots, x_m) = \sum_{j=1}^{m} A_j(x_1, \ldots, x_m)\, dx_j$$

satisfying the integrability condition

$$\omega \wedge d\omega = 0$$

and singular set $\mathrm{Sing}(\omega) = \{p \in \mathbb{C}^m : \omega(p) = 0\}$ *of codimension at least two.*

This is proved as a consequence of the solution of Cousin Problem II in $\mathbb{C}^{m+1} \setminus \{0\}$ for $m \geq 2$.

Now we recall two results from several complex variables. The first is Chow's theorem.

Theorem 2.6 (Chow's Theorem [10])**.** *Any complex analytic subvariety in a complex projective space is indeed algebraic.*

We shall also use the Remmert–Stein extension theorem.

Theorem 2.7 (Remmert-Stein extension Theorem, [12])**.** *Let M be a complex manifold, let $S \subset M$ be an analytic subset, and let $L \subset M \setminus S$ be an analytic subset (not necessarily analytic in M). Suppose that $\overline{L} \setminus L \subset S$. If* $\dim S < \dim L$*, then $\overline{L}$ is an analytic subset of M.*

As a consequence of these two results we obtain:

Proposition 2.8. *Let $\mathcal{F}$ be a codimension-one holomorphic foliation with singularities on a complex manifold M. Given a leaf L of $\mathcal{F}$, the following are equivalent:*
(i) The closure $\bar{L} \subset M$ is an analytic subset of codimension one of M
(ii) L is closed outside of the singular set of $\mathcal{F}$, i.e., $\bar{L} \subset L \cup \operatorname{sing}(\mathcal{F})$.

And for the case of foliations on complex projective spaces we have:

Proposition 2.9. *Given a codimension-one foliation $\mathcal{F}$ with singularities on $\mathbb{CP}^m$ and a leaf L of $\mathcal{F}$, the following are equivalent.*

- *(i) L is closed outside* $\operatorname{Sing}(\mathcal{F})$.
- *(ii) The closure $\bar{L} \subset \mathbb{C}P^m$ is an algebraic (invariant) subvariety (of codimension one).*

Remark 2.10. Although the closure $\overline{L} \subset \mathbb{CP}^m$ is compact, there is no compact leaf of a foliation $\mathcal{F}$ on a complex projective space: the algebraic invariant hypersurface $\overline{L}$ must intersect the singular set of $\mathcal{F}$, that is, $\operatorname{sing}(\mathcal{F}) \cap \overline{L} \neq \emptyset$.

For simplicity, a leaf L that is closed off $\operatorname{Sing}(\mathcal{F})$ will be called a *closed leaf*. A leaf L of a foliation $\mathcal{F}$ on $\mathbb{CP}^m$ whose closure $\overline{L}$ is an algebraic hypersurface will be called an *algebraic leaf* of $\mathcal{F}$.

Example 2.11. *Take a rational function $R(x_1, \dots, x_m) \in \mathbb{C}(x_1, \dots, x_m)$. Write $R = P/Q$, where $P, Q \in \mathbb{C}[x_1, \dots, x_m]$ have no common factor. Then the polynomial one-form*

$$\omega = Q\,dP - P\,dQ$$

defines a foliation $\mathcal{F}$ on $\mathbb{CP}^m$. All its leaves are algebraic; in the affine chart $\mathbb{C}^m_{(x_1,\dots,x_m)}$ they are given by

$$\lambda P(x_1, \dots, x_m) - \mu Q(x_1, \dots, x_m) = 0, \qquad [\lambda : \mu] \in \mathbb{CP}^1.$$

The converse is given by Darboux's theorem:

Theorem 2.12 (Theorem of Darboux, [28])**.** *A foliation $\mathcal{F}$ on $\mathbb{C}P^m$ admits a rational first integral R as above if, and only if, $\mathcal{F}$ admits infinitely many closed (i.e., algebraic) leaves.*

Darboux's theorem may be regarded as a type of Reeb stability theorem for foliations on $\mathbb{C}P^m$.

2.13 Proof of Darboux's theorem

Let us give a brief idea of the proof of Darboux's theorem:

- Choose an affine space $\mathbb{C}^m \subset \mathbb{C}P^m$ with affine coordinates $(x_1, \ldots, x_m)$. Then let $\omega\,(x_1, \ldots x_m)$ be a polynomial one-form defining $\mathcal{F}$ in $\mathbb{C}^m$, with $\operatorname{sing}(\omega)$ of codimension $\geqslant 2$.

Lemma 2.14. *An algebraic hypersurface $\Lambda \subset \mathbb{C}P^m$ is invariant by $\mathcal{F}$ if, and only if, $\Lambda \cap \mathbb{C}^m$ is given by an irreducible polynomial $f\,(x_1, ..., x_m)$ such that $\frac{\omega \wedge df}{f}$ is also polynomial.*

Indeed, if ω and df have $(f = 0)$ as a common invariant hypersurface, then $\omega \wedge df$ vanishes on $(f = 0)$ and is therefore divisible by f, yielding a polynomial two-form.

When $(f = 0)$ is invariant, define the polynomial two-form

$$\theta_f := \frac{\omega \wedge df}{f}.$$

Its coefficients have degree at most $\deg \omega - 1$.

Proof of Darboux's theorem. Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of algebraic leaves of $\mathcal{F}$. For each $S_\alpha \in \mathcal{S}$, choose an irreducible polynomial f_α defining $S_\alpha \cap \mathbb{C}^m$.

Let $\Omega^2_{\leq d-1}(\mathbb{C}^m)$ denote the finite-dimensional vector space of polynomial two-forms whose coefficients have degree at most $d - 1$, where $d = \deg \omega$. Consider the linear map

$$\tau \colon \mathbb{C}^{(\mathcal{A})} \longrightarrow \Omega^2_{\leq d-1}(\mathbb{C}^m), \qquad (\lambda_\alpha) \longmapsto \left(\sum_{\alpha \in \mathcal{A}} \lambda_\alpha \frac{df_\alpha}{f_\alpha} \right) \wedge \omega,$$

where $\mathbb{C}^{(\mathcal{A})}$ is the vector space of finitely supported families $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$.

Since $\mathcal{A}$ is infinite and the target is finite-dimensional, $\ker\tau$ has dimension at least two. Hence there exist polynomials $f_1, \ldots, f_\ell$ and linearly independent vectors $(\lambda_1, \ldots, \lambda_\ell)$ and $(\mu_1, \ldots, \mu_\ell)$ such that

$$\left(\sum_{j=1}^{\ell} \lambda_j \frac{df_j}{f_j}\right) \wedge \omega = 0, \qquad \left(\sum_{j=1}^{\ell} \mu_j \frac{df_j}{f_j}\right) \wedge \omega = 0.$$

Therefore the two rational one-forms are proportional: there exists a rational function $R(x_1, \ldots, x_m)$ such that

$$\sum_{j=1}^{\ell} \lambda_j \frac{df_j}{f_j} = R \sum_{j=1}^{\ell} \mu_j \frac{df_j}{f_j}.$$

Both logarithmic one-forms are closed. Differentiating the preceding identity gives

$$dR \wedge \left(\sum_{j=1}^{\ell} \mu_j \frac{df_j}{f_j}\right) = 0.$$

Since the second factor defines the same codimension-one foliation as ω, it follows that $dR \wedge \omega = 0$. Thus R is a rational first integral of $\mathcal{F}$. It is nonconstant because the vectors (λ_j) and (μ_j) are linearly independent.

□

2.15 Generalizations of Darboux's theorem

We now consider extensions of Darboux's theorem. In this direction, we have the following result of E. Ghys:

Theorem 2.16 (Ghys, [9])**.** *Let $\mathcal{F}$ be a holomorphic foliation with singularities, of codimension one on a connected compact complex manifold M. Then $\mathcal{F}$ has only a finite number of closed leaves, unless all leaves are closed and $\mathcal{F}$ has a meromorphic first integral in M.*

This result is proved in a manner similar to Darboux's theorem. The key fact is the following:

The field of meromorphic functions $\mu(M)$ of a compact complex manifold is a function field over $\mathbb{C}$.

Let now $\mathcal{F}$ be a foliation holomorphic with singularities of codimension one on a compact connected complex manifold M.

Question 2.17. *Suppose that there is a codimension-one analytic subvariety $N \subset M$ such that $\mathcal{F}|_{M\setminus N}$ has infinitely many closed leaves. Does $\mathcal{F}$ admit a meromorphic first integral on M?*

Example 2.18. The next couple of examples show the need to ask the leaves to be closed in the projective space $\mathbb{C}P^2$ in Darboux's theorem.

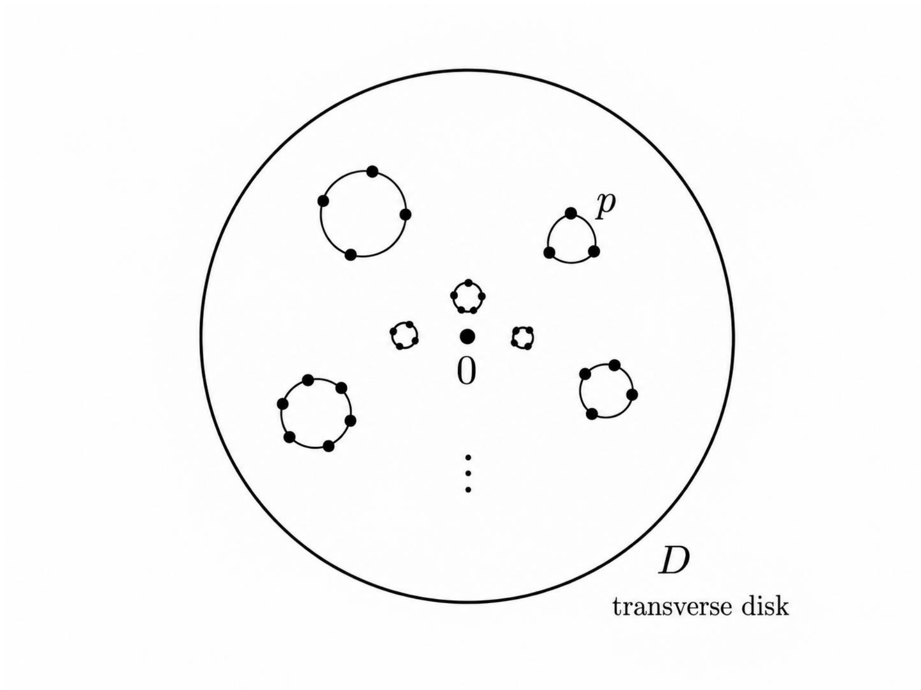
(1) We consider the foliation $\mathcal{F}$ on $\mathbb{C}P^2$ given in $\mathbb{C}^2$ by $\omega = dy - ydx$. Then $\mathcal{F}|_{\mathbb{C}^2}$ has the holomorphic first integral $f = ye^{-x}$ which is entire in $\mathbb{C}^2$. Thus all leaves of $\mathcal{F}$ are closed in $\mathbb{C}^2$ but $\mathcal{F}$ does not admit a rational (meromorphic) first integral. In order to see this one can perform the reduction of singularities of $\mathcal{F}$ in the line at infinity $l_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2$ and observe that there is some saddle-node singularity, which is incompatible. Another possibility is to observe that the function $f = ye^{-x}$ does not admit an extension to any point in l_∞ as a meromorphic function.

(2) $\omega = (x+y)dy - ydx$ (Poincaré-Dulac normal form). The foliation $\mathcal{F} = \omega = 0$ in $\mathbb{C}^2$ has closed leaves off $(y = 0)$ but admits no meromorphic first integral on $\mathbb{C}^2$.

The nature of the singularities of $\mathcal{F}$ along the curve N plays an important role in answering the above question. Let us focus on the two-dimensional case.

2.18.1 Reduction of singularities

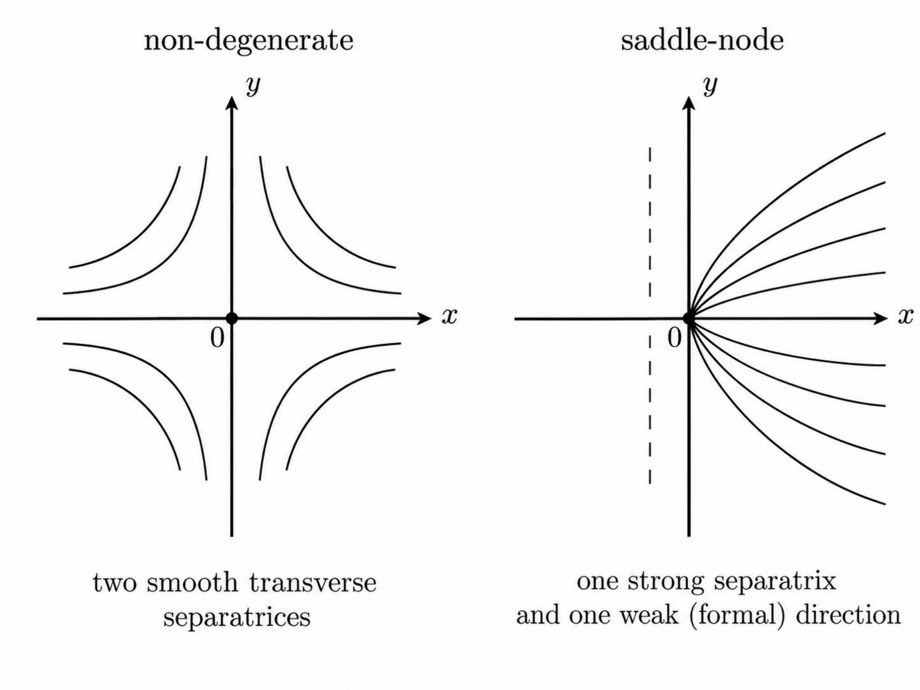
Let $\mathcal{F}$ be a germ of a holomorphic foliation with an isolated singular point p on a complex surface M^2. In local coordinates (x, y) centered at p, the *blow-up* of M^2 at p is described by the charts $y = tx$ and $x = uy$.



Seidenberg's reduction theorem [31] ensures that, after finitely many successive blow-ups, we obtain a proper holomorphic map $\pi\colon \widetilde{M}^2 \to M^2$ such that the transformed foliation $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ is defined in a neighborhood of the divisor

$$E = \pi^{-1}(p) = \bigcup_{j=1}^{r} \mathbb{P}_j.$$

The components $\mathbb{P}_j$ are projective lines meeting transversely, with no cycles or triple points, and each component is either invariant by or generically transverse to $\widetilde{\mathcal{F}}$.



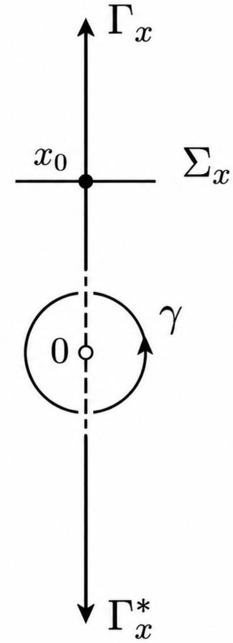
Moreover, every singularity of $\widetilde{\mathcal{F}}$ is of one of the following two types:

(i) (non-degenerate)

$$x\,dy - \lambda y\,dx + \text{higher-order terms} = 0, \qquad \lambda \in \mathbb{C} \setminus \mathbb{Q}_+;$$

(ii) (saddle-node)

$$x^{k+1}\,dy - \big(y(1+\lambda x^k) + \text{higher-order terms}\big)\,dx = 0, \qquad k \in \mathbb{N}, \quad \lambda \in \mathbb{C}.$$



Given a codimension one holomorphic foliation $\mathcal{F}$ on a complex surface M and an analytic curve $S \subset M$, we shall say that $\mathcal{F}\big|_S$ *is free of saddle-nodes* if the reduction of singularities in $\operatorname{Sing}(\mathcal{F}) \cap S$ exhibits no saddle-node in its final picture.

We may then state our main result as follows:

Theorem 2.19 ([5])**.** *Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with singularities on a connected compact complex surface N^2, and let $S \subset N$ be a smooth rational curve such that $\mathcal{F}|_S$ is free of saddle-nodes. Then $\mathcal{F}$ admits a meromorphic first integral $f\colon N^2 \dashrightarrow \mathbb{P}^1$ if and only if $\mathcal{F}$ has infinitely many closed leaves in $N \setminus S$.*

Alternatively, we may state the result as follows.

Theorem 2.20 ([5])**.** *Let $\mathcal{F}$, N, and S be as above. Assume that $\mathcal{F}$ has infinitely many closed leaves in $N \setminus S$. Then $\mathcal{F}$ admits a meromorphic first integral $f\colon N^2 \dashrightarrow \mathbb{P}^1$ if and only if $\mathcal{F}|_S$ is free of saddle-nodes.*

Finally, we have:

Theorem 2.21 ([5]). *Let $\mathcal{F}$ be a holomorphic foliation by curves in a compact complex connected surface M^2. Assume that we have a rational curve $S \subset M$ and that $\mathcal{F}$ has infinitely many closed leaves in $M \setminus S$. Then $\mathcal{F}$ admits a meromorphic first integral $f\colon M \dashrightarrow \mathbb{P}^1$ if, and only if, $\mathcal{F}\big|_S$ is free of saddle-nodes.*

If a foliation $\mathcal{F}$ on M admits a meromorphic first integral $f\colon M \dashrightarrow \mathbb{P}^1$, then, given any analytic curve $S \subset M$, the leaves of $\mathcal{F}$ are closed in $M \setminus S$ and $\mathcal{F}\big|_S$ is free of saddle-nodes (indeed, a saddle-node singularity does not admit a meromorphic first integral, since its strong separatrix has a nonperiodic local holonomy map tangent to the identity [17]).

Theorem 2.21 implies that a polynomial vector field X with isolated singularities in $\mathbb{C}^2$, having infinitely many orbits which are closed in $\mathbb{C}^2 \setminus \mathrm{Sing}(X)$, admits a rational first integral if, and only if the corresponding foliation $\mathcal{F}_X$ in $\mathbb{C}P^2$ is free of saddle nodes in the line at infinity $l_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2$.

