Rough differential equations containing path-dependent bounded variation terms

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Abstract

In this paper, we prove existence of solutions to a class of path-dependent rough differential equations. The equations contain the path-dependent bounded variation terms and can be viewed as a natural extension of reflected rough differential equations studied in the author's previous paper (SPA 2015, 125 no.9). We prove the existence of solutions by using Gubinelli's controlled paths. In the case of reflected rough differential equations, the main result in this paper extends the existence theorem in the author's previous paper. We also prove an approximate continuity property of solution mappings at smooth rough paths. Consequently, we prove a support theorem for path space measures determined by random rough paths and reflected diffusions.

Keywords: path-dependent rough differential equations, reflected stochastic differential equation, Skorohod equation, rough path, controlled path

1 Introduction

In the theory of Itô stochastic differential equations(=SDEs), path-dependent SDEs can be formulated naturally and have been studied by many researchers. The study is based on the theory of martingales. On the other hand, rough path theory ([20, 21, 16, 17, 15]) enables us to study the equations in pathwise and no martingale theory is needed in principle.

In [2], we studied reflected rough differential equations (=reflected RDEs) defined on a domain D in $\mathbb{R}^d$ and proved the existence of solutions. The boundary need not be smooth. This depends on the important results of Skorohod equation which are due to Tanaka [25], Lions-Sznitman [19] and Saisho [24]. The reflected SDE is uniquely solved in strong sense under the boundary conditions (A) and (B) ([24]). We explain conditions (A) and (B) in Section 5. The main strategy of the proof in [2] is a suitable modification of the Euler-Maruyama approximation of the solution to RDEs without reflection term which is due to Davie [8]. Unfortunately, we need a stronger assumption (H1) in addition to (A) and (B) on the boundary to prove the existence theorem.

In the case of Itô's SDE with reflection, it is well-known that the equation can be transformed to a path-dependent SDE by using the Skorohod mapping and this is used to study a large deviation results and support theorem in [5, 9] in the smooth boundary case. They use the Lipschitz continuity of the Skorohod mapping in uniform convergence topology. However, this Lipschitz continuity does not hold any more in general ([11]) and the different approach is necessary as in [25, 24, 19]. They proved the unique existence of strong solutions of the reflected SDEs and this implies the unique existence of strong solutions of the associated path-dependent SDEs under the conditions (A) and (B). Hence the path-dependent SDEs is not used in their proof for unique existence of strong solutions. We note that the rate of strong convergence of Euler-Maruyama approximation and Wong-Zakai approximation for a class of path-dependent SDEs were studied in [1].

In this paper, we propose a new approach to the reflected rough differential equations under the conditions (A) and (B) by using Gubinelli's controlled paths [17]. Actually, we consider rough differential equations containing path-dependent bounded variation terms and prove the existence of solutions. The class of path-dependent rough differential equations includes the reflected rough differential equations for which the boundary satisfies the conditions (A) and (B). Therefore, main result in this paper extends the result in [2]. In [2], we need to solve implicit Skorohod equations and thus we put the stronger conditions (H1) on the boundary. We do not need to solve the implicit equations and hence the condition (H1) is unnecessary in our new approach.

Note that the required regularity condition on the coefficient of the RDE is $\sigma \in \mathrm{Lip}^{\gamma-1}$ (see Section 2 for the definition). This is the same assumption for the existence as the RDE without boundary and the uniqueness does not hold in general ([8]). Actually, for reflected RDEs, it is possible to prove the existence theorem by the previous approach in [2] under $\sigma \in \mathrm{Lip}^{\gamma-1}$ and the stronger assumption (H1) on the boundary. By using the setting of controlled paths, we can relax the assumptions on the boundary too.

As we mentioned, the path-dependent rough differential equations in this paper includes reflected rough differential equations. In this case, further, we can prove the existence of universally measurable selection mapping of solutions as in [2]. We prove continuity property of the solution mappings at smooth rough paths under conditions (A) and (B). By using this, we prove a support theorem for path space measures determined by random rough paths. When the driving process is Brownian motion, Wong-Zakai type theorem is proved in [4, 26, 3, 12, 22]. Recently, support theorem for reflected diffusion processes under (A) and (B) are proved by Ren and Wu [23] by using the Wong-Zakai type theorem. We give another proof for this theorem by using the continuity property of solution mapping at smooth rough paths.

The structure of this paper is as follows. In Section 2, we introduce a Hölder continuous path spaces with the exponent θ based on a control function ω of a rough path. In order to study RDEs containing path-dependent bounded variation terms, we need to consider continuous paths whose q-variation norms satisfies such a Hölder continuity. Hence we introduce a family of norms $\|\cdot\|_{q-var,\theta}$ and Banach spaces $\mathcal{V}_{q,\theta}$. In Section 3, we introduce controlled path spaces based on ω. We next introduce a mapping L from a continuous path space to a continuous bounded variation path space. By using L, we introduce a path-dependent RDE. In Section 4, we prove the existence of solutions to the equations and give the estimate of them. For that purpose, we apply Schauder's fixed point theorem in the product Banach space of controlled paths and $\mathcal{V}_{q,\alpha}$. In Section 5, we consider the case of reflected RDEs on domains D under the conditions (A) and (B) on the boundary. In this case, the solution is unique when the driving rough path is smooth and the coefficient is Lipschitz. Although the solutions is not uniquely determined for generic rough paths, as in [2], we prove the existence of universally measurable selection solution mapping. We prove the continuity of the solution mapping at the smooth (rough) paths and support theorems for the path space measure and reflected diffusions in Section 6. In Section 7, we make a remark on path-dependent rough differential equations with drift term.

2 Preliminary

Let $\omega(s,t)$ $(0 \le s \le t \le T)$ be a control function. That is, $(s,t) \mapsto \omega(s,t) \in \mathbb{R}^+$ is a continuous function and $\omega(s,u) + \omega(u,t) \le \omega(s,t)$ $(0 \le s \le u \le t \le T)$ holds. We introduce a mixed norm by using ω and p-variation norm. Let E be a finite dimensional normed linear space. For a continuous path (w_t) $(0 \le t \le T)$ on E, we define for $[s,t] \subset [0,T]$,

$$\|w\|_{\infty-var,[s,t]} = \max_{s \le u \le v \le t} |w_{u,v}|, \tag{2.1}$$

$$\|w\|_{p-var,[s,t]} = \left\{ \sup_{\mathcal{P}} \sum_{k=1}^{N} |w_{t_{k-1},t_k}|^p \right\}^{1/p}, \tag{2.2}$$

where $\mathcal{P} = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s,t]$ and $w_{u,v} = w_v - w_u$. When $[s,t] = [0,T]$, we may omit denoting $[0,T]$. For $0 < \theta \le 1, q \ge 1$, $0 \le s \le t \le T$ and a continuous path w, we define

$$\|w\|_{\theta,[s,t]} = \inf \left\{ C > 0 \mid |w_{u,v}| \le C\omega(u,v)^\theta \quad s \le u \le v \le t \right\}, \tag{2.3}$$

$$\|w\|_{q-var,\theta,[s,t]} = \inf \left\{ C > 0 \;\middle|\; \|w\|_{q-var,[u,v]} \le C\omega(u,v)^\theta \quad s \le u \le v \le t \right\}. \tag{2.4}$$

When $\omega(s,t) = |t-s|$, $\|w\|_{\theta,[s,t]} < \infty$ is equivalent to that w_u $(s \le u \le t)$ is a Hölder continuous path with the exponent θ in usual sense. Hence we may say w is an ω-Hölder continuous path with the exponent θ ((ω,θ)-Hölder continuous path in short). For two parameter function $F_{s,t}$ $(0 \le s \le t \le T)$, we define $\|F\|_{\theta,[s,t]}$ and $\|F\|_{q-var,\theta,[s,t]}$ similarly.

