Stochastic ranking process with space-time dependent intensities

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Stochastic ranking processes are a model of a ranking system, such as the sales ranks found at online bookstores. We consider N particles each of which are exclusively located at $1, 2, \dots, N$. Each particle jumps to 1 according to its Poisson clock. When a jump of the particle at position i occurs, the particle moves to position 1 and the positions of the particles at $1, 2, \dots, i-1$ are sifted by $+1$. Particles whose Poisson clocks rang recently are at positions with small numbers, and the others are at positions with large numbers. We regard the number for each particle as the particle's rank. This system enables us to give ranks to N particles, and we call the time evolution of the particles given by this ranking system the *stochastic ranking process*.

In this work, we consider the case that the jumping rates of stochastic ranking processes depend not only on time, but also on the positions of particles. This is an extension of the results that have ever existed. Now we give the precise formulation of the stochastic ranking process which we consider in this paper. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\{\nu_i(d\xi ds)\}_{i=1,2,3,\dots}$ be independent Poisson random measures on $[0,\infty) \times [0,\infty)$ with the intensity measure $d\xi ds$.

Let W be a set of functions in $C^{1,0}([0,1] \times [0,\infty); [0,\infty))$ such that for each $T > 0$

$$\sup_{w \in W} \sup_{(y,t) \in [0,1]\times[0,T]} \max\left\{ w(y,t),\ \left| \frac{\partial w}{\partial y}(y,t) \right| \right\} < \infty.$$

Let w_i, $i = 1, 2, \dots$ be a sequence in W, and for a positive integer N, put

$$w_i^{(N)}(k,t) := w_i\left(\frac{k-1}{N}, t \right), \quad k = 1, 2, \dots, N,\ t \in [0,\infty),\ i = 1, 2, \dots, N.$$

Also, let $x_1^{(N)}$, $x_2^{(N)}$, $\dots$, $x_N^{(N)}$ be a rearrangement of $1, 2, \dots, N$. Define a process $X^{(N)} = \left(X_1^{(N)}, \dots, X_N^{(N)} \right)$ by, for $i = 1, 2, \dots, N$, $t \geq 0$,

$$\begin{aligned} X_i^{(N)}(t) = x_i^{(N)} &+ \sum_{j=1}^{N} \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} \mathbb{I}_{\{X_i^{(N)}(s-) < X_j^{(N)}(s-)\}} \mathbb{I}_{\{\xi \in [0, w_j^{(N)}(X_j^{(N)}(s-), s))\}} \nu_j(d\xi ds) \\ &+ \int_{s\in(0,t]} \int_{\xi\in[0,\infty)} (1 - X_i^{(N)}(s-)) \mathbb{I}_{\{\xi \in [0, w_i^{(N)}(X_i^{(N)}(s-), s))\}} \nu_i(d\xi ds). \end{aligned}$$

$X^{(N)}(t)$ is a rearrangement of $1, 2, \dots, N$ for all $t \geq 0$, which we regard as ranks or positions of particles $1, 2, \dots, N$ at time t.

Before considering the scaling limit of $X^{(N)}$, we prepare a theorem on the existence and uniqueness of the solution of an inviscid Burgers-like integral-partial differential equation with evaporation. Let $\Gamma_b := \{0\} \times [0,\infty)$, $\Gamma_i = [0,1] \times \{0\}$, $\Gamma = \Gamma_b \cup \Gamma_i$, and $\Gamma_t = \{(y_0, t_0) \in \Gamma \mid t_0 \leq t\}$ for $t \geq 0$.

Theorem 1. *Let λ be a Borel probability measure on W, and Define a Borel measure $U_0(dw, y)$ on W by*

$$U_0(dw, y) = \rho(w, y)\, \lambda(dw), \quad y \in [0,1],\ w \in W,$$

where $\rho : W \times [0,1] \to [0,1]$ is a non-negative Borel measurable function such that for $(w, y) \in W \times [0,1]$, $\dfrac{\partial \rho}{\partial y}(w, y)$ exists and continuous, and $\dfrac{\partial \rho}{\partial y}(w, y) \leq 0$, $\rho(\cdot, 0) = 1$, and $\rho(\cdot, 1) = 0$. Assume also

$$U_0(W, y) = \int_W U_0(dw, y) = 1 - y,\ 0 \leq y \leq 1.$$

Then there exists a unique pair of functions

$$y_C : \ \{(\gamma, t) \in \Gamma \times [0,\infty) \mid \gamma \in \Gamma_t\} \to [0,1],$$

and $U = U(dw, y, t)$ on $[0,1] \times [0,\infty)$ taking values in the non-negative Borel measures on W, such that,