Let us explain why Darboux's theorem is a consequence of our Theorem 2.21. First we observe that for an algebraic foliation $\mathcal{F}$ on $\mathbb{C}P^n$, a leaf $L \subset \mathbb{C}P^n$ is closed (off the singular set $\mathrm{Sing}(\mathcal{F})$) if, and only if, it has an analytic closure of codimension one in $\mathbb{C}P^n$ (this is a consequence of Remmert-Stein extension theorem for analytic sets [12]), in which case (by Chow's theorem [10]) the leaf is (contained in an) algebraic (hypersurface). Hence, an algebraic foliation $\mathcal{F}$ on $\mathbb{C}P^n$ admits infinitely many algebraic invariant hypersurfaces if, and only if, $\mathcal{F}$ admits infinitely many closed leaves. The second point is that, for an algebraic foliation $\mathcal{F}$ on $\mathbb{C}P^2$ admitting infinitely many invariant algebraic curves, given any algebraic curve $S \subset \mathbb{C}P^2$ the leaves of $\mathcal{F}$ are closed in $\mathbb{C}P^2 \setminus S$. Moreover, we may choose S in such a way that S is rational and that $\mathrm{Sing}(\mathcal{F}) \cap S = \emptyset$. Finally, for the $n \geq 3$-dimensional case we consider a linear embedding $\phi\colon \mathbb{C}P^2 \to \mathbb{C}P^n$ which is in general position with respect to $\mathcal{F}$ (in the sense of [4]) and prove the existence of a rational first integral for the restriction $\mathcal{F}\big|_{\mathbb{C}P^2} := \phi^*(\mathcal{F})$, then by induction in the final dimension n we extend this first integral to a rational first integral for $\mathcal{F}$ in $\mathbb{C}P^n$ (see [4]).

Theorem 2.21 above may be seen as an extension of Ghys' theorem 2.16 where we allow the leaves accumulate not only at points but also at some rational curve, provided it contains no saddle-nodes of the foliation. Indeed, Theorem 2.21 is a consequence of a more complete result that we shall describe below. In what follows, by a *holomorphic foliation by curves* on a complex manifold M we shall mean a pair $\mathcal{F} = (\mathcal{F}_0, \mathrm{Sing}(\mathcal{F}))$ where

$\mathrm{Sing}(\mathcal{F}) \subset M$ is a discrete set and $\mathcal{F}_0$ is a (regular) holomorphic foliation of dimension one on the open subset $M \setminus \mathrm{Sing}(\mathcal{F})$. We shall refer to $\mathrm{Sing}(\mathcal{F})$ as the *singular set* of $\mathcal{F}$. We may assume that $\mathrm{Sing}(\mathcal{F})$ is minimal in the sense that $\mathcal{F}_0$ cannot be extended to any neighborhood of a point $p \in \mathrm{Sing}(\mathcal{F})$ as a regular holomorphic foliation. The *leaves* of $\mathcal{F}$ are the leaves of the regular foliation $\mathcal{F}_0$. Given an open subset $U \subset M$, the *restriction* of $\mathcal{F}$ to U is the foliation $\mathcal{F}_U := \mathcal{F}\big|_U = i_U^*(\mathcal{F})$ given as the pull-back of $\mathcal{F}$ by the inclusion $i_U \colon U \to M$. In particular we have that the leaves of $\mathcal{F}_U$ are the connected components of the intersections $L \cap U$ where L runs through all the leaves of $\mathcal{F}$, and the singular set is $\mathrm{Sing}(\mathcal{F}_U) = \mathrm{Sing}(\mathcal{F}) \cap U$.

We are now ready to state our more general result:

Theorem 2.22. *Let $\mathcal{F}$ be a holomorphic foliation by curves in a compact complex connected surface M^2 such that $\mathcal{F}$ has infinitely many closed leaves in $M \setminus S$ for some compact analytic curve $S \subset M$. Then $\mathcal{F}$ admits a meromorphic first integral $f \colon M \dashrightarrow \mathbb{P}^1$ provided that one of the following occurs:*

1. *S is not invariant.*
2. *S is invariant by $\mathcal{F}$ and $\mathrm{Sing}(\mathcal{F}) \cap S$ contains a non-dicritical singularity p for which the reduction of $\mathcal{F}$ at p is free of saddle-nodes; or all the singularities in $\mathrm{Sing}(\mathcal{F}) \cap S$ are dicritical and S is a rational curve.*

There is one situation which is not covered by Theorem 2.22. It is the case where S is not rational, $S \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$, $\mathcal{F}$ has closed leaves in $M \setminus S$ but only finitely many fail to accumulate at S. In this case, by Camacho-Sad index theorem ([3]) the self-intersection number of S is zero $S \cdot S = 0$. This means that there is a fundamental system of neighborhoods of S in M which are foliated by smooth copies of S, each of these neighborhoods intersecting some closed leaf in $M \setminus S$.

Chapter 3

Holomorphic first integrals

3.1 Reduction of singularities – revisited

Let $\mathcal{F}$ be a germ of holomorphic foliation with singularity at $0 \in \mathbb{C}^2$. Given any small enough neighborhood U of $0 \in \mathbb{C}^2$, we have a representative $\mathcal{F}$ for the germ, given by a holomorphic 1-form $\omega = A(x,y)dx + B(x,y)dy$, with A, B holomorphic functions in U, with a common zero only at the origin. As already mentioned before, the desingularization theorem of Seidenberg ([31]) states, in few words, that after finitely many successive quadratic blow ups at singular points, the foliation $\mathcal{F}$ is transformed into a foliation $\tilde{\mathcal{F}}$ in a surface $\tilde{U}$ with a finite number of singularities, all of them *simple* and lying in the exceptional divisor. This means that around any singular point $\tilde{p} \in \mathrm{Sing}(\tilde{\mathcal{F}}) \subset \tilde{U}$ there is a local coordinate chart (z,w) centered at $\tilde{p}$ where $\tilde{\mathcal{F}}$ is given by a holomorphic 1-form $\tilde{\omega}(z,w) = \tilde{A}(z,w)dz + \tilde{B}(z,w)dw$, with $\tilde{A}(0) = \tilde{B}(0) = 0$, and $\tilde{\omega}$ belongs to one of the following categories:

1. $\tilde{\omega} = z\,dw - \lambda w\,dz + h.o.t. = 0$ and λ is not a positive rational number, i.e. $\lambda \notin \mathbb{Q}_+$ (*non-degenerate singularity*), where $h.o.t.$ stands for higher order terms.

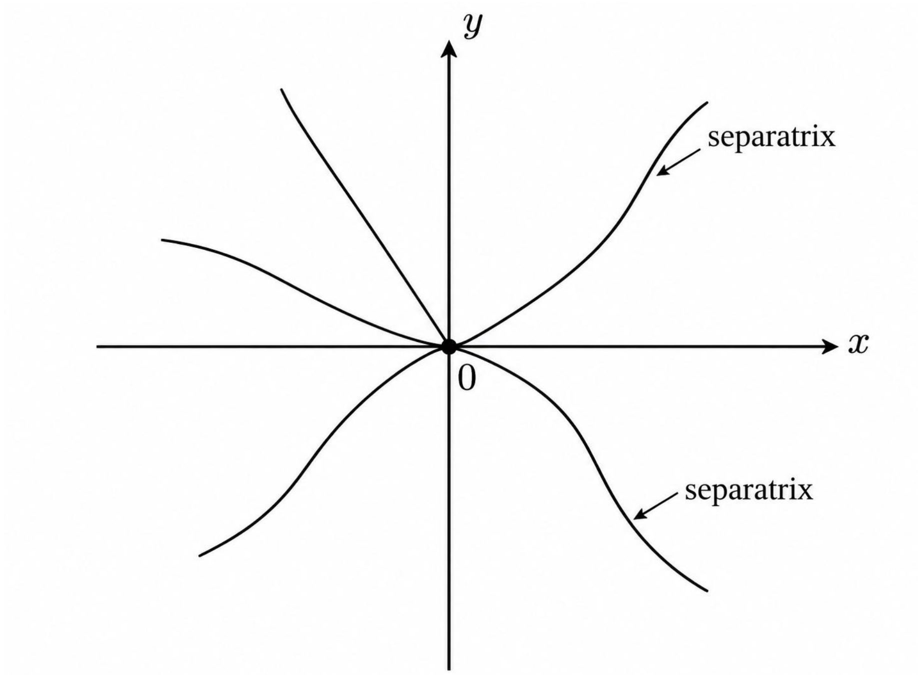
2. $\tilde{\omega} = w^{p+1}dz - [z(1+\lambda w^p) + h.o.t.]\,dw = 0$, $p \geq 1$. This case is called a *saddle-node* and it is extensively studied (cf. [17]).

A foliation singularity is *non-dicritical* if it admits only a finite number of local separatrices. This is equivalent to the fact that the *exceptional divisor*, i.e., the union of projective lines introduced by its reduction of singularities, is invariant.

Definition 3.2 (free of saddle-nodes singularity)**.** Given a foliation by curves $\mathcal{F}$ on a complex surface M, and an analytic subset $X \subset M$, we shall say that $\mathcal{F}\big|_X$ *is free of saddle-nodes* if for every singular point $p \in \operatorname{sing}(\mathcal{F}) \cap X$ its reduction of singularities exhibits no saddle-node.

3.3 The theorem of Mattei-Moussu

We consider a holomorphic function $f\colon U \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ defined in a neighborhood $U \subset \mathbb{C}^n$ of a point $p \in U$. We assume that f is not constant and consider the foliation $\mathcal{F}_0(f)$ defined by the level hypersurfaces of f in U.



If $df(x) \neq 0$ for every $x \in U$, then $\mathcal{F}_0(f)$ is a holomorphic foliation of codimension one on U, without singularities. Its leaves are the connected components of the levels $f^{-1}(c)$, $c \in \mathbb{C}$. We then say that f is a holomorphic first integral for $\mathcal{F}_0(f)$ on U. We also observe that every leaf L of $\mathcal{F}_0(f)$ is closed in U.

Assume now that p is a critical point (also called a *singularity*) of f. Since we are interested in the germ of $\mathcal{F}$ at p, we may assume that U is arbitrarily small. In particular, using the algebraic-analytic properties of germs of holomorphic functions in the ring $\mathcal{O}_{n,p} = \{$ germs of holomorphic functions at $p\}$. We may assume that:

- (i) The singular set $\operatorname{sing}(\mathcal{F}) = \{q \in U; df(q) = 0\}$ is a codimension $\geqslant 2$ analytic subset.

Using the Stein factorization lemma, we may also assume that:

- (ii) The level hypersurfaces $f^{-1}(c), c \in \mathbb{C}$ are connected and therefore correspond to the leaves of $\mathcal{F}_0(f)$ in $U \setminus \operatorname{Sing}(\mathcal{F})$.

Since for every point $q \in \operatorname{Sing}(\mathcal{F})$ we have $df(q) = 0$ and $\operatorname{codim}_q \operatorname{sing}(df) \geq 2$ we conclude that $\omega = df$ defines a holomorphic foliation $\mathcal{F}(f)$ of codimension one, with singularities in U and $\mathcal{F}(f)$ having as singular set $\operatorname{Sing}(\mathcal{F}(f)) := \operatorname{sing}(df)$.

We then write $\mathcal{F}(f) = (\mathcal{F}_0(f), \operatorname{Sing}(\mathcal{F}))$, in accordance with Definition 2.2.

Definition 3.4. A foliation $\mathcal{F}$ in U as above is said to admit a holomorphic first integral if there exists a nonconstant holomorphic function $f\colon U \subset \mathbb{C}^n \longrightarrow \mathbb{C}$ such that $\mathcal{F} = \mathcal{F}(f)$ in the sense above.

Summarizing our notations and discussion above we have:

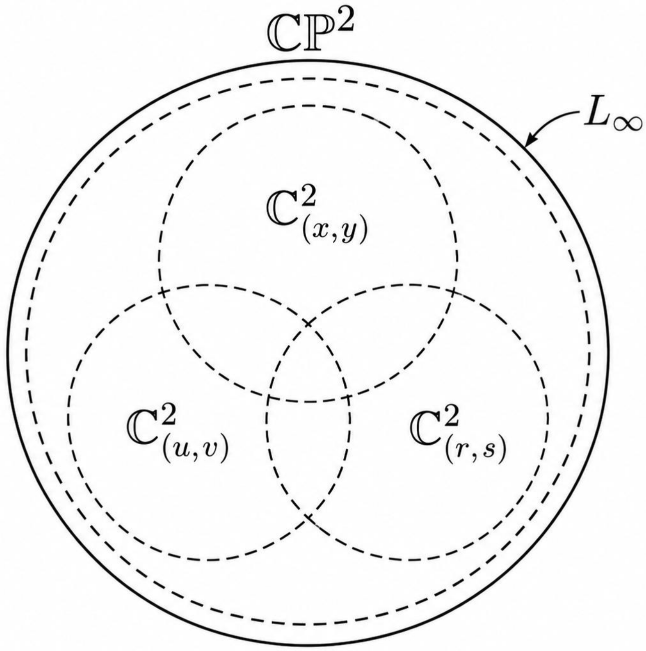
Proposition 3.5. *A germ of holomorphic foliation, of codimension one, with singularities, $\mathcal{F}$ at $p \in \mathbb{C}^n$ admits a holomorphic first integral if, and only if, there is a germ of holomorphic function $f \in \mathcal{O}_{n,p}$ such that $\mathcal{F} = \mathcal{F}(f)$ for some representatives $\mathcal{F}(f)$ and $f\colon U \to \mathbb{C}$ of the germs $\mathcal{F}$ and f in some neighborhood U of p in $\mathbb{C}^n$.*

If that is the case then:

- *(a) Every leaf of $\mathcal{F}$ is closed in $U \setminus \operatorname{sing}(\mathcal{F})$.*
- *(b) There is only a finite number of leaves L of $\mathcal{F}$ that accumulate at $\operatorname{sing}(\mathcal{F})$.*

Proof. We may assume that $\operatorname{sing}(\mathcal{F})$ is connected and contains p. For simplicity, replacing f by $f - f(p)$, we assume that $f(p) = 0$.
Given a leaf L of $\mathcal{F}$ in U, we have $L \subset f^{-1}(c)$ for some $c \in \mathbb{C}$. If $c \neq 0$, then $f^{-1}(c) \cap f^{-1}(0) = \emptyset$. Since $\operatorname{sing}(\mathcal{F}) \subset f^{-1}(0)$, it follows that $\overline{L} \cap \operatorname{sing}(\mathcal{F}) = \emptyset$.

This shows that the only leaves of $\mathcal{F}$ that accumulate at $\mathrm{Sing}(\mathcal{F})$ are those contained in the level zero $f^{-1}(0)$.
This set is a union of a finite number of irreducible (branches) of codimension one analytic subsets. $\square$



Definition 3.6. A leaf L of a foliation germ $\mathcal{F}$ such that $0 \in \bar{L} \subset \mathrm{sing}(\mathcal{F}) \cup L$ is called a separatrix of $\mathcal{F}$. In this case $L \subset \bar{L} \subset L \cup \mathrm{sing}(\mathcal{F})$ is an analytic codimension one invariant subset.

Definition 3.7. $\mathcal{F}$ is said to be non-dicritical if it has only a finite number of separatrices.

We may therefore conclude that:

Conclusion 3.8. *If a germ of foliation, holomorphic of codimension one, with singularities, admits a holomorphic first integral then it has all leaves closed off the singular set and is non-dicritical.*

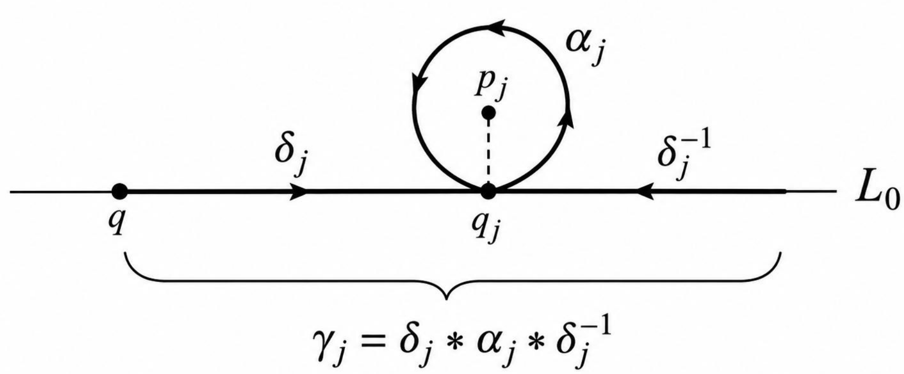
As a converse of this result we have:

Theorem 3.9 (Theorem of Mattei-Moussu, [19])**.** *A germ of holomorphic foliation $\mathcal{F}$ of codimension one at $0 \in \mathbb{C}^n, n \geqslant 2$, admits a holomorphic first integral if, and only if, $\mathcal{F}$ is non-dicritical and all its leaves are closed off* $\mathrm{Sing}(\mathcal{F}) = \{0\}$.

From the above result we reach some important conclusions:

Conclusion 3.10. *A germ holomorphic foliation topologically conjugate to a foliation admitting a holomorphic first-integral, also admits a holomorphic first integral.*

Remark 3.11. Two germs of codimension one holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ at $0 \in \mathbb{C}^n$ are *topologically conjugate* if there is a homeomorphism $\Phi\colon (U, 0) \longrightarrow (\Phi(U), 0)$ where U and $\Phi(U)$ are open neighborhoods of 0 in $\mathbb{C}^n$, such that $\Phi(0) = 0$ and Φ takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}'$.