Let $\mathcal{V}_{q,\theta,T}(E)$ denote the set of E-valued continuous paths of finite q-variation defined on $[0,T]$ satisfying $\|w\|_{q-var,\theta} := \|w\|_{q-var,\theta,[0,T]} < \infty$. Note that $\mathcal{V}_{q,\theta,T}(E)$ is a Banach space with the norm $|w_0| + \|w\|_{q-var,\theta}$. Obviously, any path $w \in \mathcal{V}_{q,\theta,T}$ satisfy $|w_{s,t}| \le \|w\|_{q,\theta}\omega(s,t)^\theta$. We denote by $\mathcal{V}_\theta$ the set of ω-Hölder continuous paths w satisfying $\|w\|_\theta = \|w\|_{\theta,[0,T]} < \infty$. $\mathcal{V}_\theta$ is a Banach space with the norm $|w_0| + \|w\|_\theta$.

We next introduce the notation for mappings between normed linear spaces. Let E, F be finite dimensional normed linear spaces. For $\gamma = n + \theta$ $(n \in \mathbb{N} \cup \{0\}, 0 < \theta \le 1)$, $\mathrm{Lip}^\gamma(E,F)$ denotes the set of bounded functions f on E with values in F which are n-times continuously differentiable and whose derivatives up to n-th order are bounded and $D^n f$ is a Hölder continuous function with the exponent θ in usual sense.

Lemma 2.1. (1) *Let $1 \le q' < q$. For a continuous path w, we have*

$$\|w\|_{q-var,[s,t]} \le \|w\|_{q'-var,[s,t]}^{q'/q} \|w\|_{\infty-var,[s,t]}^{(q-q')/q} \le \|w\|_{q'-var,[s,t]}. \tag{2.5}$$

(2) *If $\|w\|_{q-var,[s,t]} < \infty$ for some q, then $\lim_{q\to\infty} \|w\|_{q-var,[s,t]} = \|w\|_{\infty-var,[s,t]}$.*

Proof. (1) We have

$$\begin{aligned}
\|w\|_{q-var,[s,t]} &= \left\{ \sup_{\mathcal{P}} \sum_i |w_{t_{i-1},t_i}|^q \right\}^{1/q} \\
&\le \left\{ \sup_{\mathcal{P}} \sum_i |w_{t_{i-1},t_i}|^{q'} \max_i |w_{t_{i-1},t_i}|^{q-q'} \right\}^{1/q} \\
&\le \|w\|_{q'-var,[s,t]}^{q'/q} \|w\|_{\infty,[s,t]}^{(q-q')/q}.
\end{aligned} \tag{2.6}$$

The second inequality follows from the trivial bound $\|w\|_{\infty-var,[s,t]} \le \|w\|_{q'-var,[s,t]}$.
(2) We need only to prove $\limsup_{q\to\infty} \|w\|_{q-var,[s,t]} \le \|w\|_{\infty-var,[s,t]}$. Suppose $\|w\|_{q_0-var,[s,t]} < \infty$. Then for $q > q_0$,

$$\sup_{\mathcal{P}} \left(\sum_i |w_{t_{i-1},t_i}|^q\right)^{1/q} \le \sup_{\mathcal{P}} \left(\sum_i |w_{t_{i-1},t_i}|^{q_0}\right)^{1/q} \sup_{\mathcal{P}} \max_i |w_{t_{i-1},t_i}|^{(q-q_0)/q}. \tag{2.7}$$

Taking the limit $q \to \infty$, we obtain the desired estimate. □

3 Existence theorem

Let $1/3 < \beta \le 1/2$. Let $\mathbf{X}_{s,t} = (X_{s,t}, \mathbb{X}_{s,t})$ $(0 \le s \le t \le T)$ be a $1/\beta$-rough path on $\mathbb{R}^n$ with the control function ω ([15, 16, 21, 20, 6, 7]). That is, $\mathbf{X}$ satisfies Chen's relation and the path regularity conditions,

$$|X_{s,t}| \le \|X\|_\beta \omega(s,t)^\beta, \quad |\mathbb{X}_{s,t}| \le \|\mathbb{X}\|_{2\beta}\omega(s,t)^{2\beta}, \qquad 0 \le s \le t \le T, \tag{3.1}$$

where $\|X\|_\beta$ and $\|\mathbb{X}\|_{2\beta}$ denote the ω-Hölder norm which are defined in the previous section. We denote by $\mathscr{C}^\beta(\mathbb{R}^n)$ the set of $1/\beta$-rough paths. When $\omega(s,t) = |t-s|$, $\mathbf{X}_{s,t}$ is a β-Hölder rough path. If $\mathbf{X}_{s,t}$ is a rough path with finite $1/\beta$-variation, we can choose $\|X\|_\beta = \|\mathbb{X}\|_{2\beta} = 1$ and $\omega(s,t) = \|X\|^{1/\beta}_{1/\beta-var,[s,t]} + \|\mathbb{X}\|^{2/\beta}_{2/\beta-var,[s,t]}$.

Let us choose p and γ such that $2 \le 1/\beta < p < \gamma \le 3$. We use the following quantity,

$$\widetilde{|||\mathbf{X}|||_\beta} = \sum_{i=1}^3 |||\mathbf{X}|||_\beta^i, \quad |||\mathbf{X}|||_\beta = \|\mathbf{X}\|_\beta + \sqrt{\|\mathbb{X}\|_{\mathbf{2}\beta}}. \tag{3.2}$$

We introduce a set of controlled paths $\mathscr{D}^{2\theta}_X(\mathbb{R}^d)$ of $\mathbf{X}_{s,t}$, where $1/3 < \theta \le \beta$ following [17, 15]. A pair of ω-Hölder continuous paths $(Z, Z') \in \mathcal{V}_\theta([0,T], \mathbb{R}^d) \times \mathcal{V}_\theta([0,T], \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ with the exponent θ is called a controlled path of X, if the remainder term $R^Z_{s,t} = Z_t - Z_s - Z'_s X_{s,t}$ satisfies $\|R^Z\|_{2\theta} < \infty$. The set of controlled paths $\mathscr{D}^{2\theta}_X(\mathbb{R}^d)$ is a Banach space with the norm

$$\|(Z, Z')\|_{2\theta} = |Z_0| + |Z'_0| + \|Z'\|_\theta + \|R^Z\|_{2\theta} \quad (Z, Z') \in \mathscr{D}^{2\theta}_X(\mathbb{R}^d) \tag{3.3}$$

$Z'_t \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ is called a Gubinelli derivative of Z with respect to X. Note that Z' may not be uniquely determined by Z. It is easy to see that $\mathscr{D}^{2\theta}_X \ne \emptyset$ because $(f(X_t), (Df)(X_t)) \in \mathscr{D}^{2\theta}_X(\mathbb{R}^d)$ for any $f \in C^2_b(\mathbb{R}^n, \mathbb{R}^d)$.

The rough differential equation which we will study contains path dependent bounded variation term $L(w)_t$. We consider the following condition on L.