(i) $y_C(\gamma,t)$ *and* $\dfrac{\partial y_C}{\partial t}(\gamma,t)$ *are continuous,*

(ii) *for each* $t>0$, $y_C(\cdot,t): \Gamma_t \to [0,1]$ *is surjective,*

(iii) *for all bounded continuous* $h: W \to [0,\infty)$, $U(h,y,t) := \displaystyle\int_W h(w)U(dw,y,t)$ *is Lipschitz continuous in* $(y,t)\in[0,1]\times[0,T]$ *for any* $T>0$, *and non-increasing in* y, *and*

(iv) *the followings hold:*

$$y_C(\gamma,t_0)=y_0,\quad U(dw,y_0,t_0)=U_0(dw,y_0),\quad \gamma=(y_0,t_0)\in\Gamma,$$

$$U(h,y_C(\gamma,t),t)=U_0(h,y_0)-\int_{t_0}^t V(h,y_C(\gamma,s),s)\,ds,\quad t\ge t_0,\ \gamma=(y_0,t_0)\in\Gamma,$$

for all bounded continuous function $h: W\to[0,\infty)$, *where* $U(h,y,t):=\displaystyle\int_W h(w)U(dw,y,t)$,

$$V(h,y,t)=\int_W h(w)\,w(y,t)\,U(dw,y,t)+\int_y^1\int_W h(w)\,\frac{\partial w}{\partial z}(z,t)\,U(dw,z,t)\,dz,$$

and

$$\frac{\partial y_C}{\partial t}(\gamma,t)=V(\mathbb{I}_W,y_C(\gamma,t),t),\quad t\ge t_0,\ \gamma=(y_0,t_0)\in\Gamma.$$

□

By using Theorem 1, we have the scaling limit of $X^{(N)}$. Let

$$Y_i^{(N)}(t):=\frac{1}{N}(X_i^{(N)}(t)-1),\quad U^{(N)}(dw,y,t):=\frac{1}{N}\sum_{i=1}^N \mathbb{I}_{\{X_i^{(N)}(t-)\ge Ny+1\}}\delta_{w_i}(dw).$$

Then, we have the following theorem.

Theorem 2. *Assume that with probability* 1,

$$\lim_{N\to\infty}\sup_{y\in[0,1)}||U^{(N)}(\cdot,y,0)-U_0(\cdot,y)||_{\mathrm{var}}=0,$$

where $U_0(dw,y)$ *satisfies all the assumptions in Theorem 1. Then the following hold.*

(i) *With probability* 1, *for all* $T>0$, $\lim_{N\to\infty}U^{(N)}(dw,y,t)=U(dw,y,t)$, *uniformly in* $y\in[0,1)$ *and* $t\in[0,T]$, *where* U *is the solution claimed in Theorem 1.*

(ii) *Assume in addition that,*

$$\lim_{N\to\infty}\frac{1}{N}x_i^{(N)}=y_i\,,\quad i=1,2,\dots,L,$$

for a positive integer L *and* $y_i\in[0,1)$, $i=1,2,\dots,L$. *Then, with probability* 1, *for all* $T>0$, *the tagged particle system*

$$\left(Y_1^{(N)}(t),Y_2^{(N)}(t),\dots,Y_L^{(N)}(t)\right)$$

converges as $N\to\infty$, *uniformly in* $t\in[0,T]$ *to a limit* $(Y_1(t),Y_2(t),\dots,Y_L(t))$. *Here, for each* $i=1,2,\dots,L$, Y_i *is the unique solution to*

$$Y_i(t)=y_i+\int_0^t V(\mathbb{I}_W,Y_i(s-),s)ds-\int_{s\in(0,t]}\int_{\xi\in[0,\infty)} Y_i(s-)\mathbb{I}_{\{\xi\in[0,w_i(Y_i(s-),s))\}}\nu_i(d\xi ds),$$

where V *is as in Theorem 1.*

□

References

[1] T. Hattori and Sei. Kusuoka, Stochastic ranking process with space-time dependent intensities, submitted.