3.12 Irreducible singularities

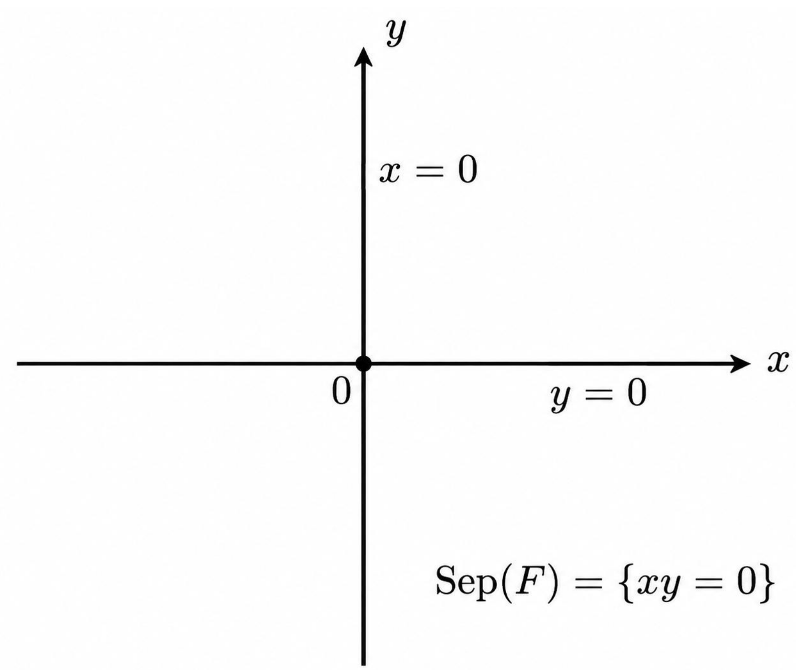
Let us examine the case of an irreducible singularity regarding the existence of a holomorphic first integral. We focus on the case $n = 2$

1st. case $\mathcal{F}$ is a non-degenerate germ.
We choose local coordinates such that $\mathcal{F}$ is given by an expression

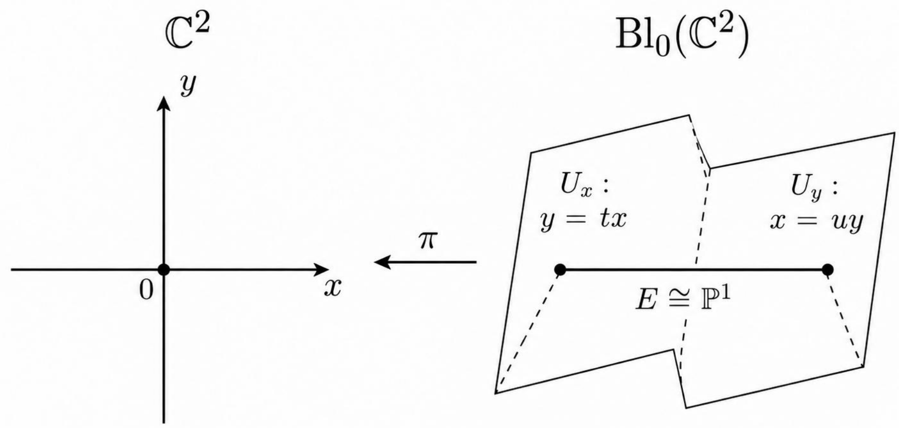
$$\omega = x(1 + A(x,y))\,dy - \lambda y(1 + B(x,y))\,dx = 0$$

where $A, B \in \mathcal{O}_2$ and $A(0,0) = B(0,0) = 0$ and $0 \neq \lambda \in \mathbb{C}\backslash\mathbb{Q}_+$.

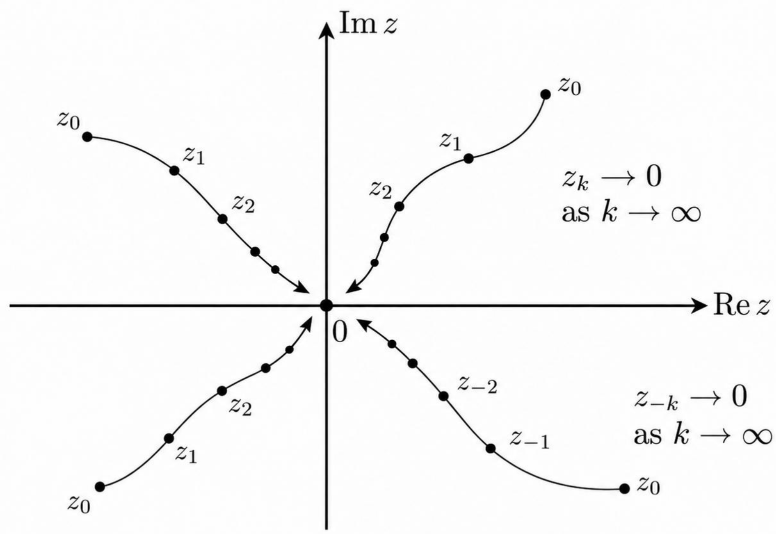
In particular the set of separatrices is given by $\text{Sep}(\mathcal{F}) = \{xy = 0\}$, it is the union of the two coordinate axes.



Let us analyze the geometry of this case.
Take a simple loop $\gamma : S^1 \to \Gamma_x^*$ where Γ_x is the separatrix $(x = 0)$ and $\Gamma_x^* = \Gamma_x\backslash 0$.



The corresponding holonomy map h_γ can be obtained as follows: Fix a transverse disk $\Sigma_x : (x = x_0)$ for some $x_0 \in \Gamma_x^*$ close enough to 0. In the solid torus $\gamma \times \Sigma_x \simeq S^1 \times \mathbb{D}^2$. We have an induced transversely holomorphic flow $\mathcal{L}$ which is transverse to the disks $\gamma(t) \times \sum_x, \quad t \in S^1$ and has a central periodic orbit $\gamma(S^1) \times \{x_0\}$.



$z \mapsto az, \quad |a| \neq 1$

All forward and backward iterates tend to 0.
There are no nontrivial periodic orbits.

The holonomy map $h_\gamma\left(\Sigma_x, x_0\right) \to \left(\Sigma_x, x_0\right)$ is then the Poincaré map of this periodic orbit of $\mathcal{L}$. It can be calculated as follows:

Parametrize $f(t) = \left(x_0 \cdot e^{it}, 0\right), 0 \leq t \leq 2\pi$. Then write $\mathcal{F}$ as

$$x(1 + A(x,y))\, dy - \lambda y(1 + B(x,y))\, dx = 0,$$
$$\frac{dy}{dx} = \frac{\lambda y(1 + B(x,y))}{x(1 + A(x,y))}.$$

The flow $\mathcal{L}$ is then given by

$$\frac{dy/dt}{dx/dt} = \frac{\lambda y\left(1 + B\left(x_0 e^{it}, y\right)\right)}{x_0 e^{it}\left(1 + A\left(x_0 e^{it}, y\right)\right)}, \qquad y = y(t),$$

and

$$\frac{dx}{dt} = \frac{d\left(x_0 \cdot e^{it}\right)}{dt} = i x_0 \cdot e^{it}.$$

Then we have

$$\mathcal{L}: \quad \frac{dy}{dt} = i\lambda y \frac{1 + B\left(x_0 e^{it}, y\right)}{1 + A\left(x_0 e^{it}, y\right)}$$

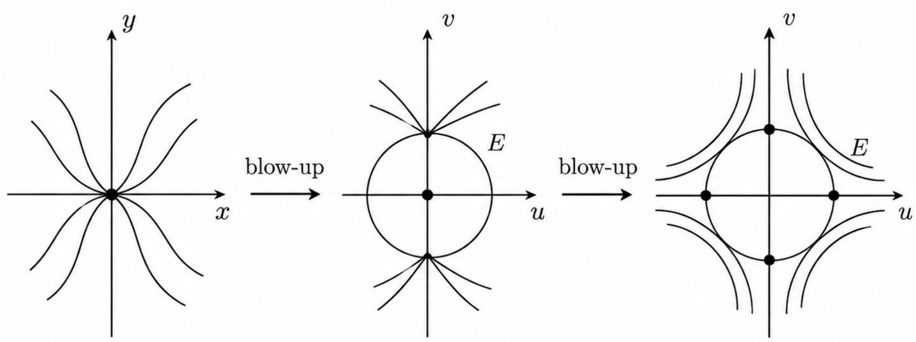
and the corresponding holonomy map is given by $h(y) = y(2\pi, y)$ where $y(t, y)$ is the solution of the ODE.

$$y' = i\lambda y \, \frac{1 + B\left(x_0 e^{it}, y\right)}{1 + A\left(x_0 e^{it}, y\right)}$$

with initial condition $\quad y(0, y) = y_0.$

Since the quotient in the differential equation is $1 + \cdots$, we conclude that $h(y) = e^{2\pi i \lambda} y + O(y^2)$.

The hypotheses on $\mathcal{F}$ imply that: The pseudo-orbits of h, as a map germ $(\Sigma_x, x_0) \to (\Sigma_x, x_0)$, are closed (recall that the only separatrix intersecting Σ_x is the axis $(y = 0)$, which corresponds to the fixed point of h).



(1) Singular germ on $(\mathbb{C}^2, 0)$

(2) After one blow-up (on the blow-up surface)

(3) Resolved foliation with simple singularities on E

In particular, given any disk $D \subset \Sigma_x$, centered at x_0, the number of iterates of any point $y \in D$ by h in the disk D is finite.

We denote by $\mathrm{Diff}(\mathbb{C}, 0)$ the group of germs of holomorphic diffeomorphisms at the origin. We say that $h \in \mathrm{Diff}(\mathbb{C}, 0)$ is periodic if there exists $k \in \mathbb{N} \setminus \{0\}$ such that $h^k = \mathrm{id}$. A very important tool in the Mattei–Moussu theorem is the following topological criterion for periodicity:

Definition 3.13. Given $h \in \text{Diff}(\mathbb{C},0)$ with a representative $h\colon U \longrightarrow h(U)$, and a subset $V \subset U$, the V-orbit of a point $x \in V$ under h is the set

$$\mathcal{O}_{V,h}(x) = \{h^k(x);\ h^j(x) \in V, \forall j = 1,...,k\} \cup \{h^{-k}(x);\ h^{-j}(x) \in V, \forall j = 1,...,k\} \cup \{x\}.$$

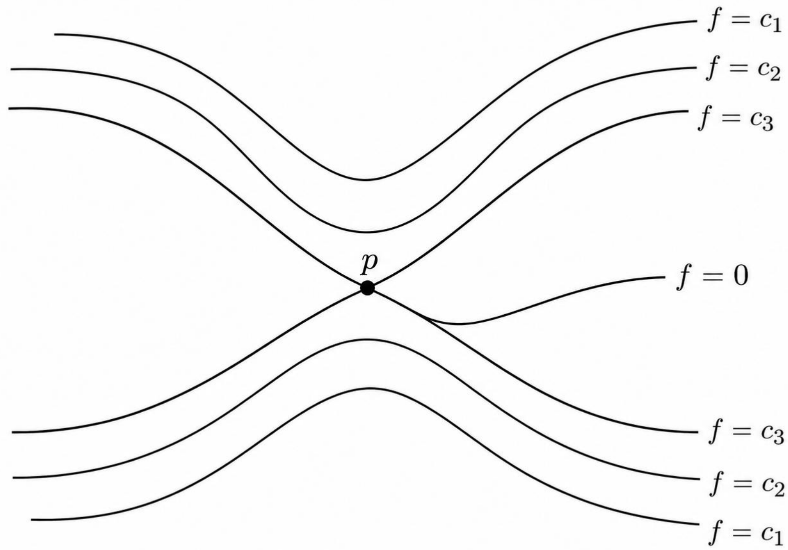
We say that h has *finite orbits* if there exist neighborhoods $V \subset U$ of the origin such that h is defined and injective on U and $\#\mathcal{O}_{V,h}(x) < \infty$ for every $x \in V$.

Proposition 3.14 ([19] Thèoréme 2 page 477)**.** *An element $h \in \text{Diff}(\mathbb{C},0)$ is periodic if, and only if, its orbits are finite.*

Proof. The proof given by Mattei-Moussu is based on a countability argument and we refer to [19] or [29] for detailed explanations. We shall offer a different proof. Let $h : \mathbb{C},0 \to \mathbb{C},0$ be a germ of a holomorphic diffeomorphism. Write $h(z) = a \cdot z + \sum_{j=2}^{+\infty} h_j \cdot z^j$ in power series, $a, h_j \in \mathbb{C}$ and $a = h'(0) \neq 0$. Write $a = e^{2\pi i\lambda}$. We analyze case by case:

1st case: $\lambda \in \mathbb{C}\backslash\mathbb{R}$ (hyperbolic case)
In this case $|a| \neq 1$, and the classical Poincaré linearization theorem ensures the existence of a holomorphic linearization: there is a coordinate $z \in (\mathbb{C},0)$ such that $f(z) = a \cdot z$.

In particular, $f^k(z) = a^k z$. Hence, if $z_0 \neq 0$, the equality $f^k(z_0) = z_0$ would imply $a^k = 1$, contradicting $|a| \neq 1$. Thus there are no nontrivial periodic orbits. Indeed, either $|a| < 1$ or $|a| > 1$, and for every $z_0 \neq 0$, either $f^k(z_0) \to 0$ or $f^{-k}(z_0) \to 0$ as $k \to +\infty$.

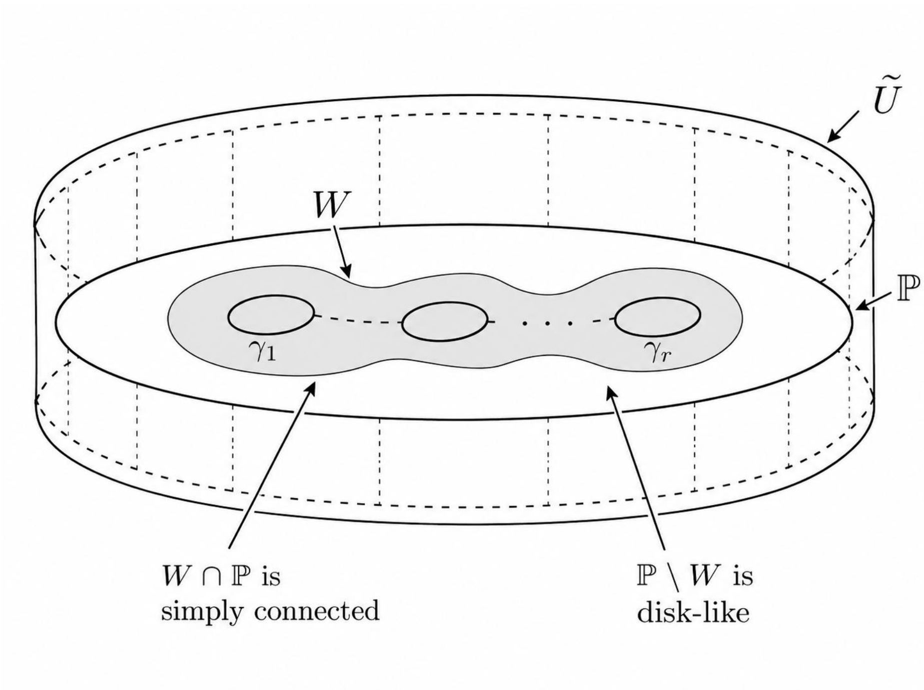


Regular level sets: $f^{-1}(c)$, $c \neq 0$
Singular level set: $f^{-1}(0)$ (contains p)

$2^{\mathbf{nd}}$ **case** : $\lambda \in \mathbb{Q}$ is a rational number.
Write $\lambda = \frac{k}{l}$ with $k, l \in \mathbb{Z}, l > 0$ and $\langle |k|, l \rangle = 1$. Put $f = h$. Then $a^l = 1$ and either

$$f(z) = h^l(z) = z + \alpha_{t+1} z^{t+1} + \alpha_{t+2} z^{t+2} + \cdots, \text{ for some } \alpha_{t+1} \neq 0, \quad t \geqslant 1 \text{ or } f(z) = h^l(z) = z, \forall z$$

If $f(z) \neq z$ then according to Camacho [2] there exist t domains called petals P_j, symmetric with respect to the t directions $\operatorname{ang}(z) = \frac{2\pi j}{t}$, $j \in \{0, \dots, t-1\}$, such that
(0) P_j is invariant by $f, \forall j$.
(1) $P_{j_1} \cap P_{j_2} = \emptyset \quad \forall j_1 \neq j_2$, each petal P_j is holomorphically equivalent to the upper half-plane $\mathbb{H}_+ = \{\operatorname{Im} z > 0\} \subset \mathbb{R}^2$
(2) For each $z \in P_j$ we have $f^m(z) \to 0$ as $m \to \infty$.
(3) For each j the map $f|_{P_j}$ is holomorphically equivalent to the translation $z \longmapsto z + i$ on $I = H'$
In particular, the dynamics of $f = h^l$ in this can be depicted as follows:



It follows that no nontrivial orbit is periodic and no nontrivial orbit is closed.

3rd case: $\lambda \in \mathbb{R} \backslash \mathbb{Q}$.

If h is analytically linearizable then it has the dynamics of an irrational rotation $h_\theta(z) = e^{2\pi i\theta} \cdot z, \theta \in \mathbb{R}\backslash\mathbb{Q}$ which means that there are concentric circles which are invariant and where the orbits are dense. In particular, no orbit is finite or closed.

Assume now that h is not analytically linearizable. We shall use following result of Cavalier:

Lemma 3.15 (Cavalier [21]). *Given a non-linearizable holomorphic map germ* $f \colon \Sigma_1, 0 \to \Sigma_1 0, f'(0) = e^{2\pi i\lambda}, \lambda \in \mathbb{R}$ *then there exists at least one infinite orbit accumulating at the origin.*

This is also a consequence of the hedgehog theorem of R. Pérez-Marco [22].

Thus we may conclude the proof of Proposition 3.14. □

Therefore:

Conclusion 3.16. *The only non-degenerate singularities that may allow holomorphic first integrals are those having a linearizable periodic holonomy map for a separatrix.*

Before going to the saddle-node case, let us study the impact of this conclusion on the foliation itself

Lemma 3.17 ([19]). *Let $\mathcal{F}$ be a germ of non-degenerate holomorphic foliation $x(1+A(x,y))dy-\lambda y(1+B(x,y))dx = 0, \quad A(0) = B(0) = 0, \lambda \in \mathbb{C}\backslash\mathbb{Q}_+$, at the origin $O \in \mathbb{C}^2$. The following conditions are equivalent:*

- *(i) $\mathcal{F}$ is analytically linearizable: there are holomorphic coordinates $(u,v) \in \mathbb{C}^2$ such that $u(0) = v(0) = 0$ and $\mathcal{F}$ is given by*

$$udv - \lambda vdu = 0$$

We may also assume that $\begin{cases} u = x \cdot u_1 \\ v = y \cdot v_1 \end{cases}$ *so that the axes are preserved.*

- *(ii) The holonomy map of some separatrix is analytically linearizable.*

Proof. We refer to [29]. The next version of these notes will include a self-contained proof. $\square$

Now we consider the saddle-node case:
The saddle-node case: There are local coordinates such that $\mathcal{F}$ writes

$$y^{k+1}\,dx - \big(x\left[1+\lambda y^k\right] + \text{higher-order terms}\big)\;dy = 0,$$

where $k \in \mathbb{N}$, $k \geq 1$, and $\lambda \in \mathbb{C}$.
The separatrix $\Gamma : (y = 0)$ is called the strong manifold of $\mathcal{F}$ and may be the unique separatrix. Its holonomy map is of the form

$$h_\Gamma(y) = y + \alpha_{k+1}y^{k+1} + \ldots \text{ for}$$

some $\alpha_{k+1} \neq 0$. In particular, as we have seen above, h_Γ has the petal dynamics depicted in the preceding figure and therefore exhibits no closed orbit. This is incompatible with the hypothesis that $\mathcal{F}$ has closed leaves outside $\operatorname{sing}(\mathcal{F}) = \{0\}$. Therefore this case is excluded.