Assumption 3.1. *Let $\xi, \eta \in \mathbb{R}^d$. Let L be a mapping from $\mathcal{V}_\beta([0,T] \to \mathbb{R}^d \mid w_0 = \xi)$ to $C([0,T] \to \mathbb{R}^d \mid w_0 = \eta)$ and satisfy the following conditions.*

(1) (*adaptedness*) *$(L(w)_s)_{0\le s\le t}$ depends only on $(w_s)_{0\le s\le t}$ for all $0 \le t \le T$.*

(2) *$L : (\mathcal{V}_\beta, \|\cdot\|_\beta) \to (C([0,T]), \|\cdot\|_{\infty-var})$ is continuous.*

(3) *There exists a non-decreasing positive continuous function F on $[0,\infty)$ such that*

$$\|L(w)\|_{1-var,[s,t]} \le F(\|w\|_{(1/\beta)-var,[s,t]})\|w\|_{\infty-var,[s,t]}. \tag{3.4}$$

The conditions (1), (2) are natural. In many cases, L is defined on continuous path spaces and is continuous with respect to the uniform norm. See the following examples. The condition (3) is strong assumption. This implies that the total variation of $L(w)$ on $[s,t]$ can be estimated by the norm of the path (w_u) on $s \le u \le t$. Note that this does not exclude the case where $L(w)_u$ $(s \le u \le t)$ depends on w_v $(v \le s)$. The condition (3) implies that for $0 < T' < T$,

$$\|L(w)\|_{1-var,[T'+s,T'+t]} \le F(\|\theta_{T'}w\|_{1/\beta-var,[s,t]})\|\theta_{T'}w\|_{\infty-var,[s,t]}, \tag{3.5}$$

where $(\theta_{T'}w)_t = w_{T'+t}$. These enable us to solve the path-dependent rough differential equation on small intervals and obtain global solution on $[0,T]$ by concatenating the solutions.

Example 3.2. (1) A typical example of L appears in the study of reflected process. We refer the reader for the detail in the following statement to Section 5. Let D be a domain in $\mathbb{R}^d$ satisfying conditions (A) and (B). Consider the Skorohod equation $y_t = w_t + \phi_t$, where w is a continuous path whose starting point is in $\bar{D}$. Also $y_t \in \bar{D}$ $(0 \le t \le T)$ and ϕ_t is the bounded variation term. The mapping $L : w \mapsto \phi$ satisfies the above conditions.
(2) Let $f = (f_1, \ldots, f_d) \in \mathrm{Lip}^1(\mathbb{R}^d, \mathbb{R}^d)$ and define

$$L(w)_t = \left(\max_{0\le s\le t} f_1(w_s), \ldots, \max_{0\le s\le t} f_d(w_s)\right). \tag{3.6}$$

Clearly this mapping is closely related to the example (1).

We now state our main theorem.

Theorem 3.3. *Let $\sigma \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ and $\xi, \eta \in \mathbb{R}^d$. Assume that the mapping $L : C([0,T] \to \mathbb{R}^d \mid w_0 = \xi) \to C([0,T] \to \mathbb{R}^d \mid w_0 = \eta)$ satisfies the condition in Assumption 3.1. Then there exists a controlled path $(Z, Z') \in \mathscr{D}_X^{2\beta}(\mathbb{R}^d)$ such that*

$$Z_t = \xi + \int_0^t \sigma(Z_s, L(Z)_s) d\mathbf{X}_s, \tag{3.7}$$

$$Z'_t = \sigma(Z_t, L(Z)_t) \tag{3.8}$$

$$\tag{3.9}$$

Further there exist positive constants κ, C_1, C_2, C_3 which depend only on σ, β, p, γ such that

$$\begin{aligned}&\|Z\|_\beta + \|R^Z\|_{2\beta} + \|L(Z)\|_{1-var,\beta} \\ &\le C_1\left\{1 + \left(1 + F(C_2\widetilde{|||\mathbf{X}|||_\beta})\right)^\kappa \left(1 + \widetilde{|||\mathbf{X}|||_\beta}\right)^\kappa \omega(0,T)\right\} \left(1 + F(C_3\widetilde{|||\mathbf{X}|||_\beta})\right) \widetilde{|||\mathbf{X}|||_\beta}.\end{aligned} \tag{3.10}$$

Remark 3.4. (1) Since $L(Z)_t$ is continuous and bounded variation, it is easy to give the meaning of the integral $\int_0^t \sigma(Z_s, L(Z)_s) d\mathbf{X}_s$ and it will be done below. The problem is to find good estimates for the integral by which we apply Schauder's fixed point theorem.
(2) It is natural to conjecture that the uniqueness of the solutions hold under a certain stronger conditions on σ and L. By checking the proof of the main theorem, one may prove the uniqueness of solutions under the assumption that $\sigma \in \mathrm{Lip}^\gamma$ and L is a Lipschitz map from $\mathcal{V}_\alpha$ to $\mathcal{V}_{q,\tilde{\alpha}}$, where α and $\tilde{\alpha}$ are constants in (3.38). However, the assumption on L is too strong and cannot be applied to interesting case, that is, reflected rough differential equations, because we cannot expect the Lipschitz continuity of L in general, see [14, 11] and the estimate in Lemma 5.4.

Recently, Falkowski and Słomiński [13] proved the Lipschitz continuity of the Skorohod mapping on a half space and proved the uniqueness of reflected SDE driven by fractional Brownian motions with the hurst parameter $H > 1/2$. This result may be useful to study the uniqueness of the solutions in this paper.
(3) We assume $L(w)$ is bounded variation in this paper. However, by checking the proof of Theorem 3.3, we may expect that similar results hold in the case where $L(w)$ is q-variation path $(q > 1)$ under suitable assumptions on $\sigma, L, \mathbf{X}$.

If we write $L(Z)_t = \Phi_t$, then the above equation reads

$$Z_t = \xi + \int_0^t \sigma(Z_s, \Phi_s) d\mathbf{X}_s, \tag{3.11}$$

$$\Phi_t = L\left(\xi + \int_0^{\cdot} \sigma(Z_s, \Phi_s) d\mathbf{X}_s\right)_t. \tag{3.12}$$

We solve this equation by using Schauder's fixed point theorem. To obtain a compactness of the embedding of the set of continuous paths of finite q-variation, we need to consider the set $\mathcal{V}_{q,\theta}$, $q \geq 1$ and $1/3 < \theta \leq 1/2$. We begin by giving a simple estimate for an integral $\int_s^t \Phi_{s,r} \otimes dw_r$ for such Φ.

Lemma 3.5. *Let $q, \tilde{\alpha}$ be positive numbers such that*

$$q \geq 1, \quad \frac{1}{p} + \frac{1}{q} \geq 1, \quad \frac{1}{3} < \tilde{\alpha} \leq \beta, \tag{3.13}$$

Let $\Phi \in \mathcal{V}_{q,\tilde{\alpha}}(\mathbb{R}^d)$ and $w \in \mathcal{V}_\beta(\mathbb{R}^n)$. Then the integral $\int_s^t \Phi_{s,r} \otimes dw_r$ converges in the sense of Young integral and

$$\left|\int_s^t \Phi_{s,r} \otimes dw_r\right| \leq 2^\beta \zeta\left(\beta p\right)^{1/p} \|\Phi\|_{q-var,\tilde{\alpha}} \|w\|_\beta \omega(s,t)^{\beta+\tilde{\alpha}}. \tag{3.14}$$

Proof. The relation $\beta p > 1$ implies that w is a continuous path of finite $p - \varepsilon$-variation for sufficiently small ε by the property of the control function ω. Since Φ is a continuous path of finite q-variation and $1/p + 1/q \geq 1$,

$$\lim_{|\mathcal{P}| \to 0} \sum_{i=0}^N \Phi_{t_i} \otimes w_{t_i, t_{i+1}} \tag{3.15}$$

converges, where $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$. Inductively we choose a point t_{i_k} $(1 \leq k \leq N-1)$ from $\{t_1, \ldots, t_{N-1}\} \setminus \{t_{i_1}, \ldots, t_{i_{k-1}}\}$ such that

$$\omega(t_{i_k - 1}, t_{i_k + 1}) \leq 2\frac{\omega(s,t)}{N-k}. \tag{3.16}$$

Let p^* be the conjugate Hölder exponent of p, that is, $1/p + 1/p^* = 1$. By the Hölder inequality and the assumptions on Φ and w, we have