Summarizing the above discussion we have:

Proposition 3.18. *Let $\mathcal{F}$ be a germ of irreducible singularity at $0 \in \mathbb{C}^2$. Then the following are equivalent:*
(i) $\mathcal{F}$ has closed leaves off $\operatorname{sing}(\mathcal{F}) = \{0\}$.
(ii) $\mathcal{F}$ admits a holomorphic first integral.
(iii) $\mathcal{F}$ is analytically linearizable and has the form $mx\,dy + ny\,dx = 0$ for some relatively prime $n, m \in \mathbb{N}$.

Proof. If $\mathcal{F}$ has closed leaves outside $\operatorname{sing}(\mathcal{F}) = \{0\}$, then the pseudo-orbits of the holonomy map of any separatrix are closed; by Proposition 3.14, this map is periodic. A periodic germ is analytically linearizable as a rational rotation $z \mapsto e^{2\pi i n/m} z$ for some relatively prime $n, m \in \mathbb{N}$. By Lemma 3.17, $\mathcal{F}$ is analytically conjugate to $mx\,dy + ny\,dx = 0$ and therefore admits the holomorphic first integral $\varphi = x^n y^m$ in these coordinates. The remaining assertions follow from the preceding considerations. □

Let us now state a well-known lemma we shall need in the case of a non-irreducible singularity.

Lemma 3.19. *Let $G < \operatorname{Diff}(\mathbb{C}, 0)$ be a finitely generated subgroup of germs of holomorphic diffeomorphisms. Assume that every element of G has closed local orbits. Then G is finite.*

Proof. Choose generators $f_1, \ldots, f_r$ with $G = \langle f_1, \ldots, f_r \rangle$. We first claim that G is abelian. Given $h_1, h_2 \in G$, their commutator

$$f = [h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$$

is tangent to the identity. If $f \neq \mathrm{id}$, then

$$f(z) = z + a_{k+1} z^{k+1} + \cdots, \qquad a_{k+1} \neq 0,$$

for some $k \geq 1$. Such a germ has nonclosed local orbits, by the tangent-to-the-identity dynamics described above. This contradicts the hypothesis. Hence every commutator is the identity and G is abelian.

The preceding one-dimensional dynamical analysis also shows that a holomorphic germ all of whose local orbits are closed must be periodic. Thus each generator f_j has finite order. Since G is finitely generated, abelian, and generated by elements of finite order, it is finite. □

Lemma 3.20. *Every finite subgroup $G < \mathrm{Diff}(\mathbb{C}, 0)$ is cyclic and analytically conjugate to a finite group of rotations. More precisely, if $\nu = |G|$, then in a suitable local coordinate*

$$G = \left\langle z \longmapsto e^{2\pi i/\nu} z \right\rangle.$$

Proof. Define
$$\Phi(z) = \frac{1}{|G|} \sum_{g \in G} \frac{g(z)}{g'(0)}.$$

Then $\Phi(0) = 0$ and $\Phi'(0) = 1$, so $\Phi \in \mathrm{Diff}(\mathbb{C}, 0)$. For any $g_0 \in G$, reindexing the sum by $\widetilde{g} = g \circ g_0$ gives

$$\begin{aligned} \Phi(g_0(z)) &= \frac{1}{|G|} \sum_{g \in G} \frac{g(g_0(z))}{g'(0)} \\ &= g_0'(0) \frac{1}{|G|} \sum_{\widetilde{g} \in G} \frac{\widetilde{g}(z)}{\widetilde{g}'(0)} \\ &= g_0'(0) \Phi(z). \end{aligned}$$

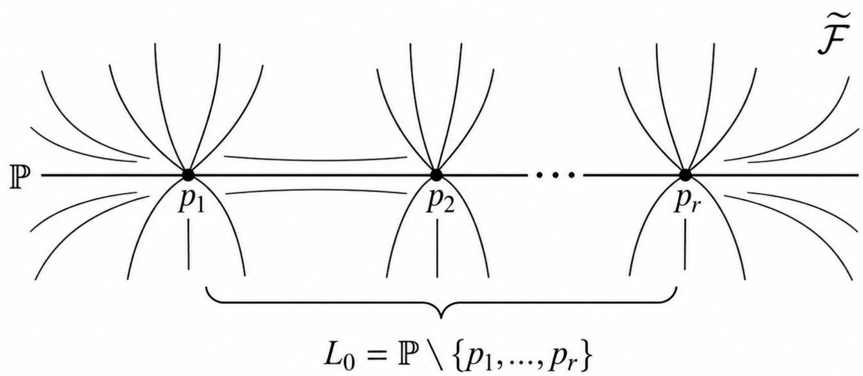
Therefore
$$\Phi \circ g_0 \circ \Phi^{-1}(z) = g_0'(0) z.$$

Thus the derivative map embeds G into the finite subgroups of $\mathbb{C}^*$, which are cyclic. The conclusion follows. □

3.21 The case of one blow-up

Let us explain the proof of Theorem 3.9 in the case where a single blow-up already gives the reduction of singularities. Suppose that $\mathcal{F}$ is non-dicritical, has closed leaves outside $\mathrm{sing}(\mathcal{F}) = \{0\}$, and can be desingularized by a single blow-up
$$\pi \colon (\widetilde{\mathbb{C}}^2, D) \longrightarrow (\mathbb{C}^2, 0).$$

Since $\mathcal{F}$ is non-dicritical, the projective line $D = \mathbb{P} = \pi^{-1}(0)$ is invariant by the pull-back foliation $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$.



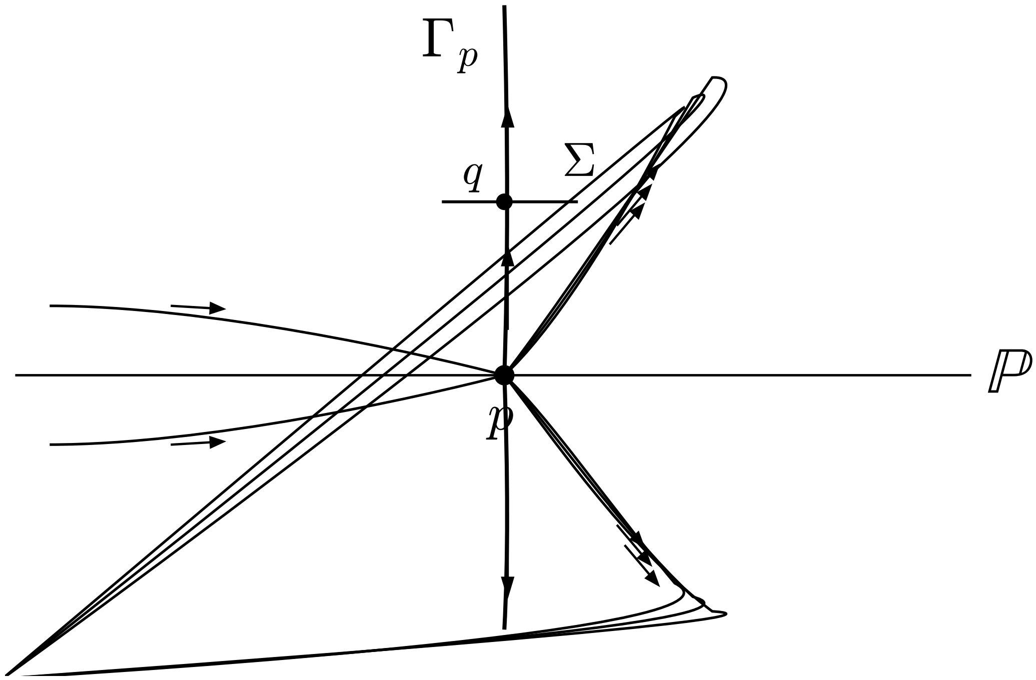
Denote by $\operatorname{Sing}(\widetilde{\mathcal{F}}) = \{p_1, \dots, p_r\}$ the singular set of $\widetilde{\mathcal{F}}$, which is contained in $\mathbb{P}$.

Claim 3.22. *Each singularity $p_j \in \operatorname{Sing}(\widetilde{\mathcal{F}})$ is nondegenerate and analytically conjugate to a form*

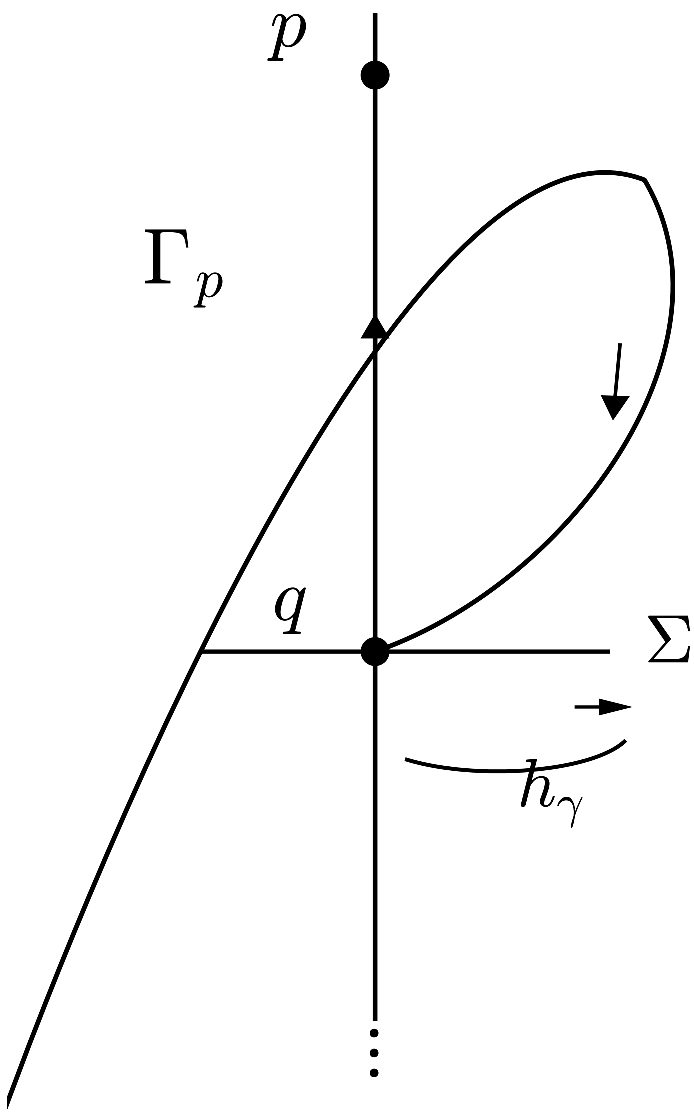
$$n_j x\, dy + m_j y\, dx = 0, \qquad m_j, n_j \in \mathbb{N}.$$

Proof of the claim. Indeed, suppose that $p \in \operatorname{Sing}(\widetilde{\mathcal{F}}) \cap \mathbb{P}$ is a saddle-node. Let Γ_p be its strong separatrix.

First case: $\Gamma_p \not\subset \mathbb{P}$.
In this case, Γ_p is transverse to $\mathbb{P}$, and the leaves of $\widetilde{\mathcal{F}}$ near a regular point $q \in \Gamma_p \setminus \{p\}$ are closed.

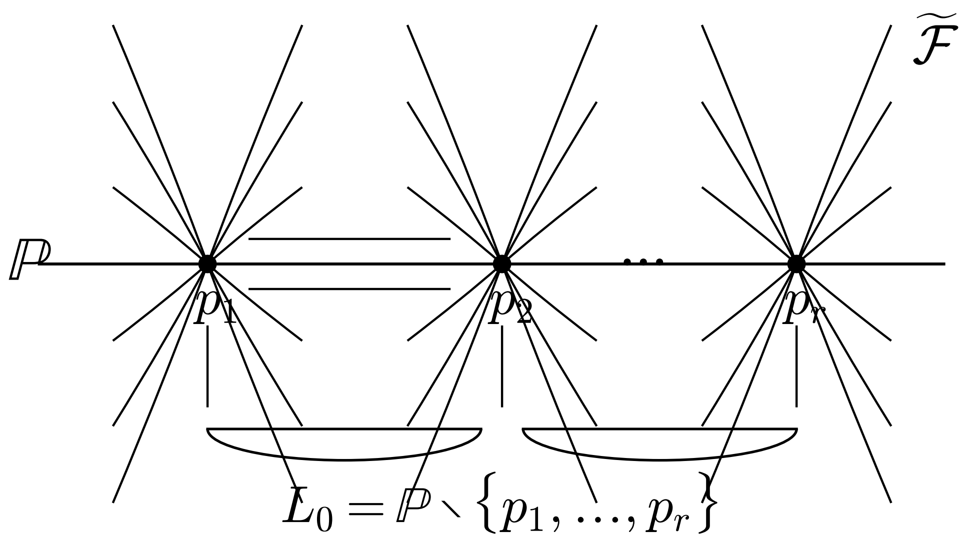


Take a transverse disk Σ centered at q, with $\Sigma \pitchfork \Gamma_p$, and a loop $\gamma\colon S^1 \to \Gamma_p \setminus \{p\}$ based at q. The corresponding holonomy map $h_\gamma\colon (\Sigma, q) \to (\Sigma, q)$ has closed orbits and is therefore periodic. This is impossible because Γ_p is the strong separatrix of the saddle-node at p.



Second case: $\Gamma_p \subset \mathbb{P}$.
Take a transverse disk Σ to $\widetilde{\mathcal{F}}$ at a regular point $q \in \Gamma_p$. Since $\mathcal{F}$ is non-dicritical, all but finitely many leaves through Σ are closed; hence the local holonomy map $h\colon (\Sigma, q) \to (\Sigma, q)$ has finite orbits and is periodic. Again, this is impossible for the strong separatrix of a saddle-node. Thus every singularity $p_j \in \mathrm{Sing}(\widetilde{\mathcal{F}}) \subset \mathbb{P}$ is nondegenerate and has the asserted normal form.

This proves Claim 3.22. □

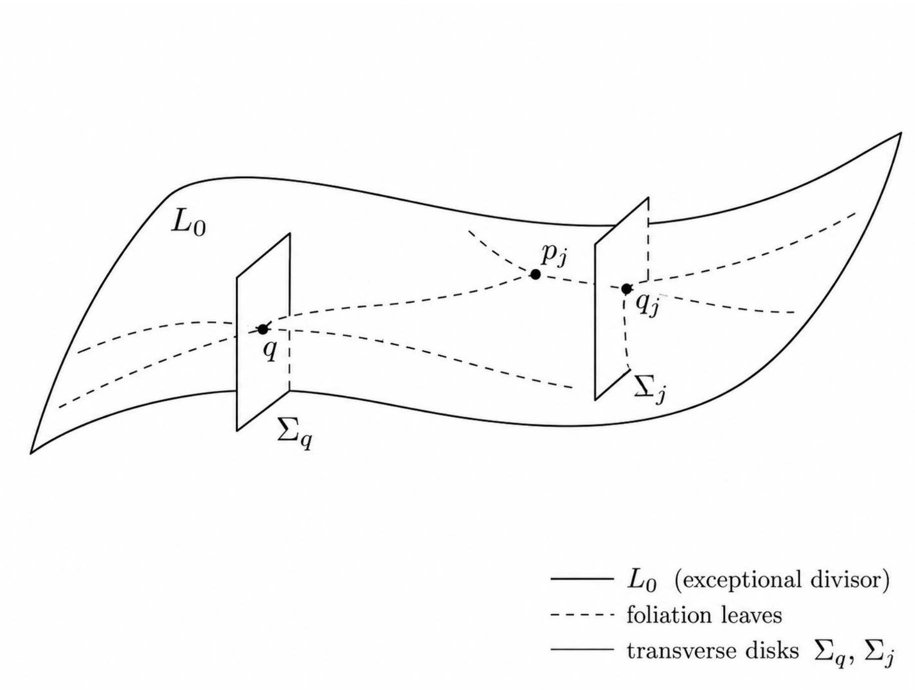


Claim 3.23. *The function constructed near the regular part of* $\mathbb{P}$ *extends holomorphically to a neighborhood of every* $p_j \in \operatorname{Sing}(\widetilde{\mathcal{F}})$*.*

Proof of the claim. Take local coordinates (x_j, y_j) on a neighborhood U_j of p_j, with $x_j(p_j) = y_j(p_j) = 0$, such that

$$\widetilde{\mathcal{F}}|_{U_j} \colon n_j x_j\, dy_j + m_j y_j\, dx_j = 0, \qquad \gcd(n_j, m_j) = 1.$$

Then $\widetilde{\mathcal{F}}|_{U_j}$ admits the local first integral $\xi_j = x_j^{m_j} y_j^{n_j}$, and $\mathbb{P} \cap U_j = (y_j = 0)$.



The local holonomy h_j of the separatrix $(y_j = 0) \subset \mathbb{P}$, calculated on the transverse disk $\Sigma_j = (x_j = x_j^0)$, is

$$h_j(y_j) = e^{-2\pi i m_j/n_j} y_j.$$

We put $L_0 = \mathbb{P} \setminus \mathrm{Sing}(\widetilde{\mathcal{F}}) = \mathbb{P} \setminus \{p_1, \ldots, p_r\}$. Then L_0 is a leaf of $\widetilde{\mathcal{F}}$. Its holonomy group $H\,(L_0) \subseteq \mathrm{Diff}(\mathbb{C}, 0)$ is called *projective holonomy* of $\mathcal{F}$.

Claim 3.24. *The projective holonomy group $H\,(L_0)$ is finite.*

Proof. Given a regular point $q \in L_0$ and a transverse disk Σ_q with $\Sigma_q \cap \mathbb{P} = \{q\}$, we identify

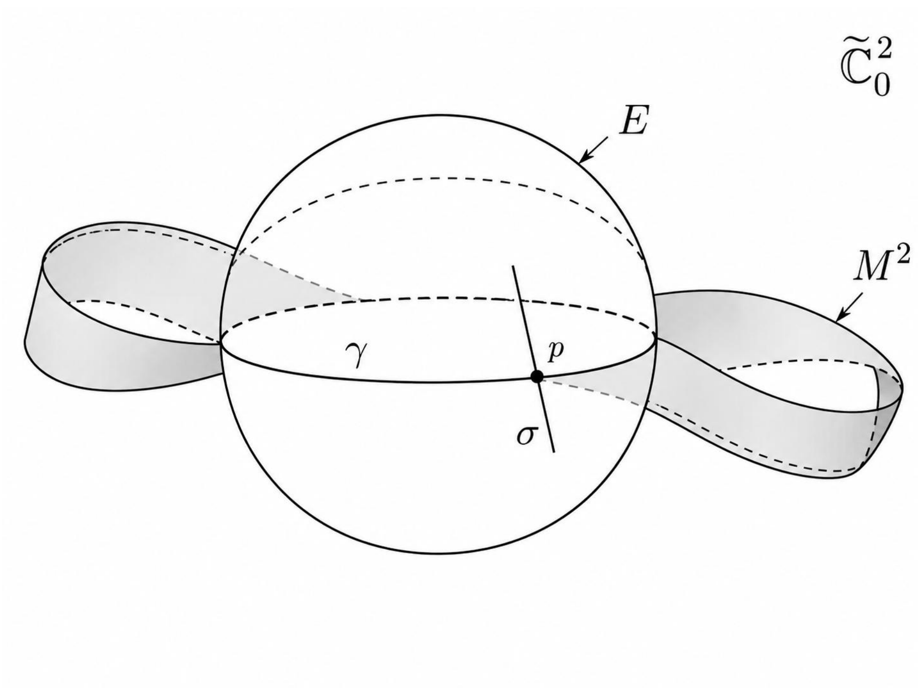
$$H(L_0) \simeq \mathrm{Hol}(\widetilde{\mathcal{F}}, L_0, \Sigma_q, q) \subset \mathrm{Diff}(\Sigma_q, q).$$

Since the leaves of $\widetilde{\mathcal{F}}$ are closed, the orbits of $H(L_0)$ are finite; hence $H(L_0)$ is finite. Consequently, there is a holomorphic coordinate y_q on Σ_q, with $y_q(q) = 0$, such that

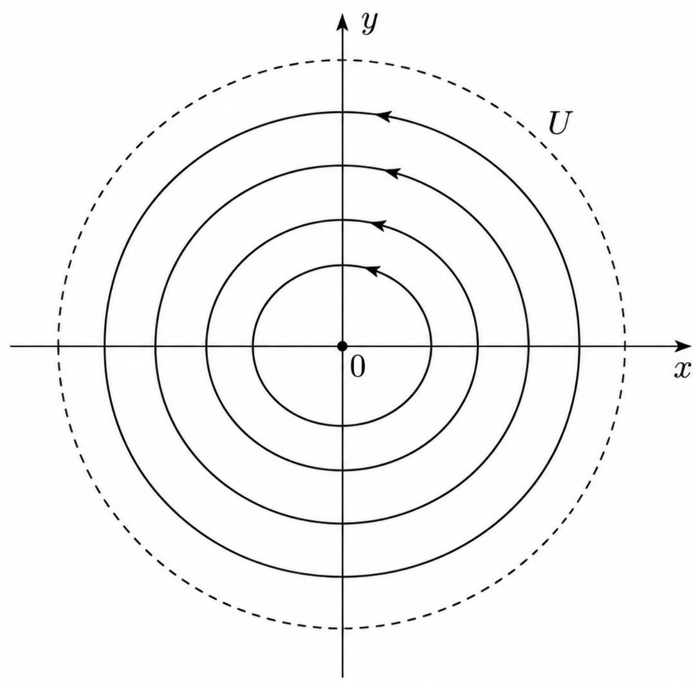
$$H(L_0) = \left\langle y_q \mapsto e^{2\pi i/\nu} y_q \right\rangle, \qquad \nu = |H(L_0)|.$$

Define $f_q(y_q) = y_q^\nu$. Then $f_q \circ h = f_q$ for every $h \in H(L_0)$. $\square$

By transversality and a local trivialization of $\widetilde{\mathcal{F}}$ near q, the function f_q extends holomorphically to a neighborhood U_q of q, constantly along the plaques of $\widetilde{\mathcal{F}}$ and invariant under the projective holonomy group.



Using the invariance of f_q under $H(L_0)$, we extend it constantly along the leaves of $\widetilde{\mathcal{F}}$ to a holomorphic function f_W on a neighborhood W of L_0.

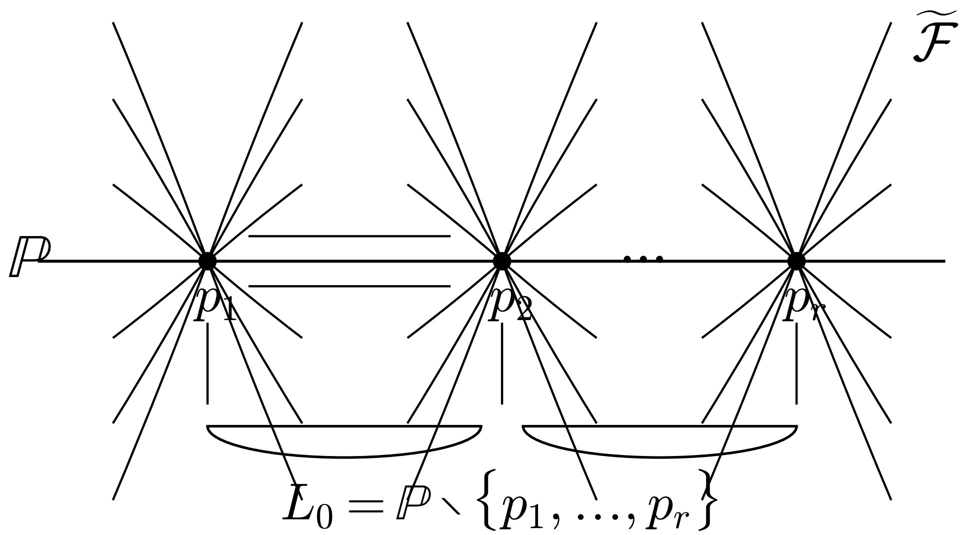


Since $h_j \in H(L_0)$, we have $h_j^\nu = \mathrm{id}$ and therefore $\frac{m_j}{n_j}\nu = l_j$ for some $l_j \in \mathbb{N}$; equivalently, $m_j\nu = n_j l_j$.

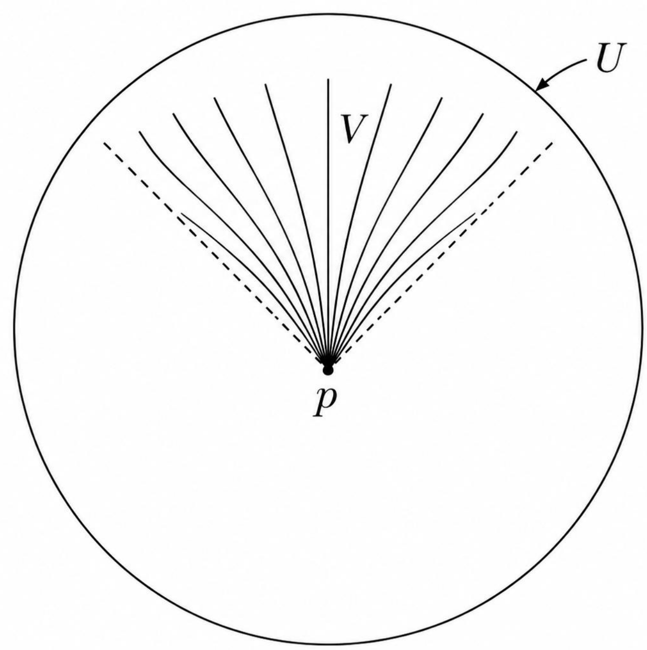
In particular $\xi_j^\nu = \left(x_j^{m_j} y_j^{n_j}\right)^\nu = x_j^{n_j l_j} y_j^{n_j \nu} = \left(x_j^{l_j} y_j^{\nu}\right)^{n_j}$.

This allows us to glue the function f_W with ξ_j^ν and therefore to extend f_W to a neighborhood of p_j. This proves Claim 3.23. $\square$

This shows that we have a holomorphic first integral $\tilde{f}$ defined for $\tilde{\mathcal{F}}$ in a neighborhood $\widetilde{W}$ of the exceptional divisor $\mathbb{P}$ in the blow-up $\mathbb{C}_0^2$.



Blowing down this function we obtain a holomorphic function $f\colon U, 0 \longrightarrow \mathbb{C}, 0$ defined in a neighborhood U of $0 \in \mathbb{C}^2$, such that f is a holomorphic first integral for $\mathcal{F}$ in U:



Indeed, f is initially defined on $U \setminus \{0\}$, using the fact that

$$\pi|_{\widetilde{\mathbb{C}}^2 \setminus \mathbb{P}} \colon \widetilde{\mathbb{C}}^2 \setminus \mathbb{P} \longrightarrow \mathbb{C}^2 \setminus \{0\}$$

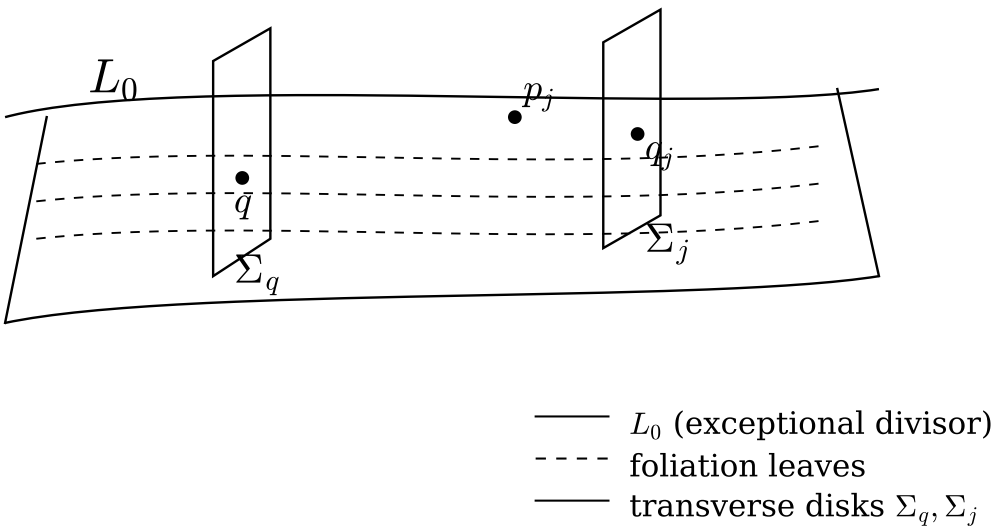
is biholomorphic. Hartogs' extension theorem [11] then shows that f extends holomorphically to the origin.

3.25 The case of several blow-ups

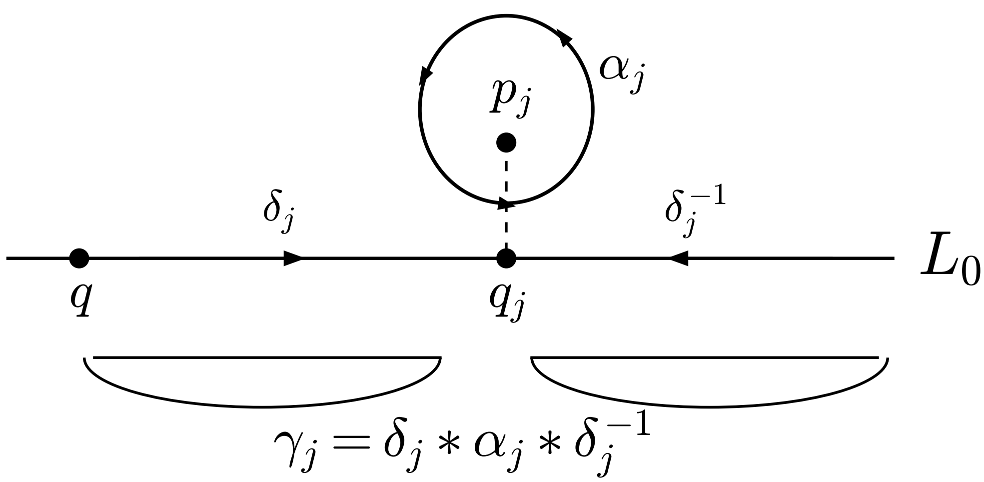
This case is quite similar to the case of one blow-up considered above. One may use induction on the number $p \in \mathbb{N}$ of necessary blow-ups in the desingularization of $\mathcal{F}$. The induction step goes as follows: Assume that the result has been proved for foliations admitting a resolution with fewer than p blow-ups, and that p is the minimum number of blow-ups in a resolution of $\mathcal{F}$. After the first blow-up, we obtain a foliation $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ with a picture similar to the one-blow-up case.

By the induction hypothesis, each singularity $p_j \in \operatorname{Sing}(\widetilde{\mathcal{F}}) \subset \mathbb{P}$ admits a holomorphic first integral on some neighborhood U_j of p_j.
Fix a base point $q \in L_0 = \mathbb{P} \setminus \operatorname{Sing}(\widetilde{\mathcal{F}})$ and choose points $q_j \in (U_j \cap \mathbb{P}) \setminus \{p_j\}$.

Let Σ_q and $\Sigma_j \subset U_j$ be disks transverse to $\widetilde{\mathcal{F}}$ and to $\mathbb{P}$, centered at q and q_j, respectively.



Choose simple paths $\delta_j \colon [0,1] \to L_0$ with $\delta_j(0) = q$ and $\delta_j(1) = q_j$, and simple local loops α_j in $\mathbb{P} \cap U_j$ around p_j, based at q_j.

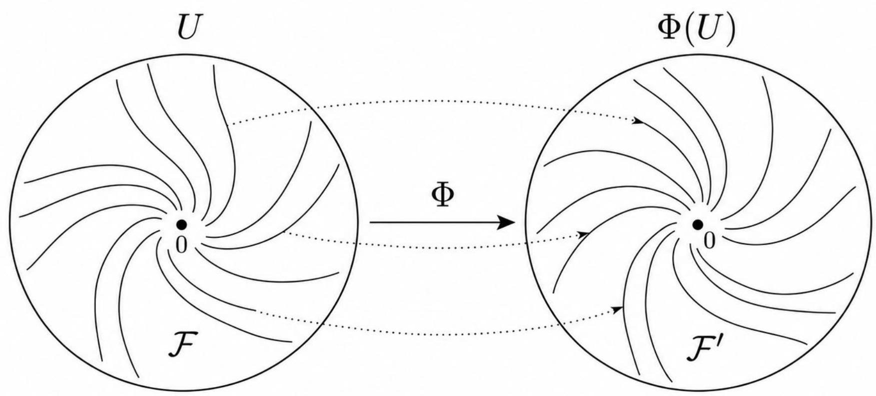


The holonomy maps $h_j \colon (\Sigma_q, q) \to (\Sigma_q, q)$ associated with the closed paths

$\gamma_j = \delta_j * \alpha_j * \delta_j^{-1}$ generate the projective holonomy group

$$H(L_0) = \langle h_1, \ldots, h_r \rangle.$$

This group preserves the leaves of $\widetilde{\mathcal{F}}$ and is therefore finite cyclic, analytically conjugate to a group generated by a rational rotation. Linearizing it yields a holomorphic first integral f_W for $\widetilde{\mathcal{F}}$ on a neighborhood W of $\bigcup_{j=1}^{r} \gamma_j$.



We may assume that $W \cap \mathbb{P}$ is simply connected and diffeomorphic to a disk. Its complement in $\mathbb{P}$ is also diffeomorphic to a disk. Hence, on a sufficiently small fibered neighborhood $\widetilde{U}$ of $\mathbb{P}$, the restriction of $\widetilde{\mathcal{F}}$ to $\widetilde{U} \setminus W$ has trivial holonomy. This allows us to extend f_W to a holomorphic first integral of $\widetilde{\mathcal{F}}$ on $\widetilde{U}$. Blowing it down and applying Hartogs' theorem as above, we obtain a holomorphic first integral of $\mathcal{F}$ near the origin in $\mathbb{C}^2$.

Chapter 4

On real analytic centers

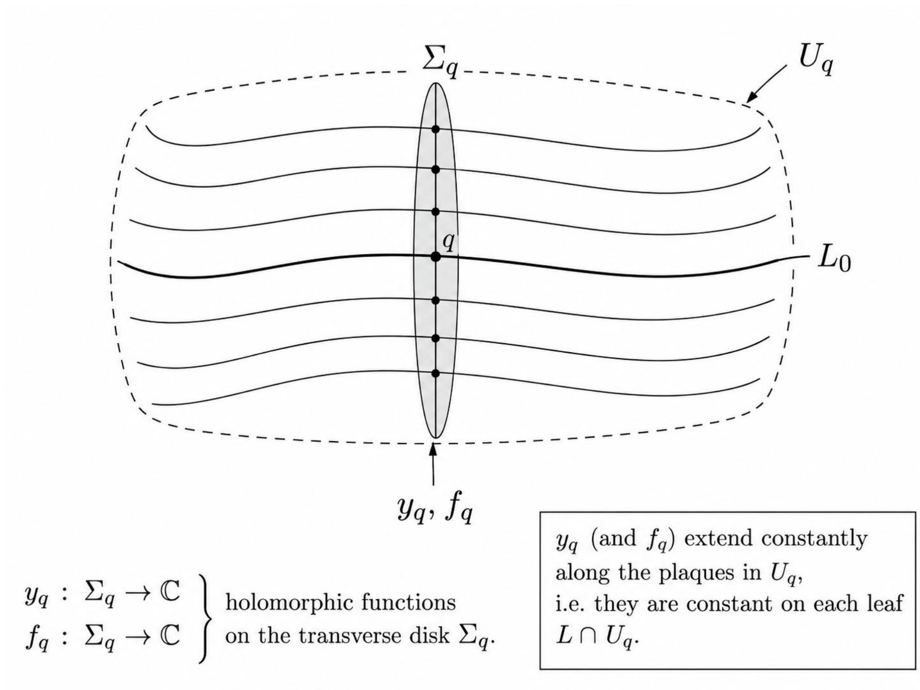
4.1 The linearization theorem of Poincaré–Lyapunov

We consider a real analytic vector field

$$\vec{X}(x,y) = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$$

defined in a neighborhood U of the origin $0 \in \mathbb{R}^2$, with an isolated singularity at the origin.