$$\begin{aligned}\left|\sum_{i=0}^{N}\Phi_{t_i}\otimes w_{t_i,t_{i+1}}-\Phi_s\otimes w_{s,t}\right| &\le \sum_{k=1}^{N-1}|\Phi_{t_{i_k-1},t_{i_k}}\otimes w_{t_{i_k},t_{i_{k+1}}}| \\ &\le \left(\sum_{k=1}^{N-1}|\Phi_{t_{i_k-1},t_{i_k}}|^{p^*}\right)^{1/p^*}\left(\sum_{k=1}^{N-1}|w_{t_{i_k},t_{i_{k+1}}}|^{p}\right)^{1/p} \\ &\le \|\Phi\|_{q-var,[s,t]}\|w\|_\alpha\left(\sum_{k=1}^{N-1}\left(\frac{2\omega(s,t)}{N-k}\right)^{\beta p}\right)^{1/p} \\ &\le 2^\beta\zeta\,(\beta p)^{1/p}\,\|\Phi\|_{q-var,\tilde{\alpha}}\|w\|_\beta\omega(s,t)^{\beta+\tilde{\alpha}}, \end{aligned} \tag{3.17}$$

which completes the proof. □

We now give a meaning of the integral (3.7). We denote the derivative of $\sigma=\sigma(x,y)$ ($x\in\mathbb{R}^d, y\in\mathbb{R}^d$) with respect to x by $D_1\sigma$ and y by $D_2\sigma$. Also we write $D\sigma(x,y)(u,v)=D_1\sigma(x,y)u+D_2\sigma(x,y)v$. Recall that we write $Y_t=(Z_t,\Phi_t)\in\mathbb{R}^d\times\mathbb{R}^d$. Let $(Z,Z')\in\mathscr{D}_X^{2\alpha}(\mathbb{R}^d)$ and $\Phi\in\mathcal{V}_{q,\tilde{\alpha}}(\mathbb{R}^d)$. We assume that $q,\alpha,\tilde{\alpha}$ satisfy the following condition.

$$q\ge 1,\quad \frac{1}{p}+\frac{1}{q}\ge 1,\quad \alpha p>1,\quad \frac{1}{3}<\alpha\le\tilde{\alpha}\le\beta. \tag{3.18}$$

Let

$$\Xi_{s,t}=\sigma(Y_s)X_{s,t}+(D_1\sigma)(Y_s)Z'_s\mathbb{X}_{s,t}+(D_2\sigma)(Y_s)\int_s^t\Phi_{s,r}\otimes dX_r. \tag{3.19}$$

By a simple calculation, we have for $s<u<t$,

$$\begin{aligned}(\delta\Xi)_{s,u,t} &:= \Xi_{s,t}-\Xi_{s,u}-\Xi_{u,t} \\ &= -\left(\int_0^1(D_1\sigma)(Y_s+\theta Y_{s,u})\right)\left(R^Z_{s,u}\otimes X_{u,t}\right) \\ &\quad +\left\{(D\sigma)(Y_s)-\int_0^1(D\sigma)(Y_s+\theta Y_{s,u})d\theta\right\}\left\{\left(Z'_sX_{s,u},\Phi_{s,u}\right)\otimes X_{u,t}\right\} \\ &\quad +\left((D_1\sigma)(Y_s)Z'_s-(D_1\sigma)(Y_u)Z'_u\right)\mathbb{X}_{u,t} \\ &\quad +\left((D_2\sigma)(Y_s)-(D_2\sigma)(Y_u)\right)\int_u^t\Phi_{u,r}\otimes dX_r. \end{aligned} \tag{3.20}$$

Thus, under the assumption on Z,Φ, using Lemma 3.5 and $(a+b+c)^{\gamma-2}\le 3^{\gamma-2}(a^{\gamma-2}+b^{\gamma-2}+c^{\gamma-2})$, we obtain

$$
\begin{aligned}
&\left|(\delta\Xi)_{s,u,t}\right| \\
&\le \|D_1\sigma\|_\infty \|R^Z\|_{2\alpha}\|X\|_\beta \omega(s,t)^{\beta+2\alpha} \\
&\quad + \|D\sigma\|_{\gamma-2}|Y_{s,u}|^{\gamma-2}\left\{\|Z'\|_\infty\|X\|_\beta\omega(s,u)^\beta + \|\Phi\|_{q-var,\tilde\alpha}\omega(s,u)^{\tilde\alpha}\right\}\|X\|_\beta\omega(u,t)^\beta \\
&\quad + \left\{\|D_1\sigma\|_\infty\|Z'\|_\alpha\omega(s,u)^\alpha + \|D_1\sigma\|_{\gamma-2}|Y_{s,u}|^{\gamma-2}\|Z'\|_\infty\right\}\|\mathbb{X}\|_{2\beta}\omega(u,t)^{2\beta} \\
&\quad + 2^\beta\zeta(\beta p)^{1/p}\|D_1\sigma\|_{\gamma-2}|Y_{s,u}|^{\gamma-2}\|\Phi\|_{q-var,\tilde\alpha}\|X\|_\beta\omega(u,t)^{\tilde\alpha+\beta} \\
&\le \|D\sigma\|_\infty\|R^Z\|_{2\alpha}\|X\|_\beta\omega(s,t)^{\beta+2\alpha} + \|D\sigma\|_\infty\|Z'\|_\alpha\|\mathbb{X}\|_{2\beta}\omega(s,t)^{\alpha+2\beta} \\
&\quad + C\|D\sigma\|_{\gamma-2}\Big\{\left(\|Z'\|_\infty\|X\|_\beta\omega(s,t)^{\beta-\alpha}\right)^{\gamma-2} + \left(\|R^Z\|_{2\alpha}\omega(s,t)^\alpha\right)^{\gamma-2} \\
&\qquad\qquad\qquad + \left(\|\Phi\|_{q-var,\tilde\alpha}\omega(s,t)^{\tilde\alpha-\alpha}\right)^{\gamma-2}\Big\}\cdot \\
&\left\{\left(\|Z'\|_\infty\|X\|_\beta\omega(s,t)^\beta + \|\Phi\|_{q-var,\tilde\alpha}\omega(s,t)^{\tilde\alpha}\right)\|X\|_\beta\omega(s,t)^\beta + \|Z'\|_\infty\|\mathbb{X}\|_{2\beta}\omega(s,t)^{2\beta}\right\} \\
&\qquad\qquad \cdot\,\omega(s,t)^{\alpha(\gamma-2)},
\end{aligned}
\tag{3.21}
$$

where $C = 3^{\gamma-2} + 2^\beta\zeta(\beta p)^{1/p}$ and $\|\cdot\|_\infty$ denotes the sup norm. Therefore, there exists a positive constant C which depends only on γ, β, p such that

$$
\begin{aligned}
\left|(\delta\Xi)_{s,u,t}\right| &\le CK_\sigma f\left(\|R^Z\|_{2\alpha}, \|Z'\|_\alpha, \|Z'\|_\infty, \|\Phi\|_{q-var,\tilde\alpha}\right)\widetilde{|||\mathbf{X}|||_\beta}\left(1+\omega(s,t)^{1/2}\right)\omega(s,t)^{\beta+(\gamma-1)\alpha} \\
&\qquad 0 \le s \le t \le T
\end{aligned}
\tag{3.22}
$$

where

$$
K_\sigma = \|D\sigma\|_{\gamma-2} + \|D\sigma\|_\infty, \tag{3.23}
$$
$$
f(x,y,z,w) = x + y + \left(x^{\gamma-2} + z^{\gamma-2} + w^{\gamma-2}\right)(z+w). \tag{3.24}
$$

Let $\mathcal{P} = \{t_k\}_{k=0}^N$ be a partition of $[s,t]$. We write $[u,v] \in \mathcal{P}$ if $u = t_{k-1}$ and $v = t_k$ for a $1 \le k \le N$. We denote $|\mathcal{P}| = \max_k |t_k - t_{k-1}|$. Since $\beta + (\gamma-1)\alpha > \beta + p\alpha - \alpha \ge p\alpha > 1$, by a standard argument in the Young integration and rough path (see the proof of Lemma 3.5 and [21, 16, 15]), the following limit exists,

$$
I((Z,Z'),\Phi)_{s,t} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}}\Xi_{u,v}. \tag{3.25}
$$

We may denote this by

$$
I(Z,\Phi)_{s,t} \quad \text{or} \quad \int_s^t \sigma(Z_u,\Phi_u)d\mathbf{X}_u \tag{3.26}
$$

simply if there are no confusion. This integral satisfies the additivity property

$$
I((Z,Z'),\Phi)_{s,u} + I((Z,Z'),\Phi)_{u,t} = I((Z,Z'),\Phi)_{s,t} \qquad 0 \le s \le u \le t \le T. \tag{3.27}
$$

The pair $(I((Z,Z'),\Phi), \sigma(Y_t))$ is actually a controlled path of X. In fact, we have the following estimates.

Lemma 3.6. *Assume* $(Z, Z') \in \mathscr{D}_X^{2\alpha}(\mathbb{R}^d)$ *and* $\Phi \in \mathcal{V}_{q,\tilde{\alpha}}(\mathbb{R}^d)$ *and* $q, \alpha, \tilde{\alpha}$ *satisfy* (3.18). *For any* $0 \le s \le t \le T$*, we have the following estimates. The constant* K *below depends only on* $\|D\sigma\|_\infty, \|D\sigma\|_{\gamma-2}, \alpha, \beta, p, \gamma$ *and may change line by line.*

(1)

$$|\Xi_{s,t}| \le \Big\{ \|\sigma\|_\infty \|X\|_\beta + \|D\sigma\|_\infty \|Z'\|_\infty \|\mathbb{X}\|_{2\beta} \omega(s,t)^\beta + 2^\beta \zeta(\beta p)^{1/p} \|D\sigma\|_\infty \|\Phi\|_{q-var,\tilde{\alpha}} \|X\|_\beta \omega(s,t)^{\tilde{\alpha}} \Big\} \omega(s,t)^\beta. \tag{3.28}$$

(2)

$$|I(Z,\Phi)_{s,t} - \Xi_{s,t}| \le K f\left(\|R^Z\|_{2\alpha}, \|Z'\|_\alpha, \|Z'\|_\infty, \|\Phi\|_{q-var,\tilde{\alpha}}\right) \widetilde{|||\mathbf{X}|||}_\beta \left(1 + \omega(s,t)^{1/2}\right) \omega(s,t)^{\gamma\alpha+\beta-\alpha} \tag{3.29}$$

and

$$\|I(Z,\Phi)\|_\beta \le K g\left(\|R^Z\|_{2\alpha}, \|Z'\|_\alpha, \|Z'\|_\infty, \|\Phi\|_{q-var,\tilde{\alpha}}\right) \left(1 + \omega(s,t)^{1/2}\right) \widetilde{|||\mathbf{X}|||}_\beta, \tag{3.30}$$

where

$$f(x,y,z,w) = x + y + (x^{\gamma-2} + z^{\gamma-2} + w^{\gamma-2})(z+w), \tag{3.31}$$

$$g(x,y,z,w) = f(x,y,z,w) + z + w. \tag{3.32}$$

(3)

$$|I(Z,\Phi)_{s,t} - \sigma(Y_s)X_{s,t}| \le \Bigg\{ K f\left(\|R^Z\|_{2\alpha}, \|Z'\|_\alpha, \|Z'\|_\infty, \|\Phi\|_{q-var,\tilde{\alpha}}\right) \widetilde{|||\mathbf{X}|||}_\beta \left(1 + \omega(s,t)^{1/2}\right) \omega(s,t)^{\gamma\alpha - 2\tilde{\alpha} + \beta - \alpha} + \|D\sigma\|_\infty \|Z'\|_\infty \|\mathbb{X}\|_{2\beta} \omega(s,t)^{2(\beta-\tilde{\alpha})} + 2^\beta \zeta(\beta p)^{1/p} \|D\sigma\|_\infty \|\Phi\|_{q-var,\tilde{\alpha}} \|X\|_\beta \omega(s,t)^{\beta-\tilde{\alpha}} \Bigg\} \omega(s,t)^{2\tilde{\alpha}}. \tag{3.33}$$

(4)

$$|\sigma(Y_t) - \sigma(Y_s)| \le \|D\sigma\|_\infty \left\{ \|Z'\|_\infty \|X\|_\beta \omega(s,t)^{\beta-\tilde{\alpha}} + \|R^Z\|_{2\alpha} \omega(s,t)^{2\alpha-\tilde{\alpha}} + \|\Phi\|_{q-var,\tilde{\alpha}} \right\} \omega(s,t)^{\tilde{\alpha}}. \tag{3.34}$$

(5) $I(Z,\Phi) \in \mathscr{D}_X^{2\tilde{\alpha}}$ *and* $I(Z,\Phi)'_t = \sigma(Z_t, \Phi_t)$.

Proof. (1) This follows from the explicit form of (3.19) and Lemma 3.5.
(2) This follows from (3.22) and the standard argument cited above.
(3) This follows from (2) and Lemma 3.5.
(4) This follows from the definition of Y_t.
(5) This follows from (3) and (4). □

We prove the existence of solutions by using Schauder's fixed point theorem. To this end, we consider the product Banach space $\mathscr{D}_X^{2\theta_1} \times \mathcal{V}_{q,\theta_2}$, where $1/3 < \theta_1 \leq 1/2$ and $0 < \theta_2 \leq 1$. The norm is defined by

$$\|((Z, Z'), \Phi)\| = |Z_0| + |Z_0'| + \|Z'\|_{\theta_1,[0,T]} + \|R^Z\|_{2\theta_1,[0,T]} + |\Phi_0| + \|\Phi\|_{q-var,\theta_2,[0,T]}. \tag{3.35}$$

Let $\xi, \eta \in \mathbb{R}^d$. Let

$$\mathcal{W}_{T,\theta_1,\theta_2,q,\xi,\eta} = \Big\{ \big((Z, Z'), \Phi\big) \in \mathscr{D}_X^{2\theta_1} \times \mathcal{V}_{q,\theta_2} \mid Z_0 = \xi, Z_0' = \sigma(\xi, \eta), \Phi_0 = \eta \Big\}. \tag{3.36}$$

The solution of RDE could be obtained as a fixed point of the mapping,

$$\mathcal{M} : \big((Z, Z'), \Phi\big) \, (\in \mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}) \mapsto ((\xi + I(Z, \Phi), \sigma(Y)), L(\xi + I(Z, \Phi))) \, (\in \mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}). \tag{3.37}$$

The following continuity property is necessary for the proof of the main result.

Lemma 3.7 (Continuity)**.** *Assume*

$$\frac{1}{3} < \alpha < \tilde{\alpha} < \beta, \quad \alpha p > 1, \quad 1 < q < \min\left(\frac{p}{p-1}, \frac{\beta}{\tilde{\alpha}}\right). \tag{3.38}$$

Then $\mathcal{M}$ is continuous.

By Lemma 2.1, $\mathcal{V}_{q',\theta} \subset \mathcal{V}_{q,\theta}$ for $1 \leq q' < q$ and $\|\Phi\|_{q-var,\theta} \leq \|\Phi\|_{q'-var,\theta}$. We also have the following compactness result. This also is necessary for applying Schauder's fixed point theorem to the proof of existence theorem.

Lemma 3.8. (1) *Let $1 \leq q' < q$. Let θ, θ' be positive numbers such that $q\theta < q'\theta'$. Then the inclusion $\mathcal{V}_{q',\theta'} \subset \mathcal{V}_{q,\theta}$ is compact.*

(2) *Let $\frac{1}{3} < \theta < \theta' \leq \beta$. Then $\mathscr{D}_X^{2\theta'} \subset \mathscr{D}_X^{2\theta}$ and the inclusion is compact.*

We prove Lemma 3.8 first.