Definition 4.2. The vector field $\vec{X}$ has a *center singularity* at 0 if there exists a neighborhood $0 \in V \subset U$ such that every orbit of $\vec{X}$ in $V \setminus \{0\}$ is periodic (and hence diffeomorphic to the circle S^1).



A basic example is the linear vector field

$$\vec{X}_0(x, y) = -2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

Another example is any vector field of the form

$$\vec{X} = u\,\vec{X}_0,$$

where $u(x, y)$ is a nonvanishing function in a neighborhood of the origin. We may also take any local diffeomorphism

$$\Phi : (U_0, 0) \longrightarrow (V_0, 0)$$

and consider $\vec{X} = \Phi_*(\vec{X}_0)$. If u and Φ are real analytic, these constructions produce real analytic examples.

One of the main problems in the study of center singularities is to decide whether a given center comes from the preceding constructions, that is, whether it is linearizable in an appropriate sense. We therefore introduce the following notion.

Definition 4.3. Let X and Y be two C^k vector fields, $1 \leq k \leq \infty$, defined respectively on open sets $U, V \subset \mathbb{R}^n$. We say that X and Y are C^k*-orbitally equivalent* if there exist a C^k diffeomorphism

$$\Phi : U \longrightarrow V$$

and a positive C^k function $u : V \to \mathbb{R}_+$ such that

$$\Phi_* X \circ \Phi^{-1} = uY.$$

The classical center theorem of Poincaré–Lyapunov can be stated in dimension two as follows.

Theorem 4.4 (Poincaré–Lyapunov; see [18, 15, 16])**.** *Let X be a germ of a real analytic vector field having a center singularity at the origin $0 \in \mathbb{R}^2$. If its linear part $DX(0)$ is nonsingular, then X is analytically orbitally linearizable.*

There are several equivalent ways of rephrasing this result, using the Morse lemma and related tools. We shall also use the following formulation in terms of one-forms.

Theorem 4.5. *Let*

$$\omega = a(x, y)\, dx + b(x, y)\, dy$$

be a germ of a real analytic one-form at the origin $0 \in \mathbb{R}^2$. Assume that the first jet $j^1\omega$ has an isolated singularity and that ω has a center at the origin. Then ω admits a first integral of Morse type: there exist real analytic function germs f, g at 0, with $g(0) \neq 0$, such that

$$\omega = g\, df,$$

and f has a Morse singularity at the origin.

The notion of center singularity for a one-form is inherited from the corresponding notion for its associated vector field. Another useful formulation is the following.

Theorem 4.6. *Let X be a real analytic plane vector field with a center singularity at the origin.*

(i) X is analytically orbitally linearizable if and only if its linear part is nondegenerate.

(ii) *If the linear part of X generates a rotation, then X admits a real analytic first integral of the form*

$$F(x,y) = Q(x,y) + \textit{higher-order terms},$$

where Q is a positive definite quadratic form, if and only if the origin is a nondegenerate center-type singularity.

We shall need the following higher-dimensional terminology.

Definition 4.7 ([1])**.** Let X be an analytic vector field in a neighborhood of the origin $0 \in \mathbb{R}^{2n}$, with $X(0) = 0$. We say that $DX(0)$ generates a *multirotation* if its eigenvalues are of the form

$$\pm i\omega_j, \qquad j = 1, \dots, n,$$

where $\omega_j \in \mathbb{R}^*$ and

$$\frac{\omega_k}{\omega_j} \in \mathbb{Q} \qquad \text{for all } j, k.$$

Equivalently, every orbit of the one-parameter group $\exp(tDX(0))$, outside the origin, is nonsingular and periodic.

We now give a proof of Theorem 4.4 using holomorphic foliations, following the basic ideas of R. Moussu [20].

Let X be a real analytic vector field in a neighborhood of the origin $0 \in \mathbb{R}^2$, having an isolated singularity at the origin. After a linear change of coordinates and a multiplication by a nonzero constant, we may write its linear part as

$$DX(0) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$

Consider the complexification $X_{\mathbb{C}}$ of X. This is a holomorphic vector field defined in a neighborhood of the origin in $\mathbb{C}^2$. In complex coordinates (x, y) it has the form

$$X_{\mathbb{C}} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + X_2,$$

where X_2 has vanishing first jet at the origin. Hence $X_{\mathbb{C}}$ generates a holomorphic foliation $\mathcal{F}_{\mathbb{C}}$ with a singularity at the origin, represented by a one-form

$$x\,dx + y\,dy + \omega_2 = 0,$$

where ω_2 has vanishing first jet.

The foliation $\mathcal{F}_{\mathbb{C}}$ is in the Siegel domain. After choosing invariant coordinate axes and blowing up the origin,

$$\pi : \widetilde{\mathbb{C}}_0^2 \longrightarrow \mathbb{C}^2,$$

we obtain a foliation $\widetilde{\mathcal{F}}_{\mathbb{C}}$ leaving invariant the exceptional divisor

$$E = \pi^{-1}(0) \simeq \mathbb{P}^1.$$

A direct computation shows that $\widetilde{\mathcal{F}}_{\mathbb{C}}$ has exactly two singular points on E, the north and south poles, and both are of Siegel type.

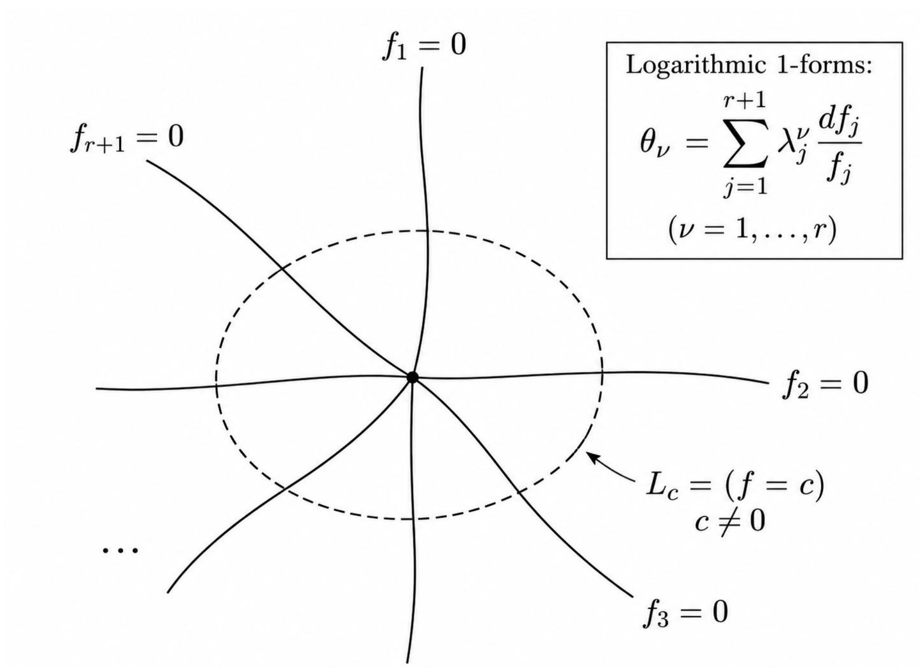
Remark 4.8. A nondegenerate singularity

$$\mathcal{F} : \quad x\,dy - \lambda y\,dx + \cdots = 0$$

is said to be of *Siegel type* when $0 \neq \lambda \in \mathbb{R}_-$. If $\lambda \notin \mathbb{R}_-$, then the singularity lies in the Poincaré domain. A Siegel-type singularity admits exactly two transverse separatrices and, in suitable coordinates, can be written as

$$x(1 + A(x,y))\,dy + \mu y(1 + B(x,y))\,dx = 0, \qquad \mu > 0,$$

with $A(0) = B(0) = 0$.



The equator $\gamma \simeq S^1$ generates the projective holonomy of $\widetilde{\mathcal{F}}_{\mathbb{C}}$, that is, the holonomy group of the leaf

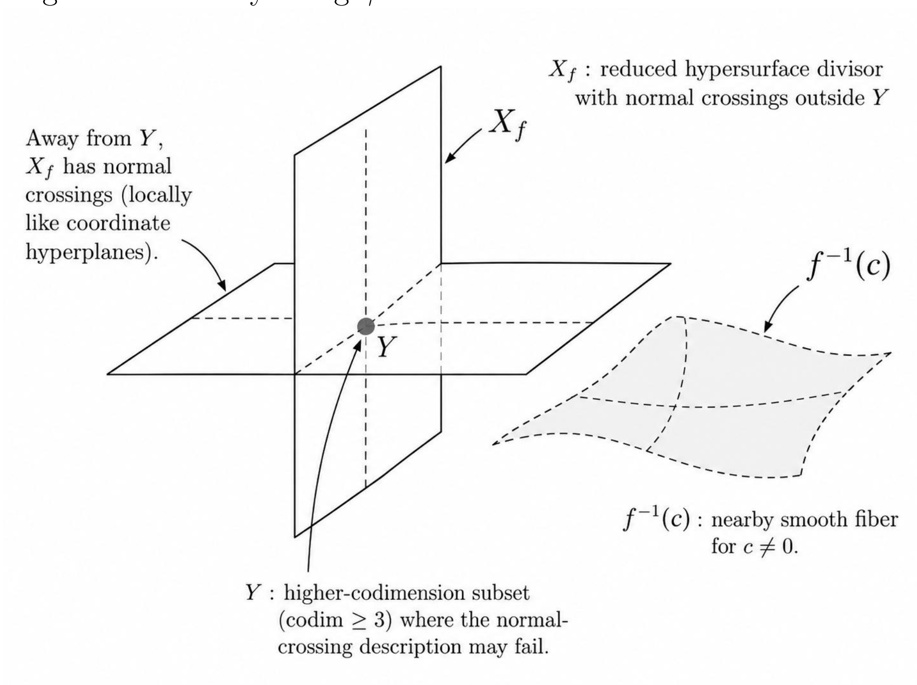
$$L_0 = E \setminus \{S, N\} \simeq \mathbb{C}^*.$$

Recall that $\pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$. Let h denote the holonomy map induced by γ.

The inverse image of $\mathbb{R}^2 \subset \mathbb{C}^2$ under the blow-up map is a Möbius band

$$M^2 \subset \widetilde{\mathbb{C}}^2_0$$

intersecting E transversely along $\gamma = M^2 \cap E$.



The Möbius band M^2 is invariant by the pull-back of the real foliation $\mathcal{F}_{\mathbb{R}}$ induced by X. The closed orbits of X near the origin lift to closed orbits of the one-dimensional foliation

$$\mathcal{L} = \widetilde{\mathcal{F}}_{\mathbb{C}}|_{M^2},$$

and γ is a periodic orbit of $\mathcal{L}$. Take an open transverse segment $\sigma \simeq (-1, 1)$ in M^2, centered at a point $p \in \gamma$. The first-return map

$$h_0 : (\sigma, p) \longrightarrow (\sigma, p)$$

associated with γ satisfies $h_0^2 = \mathrm{id}$. Let Σ be a complex disk centered at p such that $\Sigma \cap M^2 = \sigma$. The complex holonomy map

$$h : (\Sigma, p) \longrightarrow (\Sigma, p)$$

restricts to h_0 on σ. Therefore

$$(h|_\sigma)^2 = \mathrm{id},$$

and the identity principle gives

$$h^2 = \mathrm{id}$$

on (Σ, p).

Thus $\mathcal{F}_{\mathbb{C}}$ is desingularized by a single blow-up, its singularities on E are nondegenerate, and its projective holonomy group is finite. By the one-blow-up case of the Mattei–Moussu theorem proved in the preceding chapter, $\mathcal{F}_{\mathbb{C}}$ admits a holomorphic first integral

$$f_{\mathbb{C}} : (U, 0) \longrightarrow (\mathbb{C}, 0)$$

for some neighborhood U of the origin in $\mathbb{C}^2$.

Since $\mathcal{F}_{\mathbb{C}}$ is the complexification of $\mathcal{F}_{\mathbb{R}}$, the real foliation admits a real analytic first integral. Its quadratic part is positive definite, so after multiplication by a nonzero constant we may write

$$f(x_1, x_2) = x_1^2 + x_2^2 + \text{higher-order terms.}$$

The Morse lemma now finishes the proof of Theorem 4.4.

Example 4.9. Consider the real analytic one-form

$$\omega = \frac{1}{4}\, d(x^4 + y^4) - \frac{1}{2} x^2 y^2 \, dx.$$

It defines a center singularity at the origin of $\mathbb{R}^2$. Indeed, ω is invariant under the symmetry

$$\sigma(x, y) = (x, -y),$$

and the phase portrait is symmetric with respect to the x-axis.

Nevertheless, ω does not admit a real analytic first integral in the strong sense $\omega = g\,df$ with $f, g \in \mathbb{R}\{x,y\}$ and $g(0) \neq 0$. Suppose, on the contrary, that $\omega = g\,df$. Write

$$g = g_0 + g_{1,0}x + g_{0,1}y + \cdots, \qquad g_0 \neq 0.$$

Comparing the lowest-order terms gives

$$f = \frac{1}{4g_0}(x^4 + y^4) + \cdots.$$

Since $\omega = g\,df$, we have

$$d\omega = dg \wedge df.$$

Taking the terms of degree three yields

$$x^2y\,dx \wedge dy = \frac{1}{g_0}\big(g_{1,0}\,dx + g_{0,1}\,dy\big) \wedge \big(x^3\,dx + y^3\,dy\big),$$

and therefore

$$g_0x^2y = g_{1,0}y^3 - g_{0,1}x^3,$$

a contradiction.

4.10 Multidimensional versions

Let X be an analytic vector field in a neighborhood of the origin $0 \in \mathbb{R}^{2n}$, and assume that the origin is an isolated singularity of X.

Definition 4.11. The origin is called a *multicenter* of X if there exists a neighborhood V of the origin such that, for every $p \in V \setminus \{0\}$, the orbit of X through p is periodic.

If 0 is a multicenter, we may define the corresponding period function

$$T : V \setminus \{0\} \longrightarrow \mathbb{R}_+, \qquad T(p) = \text{period of the orbit through } p.$$

We shall say that $DX(0)$ generates a multirotation if every orbit of the linear equation

$$\dot{x} = DX(0)x$$

outside the origin is periodic. As observed above, this is equivalent to saying that the eigenvalues of $DX(0)$ are of the form $\pm i\omega_j$ with all quotients ω_j/ω_k rational. In this case there exists a common period $\tau \neq 0$ such that

$$e^{\tau DX(0)} = \mathrm{id}\,.$$

As a converse in this direction, we have the following theorem of Urabe and Sibuya.

Theorem 4.12 (Urabe–Sibuya, [32])**.** *Let X be an analytic vector field with a multicenter at $0 \in \mathbb{R}^{2n}$. Assume that*

(i) $DX(0)$ generates a multirotation;

(ii) the period function T is bounded on $V \setminus \{0\}$ for some neighborhood V of the origin.

Then X is C^∞-orbitally equivalent to its linear part $DX(0)$ in a neighborhood of the origin.

Recall that two vector fields X and Y are C^k-orbitally equivalent if there exist a C^k diffeomorphism Φ between neighborhoods of the corresponding singular points and a positive C^k function g such that

$$\Phi_* X \circ \Phi^{-1} = gY.$$

4.13 Singularities of holomorphic flows

We now apply the preceding techniques to the study of singularities of holomorphic flows. The theory is closely related to the work of M. Suzuki, who made fundamental contributions using methods from several complex variables and potential theory; see [23, 24, 25].

A holomorphic vector field Z on a complex manifold M is called *complete* if its local flow $Z_t(z_0)$ is defined for every complex time $t \in \mathbb{C}$. Recall that the local flow is the solution of the Cauchy problem

$$\dot{z} = Z(z), \qquad z(0) = z_0.$$

In the complete case we obtain a holomorphic action

$$Z_t : \mathbb{C} \times M \longrightarrow M, \qquad (t, z_0) \longmapsto Z_t(z_0)$$

of the additive group $(\mathbb{C},+)$ on M. For each $t\in\mathbb{C}$, the flow map

$$Z_t : M \longrightarrow M$$

is biholomorphic, and $t\mapsto Z_t$ is a group homomorphism from $(\mathbb{C},+)$ to $\mathrm{Bihol}(M)$.

Example 4.14. Every holomorphic vector field on a compact complex manifold is complete.

Example 4.15. On $\mathbb{C}^n$, a linear vector field $Z(z)=Az$, $A\in\mathrm{End}(\mathbb{C}^n)$, is complete, with flow

$$Z_t(z)=e^{tA}z, \qquad t\in\mathbb{C},\ z\in\mathbb{C}^n.$$

Given a complete holomorphic vector field Z on M and a point $p\in M$, we denote its orbit by

$$\mathcal{O}_p(Z)=\{Z_t(p) : t\in\mathbb{C}\}.$$

Lemma 4.16. *There are four possibilities for the orbit $\mathcal{O}_p(Z)$:*

(i) $\mathcal{O}_p(Z)=\{p\}$;

(ii) $\mathcal{O}_p(Z)$ *is biholomorphic to* $\mathbb{C}$;

(iii) $\mathcal{O}_p(Z)$ *is biholomorphic to the cylinder* $\mathbb{C}^*\simeq\mathbb{C}/\mathbb{Z}$;

(iv) $\mathcal{O}_p(Z)$ *is biholomorphic to a one-dimensional complex torus* $\mathbb{C}/(\mathbb{Z}\eta_1\oplus\mathbb{Z}\eta_2)$.

Proof. Define the isotropy group of p by

$$\mathrm{Fix}(p)=\{t\in\mathbb{C} : Z_t(p)=p\}.$$

It is a closed subgroup of $(\mathbb{C},+)$. If $\mathrm{Fix}(p)=\mathbb{C}$, then $Z(p)=0$ and the orbit is the single point $\{p\}$. Assume $Z(p)\neq 0$. Then $\mathrm{Fix}(p)$ is a proper discrete subgroup of $(\mathbb{C},+)$, hence one of the following occurs:

(1) $\mathrm{Fix}(p)=\{0\}$;

(2) $\mathrm{Fix}(p)=\tau\mathbb{Z}$ for some $\tau\in\mathbb{C}^*$;

(3) $\mathrm{Fix}(p) = \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$, where η_1, η_2 are linearly independent over $\mathbb{R}$.

Since $\mathcal{O}_p(Z)$ is biholomorphic to $\mathbb{C}/\,\mathrm{Fix}(p)$, the four possibilities follow. □

Orbits of types (iii) and (iv) will be called *periodic*.