Proof of Lemma 3.8. (1) By Lemma 2.1 (1), we have

$$\|\Phi\|_{q-var,[s,t]} \leq \|\Phi\|_{q'-var,\theta',[s,t]}^{q'/q} \omega(s,t)^{(\theta' q')/q-\theta} \|\Phi\|_{\infty-var,[s,t]}^{(q-q')/q} \omega(s,t)^{\theta}. \tag{3.39}$$

If $\sup_n |(\Phi_n)_0| + \|\Phi_n\|_{q'-var,\theta'} < \infty$, then by their equicontinuities, there exists a subsequence such that Φ_{n_k} converges to a certain function Φ_∞ in the uniform norm. By the above estimate, we can conclude that the convergence takes place with respect to the norm on $\mathcal{V}_{q,\theta}$.
(2) Suppose

$$\sup_n \|(Z(n), Z(n)')\|_{\theta'} = \sup_n \{|Z(n)_0| + |Z(n)_0'| + \|Z(n)'\|_{\theta'} + \|R^{Z(n)}\|_{2\theta'}\} < \infty. \tag{3.40}$$

This implies $\{Z(n)'\}$ is bounded and equicontinuous. Since $Z(n)_t - Z(n)_s = Z(n)_s' X_{s,t} + R_{s,t}^{Z(n)}$, $\{Z(n)\}$ is also bounded and equicontinuous. Hence certain subsequence $\{Z(n_k), Z(n_k)'\}$ converges uniformly. This and a similar calculation to (1) imply $\{(Z(n_k)', R^{Z(n_k)})\}$ converges in $\mathscr{D}_X^{2\theta}$. □

Proof of Lemma 3.7. First note that

$$\mathcal{M}(\mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}) \subset \mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}. \tag{3.41}$$

This follows from Lemma 3.6. We estimate $\|I(Z,\Phi)' - I(\tilde{Z},\tilde{\Phi})'\|_\alpha$. We have

$$\begin{aligned}
&\left|\Big(\sigma(Y_t) - \sigma(\tilde{Y}_t)\Big) - \Big(\sigma(Y_s) - \sigma(\tilde{Y}_s)\Big)\right| \\
&= \int_0^1 \Big\{(D\sigma)(Y_s + \theta Y_{s,t})(Y_{s,t}) - (D\sigma)(\tilde{Y}_s + \theta \tilde{Y}_{s,t})(\tilde{Y}_{s,t})\Big\} \\
&\le \|D\sigma\|_\infty |Y_{s,t} - \tilde{Y}_{s,t}| + \|D\sigma\|_{\gamma-2} 2^{\gamma-2}\left(|Y_s - \tilde{Y}_s|^{\gamma-2} + |Y_{s,t} - \tilde{Y}_{s,t}|^{\gamma-2}\right)|Y_{s,t}| \\
&\le \|D\sigma\|_\infty \left(\|Z' - \tilde{Z}'\|_\alpha \omega(0,s)^\alpha \|X\|_\beta \omega(s,t)^\beta + \|R^Z - R^{\tilde{Z}}\|_{2\alpha}\omega(s,t)^{2\alpha} + \|\Phi - \tilde{\Phi}\|_{q,\tilde{\alpha}}(t-s)^{\tilde{\alpha}}\right) \\
&\quad + 2^{\gamma-2}\|D\sigma\|_{\gamma-2}\Big\{\Big(|\sigma(\xi)|\|Z' - \tilde{Z}'\|_\alpha \|X\|_\beta \omega(0,s)^\beta + \|R^Z - R^{\tilde{Z}}\|_{2\alpha}\omega(0,s)^{2\alpha} \\
&\qquad\qquad + \|\Phi - \tilde{\Phi}\|_{q,\tilde{\alpha}}\omega(0,s)^{\tilde{\alpha}}\Big)^{\gamma-2} \\
&\quad + \left(\|Z' - \tilde{Z}'\|_\alpha \omega(0,s)^\alpha \|X\|_\beta \omega(s,t)^\beta + \|R^Z - R^{\tilde{Z}}\|_{2\alpha}\omega(s,t)^{2\alpha} + \|\Phi - \tilde{\Phi}\|_{q,\tilde{\alpha}}\omega(s,t)^{\tilde{\alpha}}\right)^{\gamma-2}\Big\} \\
&\quad\times \left((|\sigma(\xi)| + \|Z'\|_\alpha \omega(0,s)^\alpha)\|X\|_\beta \omega(s,t)^\beta + \|R^Z\|_{2\alpha}\omega(s,t)^{2\alpha} + \|\Phi\|_{q-var,\tilde{\alpha}}\omega(s,t)^{\tilde{\alpha}}\right).
\end{aligned} \tag{3.42}$$

Since $\beta > \tilde{\alpha} \ge \alpha$, this shows the continuity of the mapping $((Z,Z'),\Phi) \mapsto I(Z,\Phi)'$.

We next estimate the difference $R^{I(Z,\Phi)} - R^{I(\tilde{Z},\tilde{\Phi})}$.

$$\begin{aligned}
|R^{I(Z,\Phi)}_{s,t} - R^{I(\tilde{Z},\tilde{\Phi})}_{s,t}| &= \left|(I(Z,\Phi)_{s,t} - \sigma(Y_s)X_{s,t}) - \Big(I(\tilde{Z},\tilde{\Phi})_{s,t} - \sigma(\tilde{Y}_s)X_{s,t}\Big)\right| \\
&\le \left|(I(Z,\Phi)_{s,t} - \Xi(Z,\Phi)_{s,t}) - \Big(I(\tilde{Z},\tilde{\Phi})_{s,t} - \Xi(\tilde{Z},\tilde{\Phi})_{s,t}\Big)\right| \\
&\quad + \left|(D_1\sigma)(Y_s)(Z'_s\mathbb{X}_{s,t}) - (D_1\sigma)(\tilde{Y}_s)(\tilde{Z}'_s\mathbb{X}_{s,t})\right| \\
&\quad + \left|(D_2\sigma)(Y_s)\left(\int_s^t \Phi_{s,u}\otimes d\mathbf{X}_u\right) - (D_2\sigma)(\tilde{Y}_s)\left(\int_s^t \tilde{\Phi}_{s,u}\otimes d\mathbf{X}_u\right)\right|.
\end{aligned} \tag{3.43}$$

Let $\mathcal{P}_N = \{t_k^N = s + \frac{k(t-s)}{2^N}\}$ be a dyadic partition of $[s,t]$. Recall that $[u,v] \in \mathcal{P}$ is equivalent to $u = t_{k-1}, v = t_k$ for a $1 \le k \le N$. Then

$$\begin{aligned}
&\left|(I(Z,\Phi)_{s,t} - \Xi(Z,\Phi)_{s,t}) - \Big(I(\tilde{Z},\tilde{\Phi})_{s,t} - \Xi(\tilde{Z},\tilde{\Phi})_{s,t}\Big)\right| \\
&\le \left|\left(\sum_{[u,v]\in\mathcal{P}_N} \Xi(Z,\Phi)_{u,v} - \Xi(Z,\Phi)_{s,t}\right) - \left(\sum_{[u,v]\in\mathcal{P}_N} \Xi(\tilde{Z},\tilde{\Phi})_{u,v} - \Xi(\tilde{Z},\tilde{\Phi})_{s,t}\right)\right| \\
&\quad + \left|\left(I(Z,\Phi)_{s,t} - \sum_{[u,v]\in\mathcal{P}_N} \Xi(Z,\Phi)_{u,v}\right)\right| \\
&\quad + \left|\left(I(\tilde{Z},\tilde{\Phi})_{s,t} - \sum_{[u,v]\in\mathcal{P}_N}^{N} \Xi(\tilde{Z},\tilde{\Phi})_{u,v}\right)\right|.
\end{aligned} \tag{3.44}$$