Lemma 4.17. *A periodic orbit of a complete holomorphic vector field is closed. More precisely, if $\mathcal{O}_p(Z)$ is periodic, then its closure is an analytic subset of complex dimension one.*

Proof. If $\mathrm{Fix}(p) \simeq \mathbb{Z} \oplus \mathbb{Z}$, then $\mathcal{O}_p(Z)$ is a complex torus, hence compact and analytic. Assume $\mathrm{Fix}(p) = \tau\mathbb{Z}$. Then $Z_\tau(p) = p$. For every $z = Z_t(p) \in \mathcal{O}_p(Z)$ we have

$$Z_\tau(z) = Z_\tau(Z_t(p)) = Z_t(Z_\tau(p)) = Z_t(p) = z.$$

Thus

$$\mathcal{O}_p(Z) \subset \{z \in M : Z_\tau(z) = z\},$$

and the right-hand side is an analytic subset. The orbit is an open one-dimensional component of this fixed-point set, hence its closure is analytic of dimension one. □

Corollary 4.18. *Let Z be a complete holomorphic vector field on a complex surface N^2. Suppose that every orbit of Z is periodic. If $p \in \mathrm{Sing}(Z)$ is an isolated non-dicritical singularity, then Z admits a holomorphic first integral in a neighborhood of p.*

Proof. By Lemma 4.17, the leaves of the local foliation $\mathcal{F}_Z$ are closed off the singular set. The conclusion therefore follows from the Mattei–Moussu theorem. □

The following proposition collects two further consequences.

Proposition 4.19. *Let Z be a complete holomorphic vector field on a connected compact complex surface N^2.*

(i) If Z has infinitely many periodic orbits, then it admits a meromorphic first integral

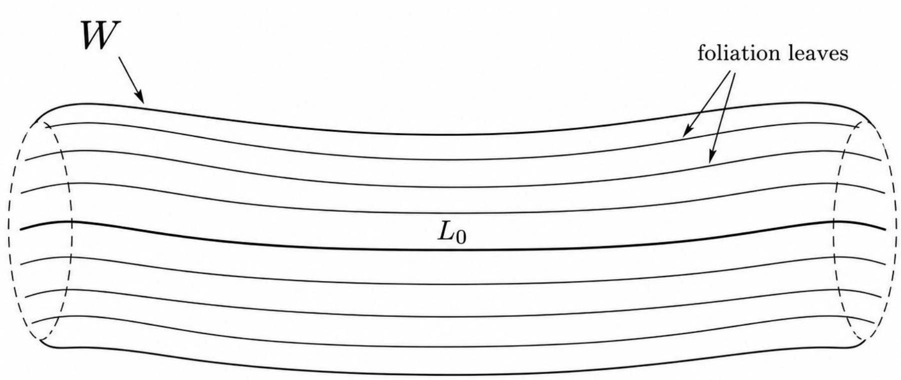
$$f : N^2 \dashrightarrow \mathbb{P}^1.$$

(ii) *If Z has an isolated dicritical singularity $p \in \mathrm{Sing}(Z)$, then it admits a meromorphic first integral*

$$f : N^2 \dashrightarrow \mathbb{P}^1.$$

Proof. If $\mathcal{O}_p(Z)$ is periodic, then $\overline{\mathcal{O}_p(Z)}$ is a compact invariant analytic curve by Lemma 4.17. Hence (i) follows from Ghys' theorem [9].

For (ii), let $p \in \mathrm{Sing}(Z)$ be an isolated dicritical singularity. Given a sufficiently small neighborhood U of p, there exists a nonempty open set $V \subset U$, with $p \in \overline{V}$, such that every local leaf of $\mathcal{F}_Z|_U$ through a point of V is contained in a separatrix through p.



A local function extends constantly along leaves to a holomorphic function f_W on W.

For $q \in V$, consider the global orbit $\mathcal{O}_q(Z)$. If it is periodic, its closure is a compact analytic curve. If it is nonperiodic, then $\mathcal{O}_q(Z) \simeq \mathbb{C}$. Following the argument of Cerveau [6], the local separatrix through p gives a removable end for the orbit parametrization. Thus the parametrization extends holomorphically to $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, with ∞ mapped to p. Hence $\mathcal{O}_q(Z) \cup \{p\}$ is a compact analytic curve. Thus Z has infinitely many compact invariant analytic curves. Ghys' theorem again yields a meromorphic first integral on N^2. $\square$

Chapter 5

Integrable deformations of holomorphic functions

5.1 Ilyashenko's thesis

In 1969, Ilyashenko published his PhD thesis on limit cycles of analytic planar vector fields [13]. In his work, Ilyashenko studies perturbations of the Hamiltonian system associated with a generic complex polynomial in the complex plane $\mathbb{C}^2$. Let us recall the result that will motivate this chapter.

We use affine complex coordinates $(z, w) \in \mathbb{C}^2$. Let $A(z, w)$ and $B(z, w)$ be complex polynomials of degree at most m, and let $R(z, w)$ be a complex polynomial of degree $m + 1$. We consider the Hamiltonian equation

$$(\varepsilon_0) \qquad \frac{dw}{dz} = -\frac{R_z}{R_w},$$

and, for a complex parameter $t \in \mathbb{C}$, the deformation

$$(\varepsilon_t) \qquad \frac{dw}{dz} = -\frac{R_z + tA}{R_w + tB}.$$

Remark 5.2. In terms of differential forms,

$$(\varepsilon_0) \qquad R_z\,dz + R_w\,dw = 0,$$

that is, $dR = 0$, whereas

$$(\varepsilon_t) \qquad (R_z + tA)\,dz + (R_w + tB)\,dw = 0,$$

that is,

$$dR + t(A\,dz + B\,dw) = 0.$$

Let B_{m+1} denote the space of complex polynomials of degree at most $m+1$ in $\mathbb{C}[z,w]$. We denote by $B''_{m+1} \subset B_{m+1}$ the set of those polynomials R for which (ε_0) has exactly m^2 singular points, and by $B'_{m+1} \subset B''_{m+1}$ the set of those R for which these singular points lie on distinct level sets of R.

Remark 5.3. The set B'_{m+1} is a Zariski open dense subset of B_{m+1}.

Definition 5.4 (Ilyashenko)**.** A singular point of (ε_0) is said to be a *center* if the foliation of a neighborhood of the point by solutions of (ε_0) is topologically equivalent to the foliation of a neighborhood of the origin in $\mathbb{C}^2$ by the curves

$$zw = \text{constant},$$

equivalently, in suitable real coordinates, by the curves $x^2 + y^2 = \text{constant}$. By the Mattei–Moussu theorem, the topological equivalence may in fact be taken holomorphic.

Denote by Ω_m the space of polynomial 1-forms

$$\omega = A\,dz + B\,dw, \qquad A, B \in B_m.$$

Theorem 5.5 (Ilyashenko [13])**.** *Let $R \in B'_{m+1}$ and let $p \in \mathbb{C}^2$ be a singular point of (ε_0) such that $A(p) = B(p) = 0$, so that p is also a singular point of (ε_t) for every t. Suppose that p is a center for (ε_t) for every t in a neighborhood of $0 \in \mathbb{C}$. Then*

$$\omega = A\,dz + B\,dw \in \Omega_m$$

is exact. In particular, (ε_t) admits a polynomial first integral.

The theorem is based on the following integration lemma.

Theorem 5.6 (Ilyashenko [13])**.** *Let $R \in B''_{m+1}$ and let $\omega \in \Omega_m$. Then ω is exact, $\omega = dS$ for some $S \in B_{m+1}$, if and only if*

$$\int_\gamma \omega = 0$$

for every closed curve γ contained in a regular fiber $\{R = c\}$.

In short, a form $\omega \in \Omega_m$ is exact provided that its restrictions to the regular fibers of $R \in B''_{m+1}$ are exact.

Remark 5.7. Every holomorphic 1-form on a Riemann surface is closed.

In what follows we investigate natural extensions of Ilyashenko's result to higher dimension. We first rewrite the planar deformation in a form that will be useful later. Put

$$\omega_0 := dR, \qquad \omega^t := dR + t(A\,dz + B\,dw) = \omega_0 + t\omega_1,$$

where $\omega_1 := A\,dz + B\,dw$. Thus (ε_t) corresponds to the foliation $\omega^t = 0$. This is a degree-one deformation, in the parameter t, of $\omega_0 = dR$ by polynomial 1-forms of degree at most m.

With this notation, Ilyashenko's theorem can be rephrased as follows.

Theorem 5.8. *Let*

$$\omega^t = dR + t\omega$$

be a degree-one analytic deformation of $\omega_0 = dR$, where $R \in B''_{m+1}$ and each ω^t is polynomial of degree at most m. Then the following conditions are equivalent.

(i) The deformation is of Hamiltonian type: for every t sufficiently close to 0,

$$\omega^t = dR_t$$

for some polynomial R_t of degree at most $m+1$.

(ii) Given any singular point p of $\omega_0 = dR$, there exists an analytic curve

$$\gamma_p : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, p), \qquad \gamma_p(0) = p,$$

such that $\gamma_p(t)$ is a center-type singularity of ω^t for t sufficiently close to 0.

(iii) There exists a singular point p of $\omega_0 = dR$ and an analytic curve

$$\gamma_p : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, p), \qquad \gamma_p(0) = p,$$

such that $\gamma_p(t)$ is a center-type singularity of ω^t for t sufficiently close to 0.

Remark 5.9. We draw the reader's attention to the difference between conditions (ii) and (iii): condition (ii) concerns every singular point of ω_0, whereas condition (iii) requires the persistence of only one center.

We now state the higher-dimensional deformation theorem that will be proved below.

Theorem 5.10 (Theorem A: deformation theorem; cf. [8])**.** *Let*

$$f = f_1 \cdots f_{r+1}$$

be a product of pairwise relatively prime irreducible homogeneous polynomials $f_j \in \mathbb{C}[x_1, \ldots, x_m]$, $m \geq 3$. *Assume that the germ of the hypersurface* $(f = 0)$ *has only normal-crossing singularities outside an analytic subset of codimension at least* 3. *Let*

$$\omega^t = df + \sum_{j=1}^{\infty} t^j \omega_j$$

be an analytic deformation of $\omega_0 = df$ *such that*

(i) *each* ω^t *is a polynomial integrable* 1*-form;*

(ii) $\deg \omega^t \leq \deg \omega_0$ *for every* t *sufficiently close to* 0*.*

Then the following conditions are equivalent.

(a) ω^t *is exact for every* t *sufficiently close to* 0*;*

(b) *there is an analytic family of polynomials* $F_t \in \mathbb{C}[x_1, \ldots, x_m]$*, with*

$$\deg F_t \leq \deg f,$$

such that

$$dF_t = \omega^t$$

for every t *sufficiently close to* 0*;*

(c) *for every* $c \neq 0$ *and every* t *sufficiently close to* 0*, the restriction* $\omega^t|_{L_c}$ *vanishes in*

$$H^1(L_c, \mathbb{C}), \qquad L_c = (f = c);$$

(d) *for every* $c \neq 0$ *and every* t *sufficiently close to* 0*, the restriction of* ω^t/f *to* L_c *vanishes in* $H^1(L_c, \mathbb{C})$*.*

Remark 5.11. If $f = f_1$ is irreducible, reduced, and has only normal-crossing singularities outside an analytic subset of codimension at least 3, then the nonsingular fibers $L_c = (f = c)$, $c \neq 0$, are simply connected by the Lê–Saito theorem. Hence the period condition

$$\oint_\gamma \omega^t = 0$$

is automatically satisfied.

We immediately obtain the following consequence.

Corollary 5.12. *Let $P_{\nu+1} \in \mathbb{C}[x_1, \dots, x_m]$ be an irreducible homogeneous polynomial of degree $\nu + 1$, with $m \geq 3$. Assume that the hypersurface $(P_{\nu+1} = 0)$ has only normal-crossing singularities outside an analytic subset of codimension at least 3. Then any analytic deformation ω^t of*

$$\omega_0 = dP_{\nu+1}$$

by polynomial integrable 1-forms satisfying

$$\deg \omega^t \leq \nu$$

admits polynomial first integrals for t sufficiently close to 0. More precisely, there is an analytic family of polynomials P^t of degree at most $\nu + 1$, with $P^0 = P_{\nu+1}$, such that

$$\omega^t = dP^t.$$

5.13 Equations of a deformation

Let ω_0 be a holomorphic 1-form in a neighborhood of the origin in $\mathbb{C}^m$. By an *analytic one-parameter deformation* of ω_0 we mean a family

$$\omega^t = \omega_0 + \sum_{j=1}^{\infty} t^j \omega_j,$$

where the 1-forms ω_j are holomorphic in a common neighborhood U of the origin and the parameter t belongs to a small disk $D \subset \mathbb{C}$ centered at 0. We say that the deformation is *by integrable forms* if

$$\omega^t \wedge d\omega^t = 0$$

for every $t \in D$. In particular, ω_0 is integrable.

Writing

$$\omega^t = \omega_0 + t\omega_1 + t^2\omega_2 + \cdots , \qquad d\omega^t = d\omega_0 + t\, d\omega_1 + t^2 d\omega_2 + \cdots ,$$

and comparing equal powers of t in $\omega^t \wedge d\omega^t = 0$, we obtain

$$\begin{aligned}
&\omega_0 \wedge d\omega_0 = 0, \\
&\omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0 = 0, \\
&\omega_2 \wedge d\omega_0 + \omega_1 \wedge d\omega_1 + \omega_0 \wedge d\omega_2 = 0, \\
&\qquad\qquad \vdots
\end{aligned}$$

These are the *equations of the deformation.*

We shall consider the case in which ω_0 admits a holomorphic first integral,

$$\omega_0 = df$$

for some holomorphic function $f : U, 0 \to \mathbb{C}, 0$. Since $d\omega_0 = 0$, the deformation equations become

$$\begin{aligned}
&df \wedge d\omega_1 = 0, \\
&\omega_1 \wedge d\omega_1 + df \wedge d\omega_2 = 0, \\
&df \wedge d\omega_3 + \omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = 0, \\
&\qquad\qquad \vdots
\end{aligned}$$

Remark 5.14. The equation

$$df \wedge d\omega_1 = 0$$

means that ω_1 is closed when restricted to each leaf of $\omega_0 = df$, that is, to each regular fiber of f.

5.15 Homology of the fibers

Let us now study the topology of the fibers of

$$f : U, 0 \longrightarrow \mathbb{C}, 0$$

with the aim of solving the deformation equations above. We assume $m \geq 3$ and denote by

$$X_f = (f = 0)$$

the germ of hypersurface defined by f. We assume that f is reduced. Thus

$$\mathrm{Sing}(X_f) = \{p \in X_f : df(p) = 0\}.$$

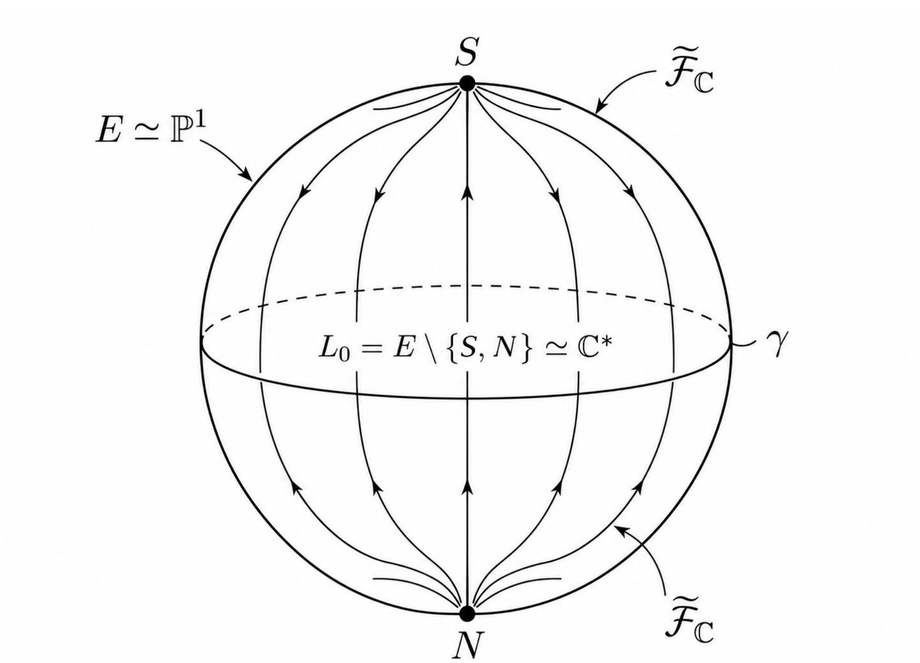
Recall that, after shrinking the representative if necessary, the critical points of f are contained in the special fiber $f^{-1}(0)$; see, for instance, [30].

Theorem 5.16 (Lê–Saito [14])**.** *Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a reduced germ, with $m \geq 3$. Assume that, outside an analytic subset $(Y, 0) \subset (X_f, 0)$ of codimension at least 3, the singularities of X_f are normal crossings. Then the local fundamental group of the complement of $(X_f, 0)$ in $(\mathbb{C}^m, 0)$ is abelian. Moreover, the fundamental group of a Milnor fiber of f is free abelian of rank r, where $r + 1$ is the number of irreducible components of X_f at the origin.*

In particular, if X_f is irreducible, then the local fiber

$$f^{-1}(c), \qquad c \neq 0,$$

is simply connected.



5.17 Generic systems of logarithmic forms

Write

$$f = f_1 \cdots f_{r+1},$$

where each $f_j \in \mathcal{O}_m$ is irreducible and each irreducible component of X_f corresponds to exactly one hypersurface $(f_j = 0)$. Consider logarithmic 1-forms

$$\theta_\nu = \sum_{j=1}^{r+1} \lambda_j^\nu \frac{df_j}{f_j}, \qquad \nu = 1, \ldots, r,$$

with $\lambda_j^\nu \in \mathbb{C}$.

Definition 5.18. The system $\{\theta_1, \ldots, \theta_r\}$ is said to be *generic with respect to f* if it is completely independent from

$$\frac{df}{f} = \sum_{j=1}^{r+1} \frac{df_j}{f_j},$$

in the following sense: if

$$\sum_{\nu=1}^{r} a_\nu \theta_\nu + b\frac{df}{f} = 0$$

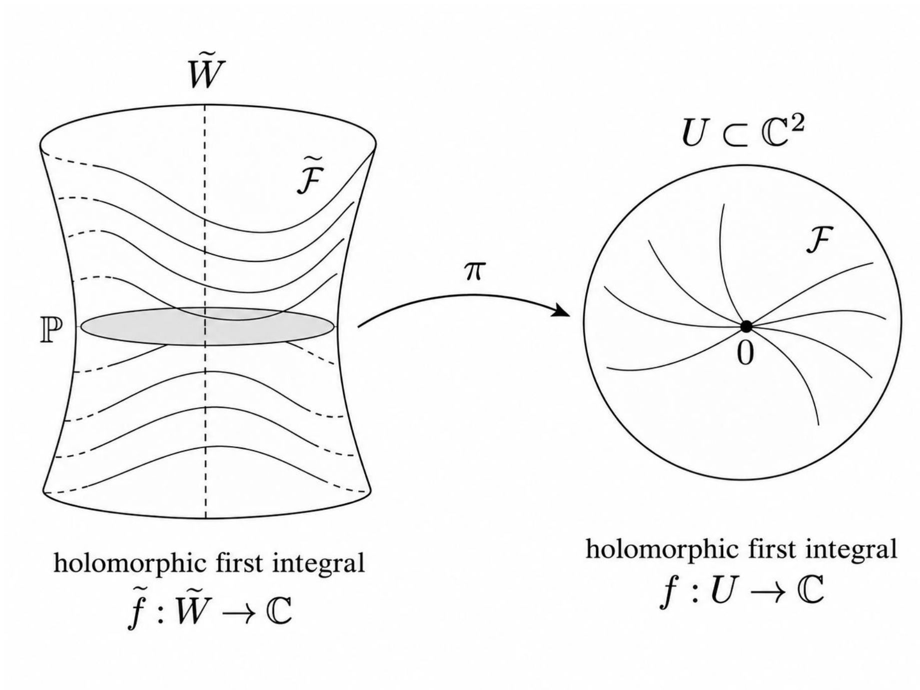
for constants $a_1, \ldots, a_r, b \in \mathbb{C}$, then

$$a_1 = \cdots = a_r = b = 0.$$

Equivalently, the matrix

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1^1 & \cdots & \lambda_{r+1}^1 \\ \vdots & & \vdots \\ \lambda_1^r & \cdots & \lambda_{r+1}^r \end{pmatrix}$$

has nonzero determinant.



Lemma 5.19. *Let f be as in Theorem 5.16. For each $c \neq 0$, the first cohomology group of the local fiber*

$$L_c = (f = c)$$

is generated by the classes of the restrictions

$$\theta_1|_{L_c}, \ldots, \theta_r|_{L_c}.$$

Proof. For simplicity, suppose $r = 1$ and $f = f_1 f_2$. Write

$$\theta = \lambda_1 \frac{df_1}{f_1} + \lambda_2 \frac{df_2}{f_2}, \qquad \lambda_1 \neq \lambda_2.$$

On a fiber $L_c = (f = c)$, $c \neq 0$, we have

$$\left. \frac{d(f_1 f_2)}{f_1 f_2} \right|_{L_c} = 0,$$

and hence

$$\left. \frac{df_1}{f_1} \right|_{L_c} = - \left. \frac{df_2}{f_2} \right|_{L_c}.$$

Therefore

$$\theta|_{L_c} = (\lambda_1 - \lambda_2)\, \frac{df_1}{f_1}\bigg|_{L_c}.$$

Since $(f_1 f_2 = 0)$ has normal crossings outside the exceptional analytic subset, we may choose a transverse disk Σ near a regular point of $(f_1 = 0)$, contained in a level $(f_2 = \varepsilon)$. For $c \neq 0$ sufficiently small, the intersection with L_c determines a cycle γ_c. Its period is

$$\int_{\gamma_c} \theta = (\lambda_1 - \lambda_2) \int_{\gamma_c} \frac{df_1}{f_1} = (\lambda_1 - \lambda_2)\, 2\pi i\, n,$$

where n is the winding number of γ_c about $(f_1 = 0) \cap \Sigma$. Thus this period is nonzero.

By Theorem 5.16, $\pi_1(L_c) \simeq \mathbb{Z}$ in the case $r = 1$. Hence $H_1(L_c, \mathbb{Z}) \simeq \mathbb{Z}$, and the closed form $\theta|_{L_c}$ represents a generator of $H^1(L_c, \mathbb{C})$. The general case is analogous. □

Remark 5.20. For most applications we may take

$$\theta_\nu = \frac{df_\nu}{f_\nu}, \qquad \nu = 1, \ldots, r.$$

Together with df/f, these forms give a generic system.

We now come to a key lemma.

Lemma 5.21. *Let ω_1 be a germ of holomorphic 1-form at $0 \in \mathbb{C}^m$, $m \geq 3$, and suppose that*

$$d\omega_1 \wedge df = 0,$$

where $f = f_1 \cdots f_{r+1}$ is as in Theorem 5.16. Then there exist germs $a_1, h_1 \in \mathcal{O}_m$ and germs $\psi_j \in \mathcal{O}_1$, $j = 1, \ldots, r$, such that

$$\omega_1 = a_1\, df + dh_1 + f \sum_{j=1}^{r} \psi_j(f)\theta_j.$$

Equivalently, the coefficient of each logarithmic class is divisible by f.

Proof. Using the description of the first cohomology of the fibers $L_c = (f = c)$, $c \neq 0$, define holomorphic functions $\alpha_j(c)$ by

$$[\omega_1|_{L_c}] = \sum_{j=1}^{r} \alpha_j(c)[\theta_j|_{L_c}] \quad \text{in } H^1(L_c, \mathbb{C}).$$

Fix a regular point $p_1 \in (f = 0)$ and choose a small transversal to the fibers through p_1, parametrized by the value $c = f$. On a punctured neighborhood of p_1, define h_1 by integrating, along the fibers,

$$\omega_1 - \sum_{j=1}^{r} \alpha_j(f)\theta_j.$$

The vanishing of the corresponding cohomology class makes this definition independent of the chosen path. Thus, away from X_f, we have

$$\left(\omega_1 - \sum_{j=1}^{r} \alpha_j(f)\theta_j - dh_1\right) \wedge df = 0,$$

and consequently

$$\omega_1 = a_1\, df + dh_1 + \sum_{j=1}^{r} \alpha_j(f)\theta_j$$

for a holomorphic function a_1 on the complement of X_f.

Choose generators $\gamma_c^1, \ldots, \gamma_c^r$ of $H_1(L_c, \mathbb{Z})$ normalized by

$$\int_{\gamma_c^i} \theta_j = 2\pi i\, \delta_{ij}.$$

Then

$$\alpha_j(c) = \frac{1}{2\pi i} \int_{\gamma_c^j} \omega_1.$$

Since ω_1 is holomorphic, the functions $\alpha_j(c)$ are holomorphic and bounded for $c \neq 0$. By Riemann's extension theorem they extend holomorphically to $c = 0$.

The extension of h_1 across X_f is the nontrivial point here. It is precisely the extension argument used in the final part of Proposition 5.2 of Cerveau–Scárdua [7]; applied to

$$\left(\omega_1 - \sum_{j=1}^{r} \alpha_j(f)\theta_j\right) \wedge df = dh_1 \wedge df,$$

it shows that h_1 extends holomorphically across X_f. The identity above then implies that a_1 extends holomorphically as well.

Finally, each θ_j has only simple poles along components of X_f, whereas ω_1, $a_1 df$, and dh_1 are holomorphic. Hence f divides each $\alpha_j(f)$. Thus

$$\alpha_j(f) = f\psi_j(f)$$

for some $\psi_j \in \mathcal{O}_1$, which gives the asserted expression. □

5.22 Relative cohomology: polynomial case

The division phenomena used below are closely related to de Rham–Saito type lemmas; see [26]. We now study the cohomological equation

$$d\omega_1 \wedge df = 0$$

in the polynomial case.

Proposition 5.23. *Let*

$$f = f_1 \cdots f_{r+1}$$

be a product of pairwise relatively prime homogeneous polynomials in $\mathbb{C}[x_1, \ldots, x_m]$, *with*

$$\deg f = \nu + 1.$$

Assume that the germ $(f = 0)$ *has only normal-crossing singularities outside an analytic subset of codimension at least* 3, *and let* $\theta_1, \ldots, \theta_r$ *be a generic system as above. If* ω_1 *is a polynomial* 1*-form of degree at most* ν *satisfying*

$$d\omega_1 \wedge df = 0,$$

then

$$\omega_1 = a_0\, df + dh + f \sum_{j=1}^{r} \lambda_j \theta_j$$

for constants $a_0, \lambda_1, \ldots, \lambda_r \in \mathbb{C}$ *and a polynomial* h *of degree at most* $\nu + 1$.

Moreover, h *is homogeneous of degree* $\nu + 1$ *if and only if* ω_1 *is homogeneous of degree* ν.

Proof. By Lemma 5.21,

$$\omega_1 = a\, df + dh + f \sum_{j=1}^{r} \psi_j(f)\theta_j$$

for holomorphic germs a, h and one-variable germs ψ_j. Expanding a, h, and the ψ_j into homogeneous components, and using that both df and $f\theta_j$ are homogeneous of degree ν, while $\deg \omega_1 \leq \nu$, only the constant term $\lambda_j := \psi_j(0)$ of each ψ_j can contribute; all higher powers of f have degree strictly larger than ν. Thus only the terms displayed in the statement can occur. The last assertion follows from the same homogeneous-degree comparison. □

Denote by $\mathbb{C}[x_1, \dots, x_m]_\ell$ the vector space of homogeneous polynomials of degree ℓ.

Remark 5.24. Let $h, f \in \mathbb{C}[x_1, \dots, x_m]_{\nu+1}$, with

$$f = f_1 \cdots f_{r+1}$$

reduced and with the f_j irreducible and pairwise relatively prime. Then

$$f \mid dh \wedge df \quad \iff \quad h = \lambda f$$

for some $\lambda \in \mathbb{C}$.

Proof of the remark. Suppose that $f \mid dh \wedge df$. Then $(f = 0)$ is invariant by the foliation $dh = 0$. Since h and f are homogeneous of the same degree and the irreducible components of $(f = 0)$ are invariant, we obtain $(f = 0) \subset (h = 0)$; hence $h = pf$ for some polynomial p. Equality of degrees forces p to be constant. The converse is immediate. □

5.25 Proof of the Deformation theorem

We now prove Theorem 5.10.

Proof of Theorem 5.10. Consider

$$\omega^t = df + \sum_{j \geq 1} t^j \omega_j,$$

where $f = f_1 \cdots f_{r+1}$ is as in the statement. The forms ω^t are polynomial, but they need not be homogeneous.

Assume condition (c), namely that

$$[\omega^t|_{L_c}] = 0 \in H^1(L_c, \mathbb{C})$$

for every $c \neq 0$ and every sufficiently small t. We prove (b). For simplicity, first suppose $r = 1$, so $f = f_1 f_2$. By Proposition 5.23,

$$\omega_1 = a_1 df + dh_1 + \lambda_1 f \theta_1$$

for some constants $a_1, \lambda_1 \in \mathbb{C}$ and a polynomial h_1 of degree at most $\nu + 1$. Since all periods of ω^t on the fibers vanish, the coefficient of t gives

$$\int_{\gamma_c} \omega_1 = 0,$$

and therefore $\lambda_1 = 0$. Thus $\omega_1 = a_1 df + dh_1$ is closed.

The second deformation equation now yields

$$d\omega_2 \wedge df = 0.$$

Applying Proposition 5.23 again gives

$$\omega_2 = a_2 df + dh_2 + \lambda_2 f \theta_1,$$

with $a_2, \lambda_2 \in \mathbb{C}$ and $\deg h_2 \leq \nu + 1$. The vanishing of the periods of ω_2 gives $\lambda_2 = 0$. Proceeding inductively, we obtain constants a_j and polynomials h_j, $\deg h_j \leq \nu + 1$, such that

$$\omega_j = a_j df + dh_j \qquad \text{for every } j \geq 1.$$

Hence the formal series

$$\widehat{F} = f + \left(\sum_{j=1}^{\infty} a_j t^j \right) f + \sum_{j=1}^{\infty} t^j h_j$$

satisfies

$$\omega^t = d_x \widehat{F},$$

where d_x denotes differentiation with respect to $x = (x_1, \dots, x_m)$. Since ω^t is convergent and all h_j have uniformly bounded degree, $\widehat{F}$ is convergent; indeed,

$$\widehat{F} \in \mathbb{C}[x_1, \dots, x_m]_{\leq \nu+1} \otimes \mathcal{O}_1.$$

Thus ω^t is exact and admits an analytic family of polynomial first integrals F_t of degree at most $\nu + 1$.

The case $r \geq 2$ is identical, with the full generic system $\theta_1, \dots, \theta_r$.

This proves $(c) \Rightarrow (b)$. The implication $(b) \Rightarrow (a)$ is immediate, and $(a) \Rightarrow (c)$ is clear. Finally, since $f = c \neq 0$ on L_c, conditions (c) and (d) are equivalent. $\square$

We shall also use the local analytic version of the deformation result proved in Cerveau–Scárdua [7].

Theorem 5.26 (Local analytic deformation theorem)**.** *Let $f \in \mathcal{O}_m$, $m \geq 3$, be irreducible and reduced. Assume that the hypersurface germ $(f = 0)$ has only normal-crossing singularities outside an analytic subset of codimension at least 3. Let*

$$\eta^t = df + \sum_{j \geq 1} t^j \eta_j$$

be an analytic deformation by integrable holomorphic 1-forms, defined for (x, t) near $(0, 0)$. Then, after shrinking the neighborhoods if necessary, the foliations defined by η^t admit holomorphic first integrals depending analytically on t.

Remark 5.27. The distinction between this result and Theorem 5.10 is important. Theorem 5.10 is a polynomial integration theorem with a uniform degree bound and yields exactness. The theorem above is local analytic and is the result needed when the coefficients of the deformation are arbitrary holomorphic germs.

We are now in a position to state a second consequence.

Theorem 5.28 (Theorem B)**.** *Let Ω be a germ of an integrable holomorphic 1-form at the origin $0 \in \mathbb{C}^m$, $m \geq 3$. Assume that the first nonzero homogeneous component of Ω is*

$$\Omega_\nu = dP_{\nu+1},$$

where $P_{\nu+1} \in \mathbb{C}[x_1, \ldots, x_m]$ is an irreducible homogeneous polynomial of degree $\nu + 1$ whose zero set has only normal-crossing singularities outside an analytic subset of codimension at least 3. Then Ω admits a holomorphic first integral in a neighborhood of the origin.

Proof. Write

$$\Omega = dP_{\nu+1} + \sum_{j=\nu+1}^{\infty} \Omega_j,$$

where Ω_j is homogeneous of degree j as a 1-form. Consider the homotheties

$$\sigma_t : \mathbb{C}^m \longrightarrow \mathbb{C}^m, \qquad \sigma_t(z) = tz,$$

and define

$$\omega^t := \frac{1}{t^{\nu+1}}\sigma_t^*(\Omega) \qquad (t \neq 0).$$

Since $\Omega \wedge d\Omega = 0$, we also have

$$\omega^t \wedge d\omega^t = 0.$$

Moreover,

$$\sigma_t^*(\Omega) = t^{\nu+1}dP_{\nu+1} + \sum_{j=\nu+1}^{\infty} t^{j+1}\Omega_j,$$

so that

$$\omega^t = dP_{\nu+1} + \sum_{j=\nu+1}^{\infty} t^{j-\nu}\Omega_j.$$

Consequently the family extends analytically to $t = 0$, with

$$\omega^0 = dP_{\nu+1}, \qquad \omega^1 = \Omega.$$

Claim 5.29. *The family $\{\omega^t\}$ is an analytic deformation of $dP_{\nu+1}$ by integrable holomorphic 1-forms. If Ω is polynomial, then each ω^t is polynomial as well.*

The claim follows directly from the preceding expansion. The family $\{\omega^t\}$ is generally not polynomial when Ω is an arbitrary holomorphic germ, so Theorem 5.10 cannot be invoked here. Instead we apply the local analytic deformation theorem, Theorem 5.26, to $f = P_{\nu+1}$. Its hypotheses hold because $P_{\nu+1}$ is irreducible, reduced, and has the prescribed normal-crossing behavior. We obtain a holomorphic first integral $F(z,t)$ for the family, depending analytically on t, with

$$d_zF(z,t) \wedge \omega^t = 0.$$

After the harmless normalization $F(z,0) = P_{\nu+1}(z)$, evaluation at $t = 1$ gives a holomorphic first integral for $\omega^1 = \Omega$ in a neighborhood of the origin. $\square$

5.30 The real analytic case

We recall a classical result due to G. Reeb.

Theorem 5.31 (Reeb, Theorem 5 in [27])**.** *Let ω be a real analytic integrable 1-form defined in a neighborhood of the origin $0 \in \mathbb{R}^m$, $m \geq 3$. Suppose that $\omega(0) = 0$ and that its linear part is nondegenerate,*

$$\omega_1 = dQ,$$

where Q is a quadratic form of maximal rank, not necessarily definite. Then there exist a real analytic local diffeomorphism

$$h : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^m, 0)$$

and a real analytic function g such that

$$h^*\omega = g\, dQ.$$

In particular, ω admits a real analytic first integral in a neighborhood of the origin.

Connected with Reeb's result, we have the following.

Theorem 5.32 (Theorem C)**.** *Let ω be an integrable real analytic 1-form defined in a neighborhood of the origin $0 \in \mathbb{R}^m$, $m \geq 3$. Suppose that $\omega(0) = 0$ and that its first nonzero homogeneous component is*

$$\omega_\nu = dP_{\nu+1},$$

where $P_{\nu+1}$ is a real homogeneous polynomial of degree $\nu + 1 \geq 2$. Assume that:

(1) the complexification

$$P^{\mathbb{C}}_{\nu+1} \in \mathbb{C}[z_1, \ldots, z_m]$$

has only normal-crossing singularities outside an analytic subset of codimension at least 3;

(2) $P^{\mathbb{C}}_{\nu+1}$ is irreducible in $\mathbb{C}[z_1, \ldots, z_m]$.

Then ω admits a real analytic first integral

$$f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}, 0)$$

whose first nonzero homogeneous term is $P_{\nu+1}$.

Remark 5.33. The proof proceeds by complexifying ω, applying Theorem 5.28, and then returning to the real analytic setting. In particular, the theorem gives another proof of Reeb's result above.

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