By (3.29),

$$\begin{aligned}
&\left|\left(I(Z,\Phi)_{s,t}-\sum_{[u,v]\in\mathcal{P}_N}\Xi(Z,\Phi)_{u,v}\right)\right|+\left|\left(I(\tilde{Z},\tilde{\Phi})_{s,t}-\sum_{[u,v]\in\mathcal{P}_N}\Xi(\tilde{Z},\tilde{\Phi})_{u,v}\right)\right|\\
&\le K\left\{f(\|R^Z\|_{2\alpha},\|Z'\|_\alpha,\|Z'\|_\infty,\|\Phi\|_{q-var,\tilde{\alpha}})+f(\|R^{\tilde{Z}}\|_{2\alpha},\|\tilde{Z}'\|_\alpha,\|\tilde{Z}'\|_\infty,\|\Phi'\|_{q-var,\tilde{\alpha}})\right\}\cdot\\
&\qquad\widetilde{\|\!|\mathbf{X}|\!\|_\beta}\max_{[u,v]\in\mathcal{P}_N}\omega(u,v)^{(\gamma-1)\alpha+\beta-1}K(T)\omega(s,t). \qquad (3.45)
\end{aligned}$$

Hence this term is small in the ω-Hölder space $\mathcal{V}_{2\alpha}$ on a bounded set of $\mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}$ if N is large. We fix a partition so that this term is small. Although the partition number may be big,

$$\begin{aligned}
&\left(\sum_{[u,v]\in\mathcal{P}_N}\Xi(Z,\Phi)_{u,v}-\Xi(Z,\Phi)_{s,t}\right)-\left(\sum_{[u,v]\in\mathcal{P}_N}\Xi(\tilde{Z},\tilde{\Phi})_{u,v}-\Xi(\tilde{Z},\tilde{\Phi})_{s,t}\right)\\
&=\sum_{k=0}^{N}\sum_{[u,v]\in\mathcal{P}_k}\left(\delta\Xi(Z,\Phi)_{u,(u+v)/2,v}-\delta\Xi(\tilde{Z},\tilde{\Phi})_{u,(u+v)/2,v}\right) \qquad (3.46)
\end{aligned}$$

is a finite sum, and by the explicit form of $\delta\Xi$ as in (3.20), we see that this difference is small in $\mathcal{V}_{2\alpha}$ if $((Z,Z'),\Phi)$ and $((\tilde{Z},\tilde{Z}'),\tilde{\Phi})$ are sufficiently close in $\mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta}$. The estimate of the other terms are similar to the above and we obtain the continuity of the mapping

$$((Z,Z'),\Phi)(\in\mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta})\mapsto(\xi+I(Z,\Phi),\sigma(Y))\,(\in\mathscr{D}_X^{2\alpha}). \qquad (3.47)$$

We next prove the continuity of the mapping

$$((Z,Z'),\Phi)(\in\mathcal{W}_{T,\alpha,\tilde{\alpha},q,\xi,\eta})\mapsto L(\xi+I(Z,\Phi))(\in\mathcal{V}_{q,\tilde{\alpha}}). \qquad (3.48)$$

Applying (3.39) to the case $q'=1,\theta'=\beta,\theta=\tilde{\alpha}$ under the assumption $q<\beta/\tilde{\alpha}$, we get

$$\|\Phi\|_{q-var,\tilde{\alpha},[s,t]}\le\omega(s,t)^{\frac{\beta-\tilde{\alpha}q}{q}}\|\Phi\|_{1-var,\beta,[s,t]}^{1/q}\|\Phi\|_{\infty-var,[s,t]}^{(q-1)/q}. \qquad (3.49)$$

Therefore, by Assumption 3.1 (3) on L, we obtain

$$\begin{aligned}
&\|L(\xi+I(Z,\Phi))-L(\xi+I(\tilde{Z},\tilde{\Phi}))\|_{q-var,\tilde{\alpha}}\\
&\le 2^{1/q}\omega(0,T)^{\frac{\beta-\tilde{\alpha}q}{q}}\left(\|L(\xi+I(Z,\Phi))\|_{1-var,\beta}^{1/q}+\|L(\xi+I(\tilde{Z},\tilde{\Phi}))\|_{1-var,\beta}^{1/q}\right)\\
&\qquad\times\|L(\xi+I(Z,\Phi))-L(\xi+I(\tilde{Z},\tilde{\Phi}))\|_{\infty-var}^{(q-1)/q}\\
&\le\left\{H\left(\|I(Z,\Phi)\|_\beta\omega(0,T)^\beta\right)^{1/q}+H\left(\|I(\tilde{Z},\tilde{\Phi})\|_\beta\omega(0,T)^\beta\right)^{1/q}\right\}\\
&\qquad\qquad\times\|L(\xi+I(Z,\Phi))-L(\xi+I(\tilde{Z},\tilde{\Phi}))\|_{\infty-var}^{(q-1)/q}, \qquad (3.50)
\end{aligned}$$

where $H(x)=F(x)x$. Hence the continuity of the mappings $((Z,Z'),\Phi)\mapsto I((Z,Z'),\Phi)$ which we have proved above and the continuity of the mapping $w\mapsto L(w)$ and (3.30) implies the desired continuity. □

4 Proof of Theorem 3.3

First, we prove the existence of solutions on small interval $[0,T']$ by using Schauder's fixed point theorem. Since the interval can be chosen independent of the initial condition, we obtain the global existence of solutions and the estimate for solutions. To this end, take $0<T'<T$ such that $\omega(0,T')\le 1$ and consider $\mathcal{W}_{T',\alpha,\tilde{\alpha},q,\xi,0}$. In this section, we assume that $q,\alpha,\tilde{\alpha}$ satisfy

$$\frac{1}{3}<\alpha<\tilde{\alpha}<\beta,\quad \alpha p>1,\quad 1<q<\min\left(\frac{p}{p-1},\ \frac{\beta}{\tilde{\alpha}}\right). \tag{4.1}$$

We consider balls with radius 1 centered at $((\xi+\sigma(\xi,\eta)X_t,\sigma(\xi,\eta)),\eta)\quad(0\le t\le T')$,

$$\mathcal{B}_{T',\theta_1,\theta_2,q}=\left\{((Z,Z'),\Phi)\in\mathcal{W}_{T',\theta_1,\theta_2,q,\xi,0}\mid \|Z'\|_{\theta_1,[0,T']}+\|R^Z\|_{\theta_1,[0,T']}+\|\Phi\|_{q-var,\theta_2,[0,T']}\le 1\right\}. \tag{4.2}$$

Below, we consider two balls $\mathcal{B}_{T',\underline{\alpha},\tilde{\alpha},q}$ and $\mathcal{B}_{T',\tilde{\alpha},\bar{\alpha},q}$, where $\bar{\alpha}$ is any positive number such that $\tilde{\alpha}<\bar{\alpha}<\beta$.

Lemma 4.1 (Invariance and compactness). *Assume* (4.1) *and let $\alpha<\underline{\alpha}<\tilde{\alpha}<\bar{\alpha}<\beta$. For sufficiently small T', we have*

$$\mathcal{M}(\mathcal{B}_{T',\alpha,\tilde{\alpha},q})\subset\mathcal{B}_{T',\underline{\alpha},\bar{\alpha},q}\subset\mathcal{B}_{T',\alpha,\tilde{\alpha},q}. \tag{4.3}$$

Moreover T' does not depend on ξ and η.

Proof. The second inclusion is immediate because $\omega(0,T')\le 1$ and the definition of the norms. We prove the first inclusion. Recall that $I(Z,\Phi)'_t=\sigma(Z_t,\Phi_t)$. From Lemma 3.6 (4), we have

$$\begin{aligned}\|I(Z,\Phi)'\|_{\underline{\alpha},[0,T']}\le\|D\sigma\|_\infty\Big\{&\|Z'\|_{\infty,[0,T']}\|X\|_\beta\omega(0,T')^{\beta-\underline{\alpha}}+\|R^Z\|_{2\alpha,[0,T]}\omega(0,T')^{2\alpha-\underline{\alpha}}\\&+\|\Phi\|_{q-var,\tilde{\alpha},[0,T']}\omega(0,T')^{\tilde{\alpha}-\underline{\alpha}}\Big\}\end{aligned} \tag{4.4}$$

We next estimate $R^{I(Z,\Phi)}$. Let $0<s<t<T'$. By Lemma 3.6 (3), we have

$$\begin{aligned}&|R^{I(Z,\Phi)}_{s,t}|\\&\quad\le\|D\sigma\|_\infty\|Z'\|_{\infty,[0,T']}\|\mathbb{X}\|_{2\beta}\omega(s,t)^{2\beta}\\&\quad+2^\beta\zeta(\beta p)^{1/p}\|D\sigma\|_\infty\|\Phi\|_{q-var,\tilde{\alpha},[0,T']}\|X\|_\beta\omega(s,t)^{\tilde{\alpha}+\beta}\\&\quad+Kf\left(\|R^Z\|_{2\alpha,[0,T']}\|Z'\|_{\alpha,[0,T']},\|Z'\|_{\infty,[0,T']},\|\Phi\|_{q-var,\tilde{\alpha},[0,T']}\right)\widetilde{\|\!|\mathbf{X}|\!\|_\beta}\,\omega(s,t)^{\gamma\alpha+\beta-\alpha}\end{aligned} \tag{4.5}$$

which implies

$$\begin{aligned}\|R^{I(Z,\Phi)}\|_{2\underline{\alpha},[0,T']}\le&\|D\sigma\|_\infty\|Z'\|_{\infty,[0,T']}\|\mathbb{X}\|_{2\beta}\omega(0,T')^{2(\beta-\underline{\alpha})}\\&+2^\beta\zeta(\beta p)^{1/p}\|D\sigma\|_\infty\|\Phi\|_{q-var,\tilde{\alpha},[0,T']}\|X\|_\beta\omega(0,T')^{\tilde{\alpha}+\beta-2\underline{\alpha}}\\&+Kf\left(\|R^Z\|_{2\alpha,[0,T']}\|Z'\|_{\alpha,[0,T']},\|Z'\|_{\infty,[0,T']},\|\Phi\|_{q-var,\tilde{\alpha},[0,T']}\right)\\&\quad\times\widetilde{\|\!|\mathbf{X}|\!\|_\beta}\,\omega(0,T')^{\gamma\alpha+\beta-\alpha-2\underline{\alpha}}.\end{aligned} \tag{4.6}$$

We turn to the estimate of $L(\xi+I(Z,\Phi))$. By Assumption 3.1 (3) on L and applying Lemma 2.1 (1) and Lemma 3.6 (2), we have

$$\begin{aligned}\|L(\xi+I(Z,\Phi))\|_{q-var,\bar{\alpha},[0,T']} &\le F\left(\|I(Z,\Phi)\|_{\beta,[0,T']}\omega(0,T')^{\beta}\right)\|I(Z,\Phi)\|_{\beta,[0,T']}\omega(0,T')^{\beta-\bar{\alpha}} \\ &\le F\left(K\widetilde{\|\!|\mathbf{X}|\!\|_{\beta}}\right)K\widetilde{\|\!|\mathbf{X}|\!\|_{\beta}}\omega(0,T')^{\beta-\bar{\alpha}}. \end{aligned} \tag{4.7}$$

Here we have used $\omega(0,T')\le 1$. This term is small if T' is small. All the estimates above implies the desired inclusion for small T'. To obtain explicit, T', let

$$\kappa_0=\min\left\{\beta-\underline{\alpha},2\alpha-\tilde{\alpha},\tilde{\alpha}-\underline{\alpha},\tilde{\alpha}+\beta-2\underline{\alpha},\gamma\alpha+\beta-\alpha-2\underline{\alpha}\right\}^{-1}. \tag{4.8}$$

Then $\kappa_0=\min\left\{\beta-\underline{\alpha},\frac{1}{6},\tilde{\alpha}-\underline{\alpha}\right\}^{-1}$. Let $G(x)=1/\left(xF(x)\right)^{\kappa_0}$. We can take T' such that $\omega(0,T')\le\min\left(1,G(K\widetilde{\|\!|\mathbf{X}|\!\|_{\beta}})\right)$. □

Proof of Theorem 3.3. In the proof of Lemma 3.7 and Lemma 4.1, technical constants $\alpha,\tilde{\alpha},\underline{\alpha},\bar{\alpha}$ appeared. We fix them as follows,

$$\alpha=\frac{1}{p}\left(1+\frac{\beta p-1}{2}\right),\quad \beta-\bar{\alpha}=\bar{\alpha}-\tilde{\alpha}=\tilde{\alpha}-\underline{\alpha}=\underline{\alpha}-\alpha=\frac{\beta-\alpha}{4}. \tag{4.9}$$

For these constants, let $\kappa=\kappa_0$. By Lemma 3.7 and Lemma 4.1, applying Schauder's fixed point theorem, we obtain a fixed point for small interval $[0,T']$ with $\omega(0,T')\le\min\left(1,G(K\widetilde{\|\!|\mathbf{X}|\!\|_{\beta}})\right)$. That is, there exists a solution on $[0,T']$. We now consider the equation on $[T',T]$. We can rewrite the equation as

$$Z_{T'+t}=Z_{T'}+\int_{T'}^{T'+t}\sigma(Z_u,\Phi_u)d\mathbf{X}_u \qquad 0\le t\le T-T', \tag{4.10}$$

$$\Phi_{T'+t}=L\left(\xi+\int_0^{\cdot}\sigma(Z_u,\Phi_u)d\mathbf{X}_u\right)_{T'+t} \qquad 0\le t\le T-T'. \tag{4.11}$$

Therefore $\tilde{Z}_t=Z_{T'+t}$ and $\tilde{\Phi}_t=\Phi_{T'+t}$ $(0\le t\le T-T')$ is a solution to

$$\tilde{Z}_t=\tilde{Z}_0+\int_0^t\sigma(\tilde{Z}_u,\tilde{\Phi}_u)d\mathbf{X}_{T'+u} \qquad 0\le t\le T-T', \tag{4.12}$$

$$\tilde{\Phi}_t=\tilde{L}_{T'}\left(\int_0^{\cdot}\sigma(\tilde{Z}_u,\tilde{\Phi}_u)d\mathbf{X}_{T'+u}\right)_t \qquad 0\le t\le T-T'. \tag{4.13}$$

where

$$\tilde{L}_{T'}(w)_t=L(\tilde{w})_{T'+t},\quad w\in C([0,T-T'],\mathbb{R}^d;w_0=0),\quad 0\le t\le T-T' \tag{4.14}$$

and

$$\tilde{w}_s=\begin{cases}\xi+\int_0^s\sigma(Z_u,\Phi_u)d\mathbf{X}_u & s\le T'\\ \xi+\int_0^{T'}\sigma(Z_u,\Phi_u)d\mathbf{X}_u+w_{s-T'} & T'\le s\le T.\end{cases} \tag{4.15}$$

For two paths, $w_1,w_2\in C([0,T-T'],\mathbb{R}^d;w_0=0)$, note that $\|\tilde{w_1}-\tilde{w_2}\|_{\beta}$ is equal to the $(\omega(T'+\cdot,T'+\cdot),\beta)$-Hölder norm of w_1-w_2 . By the assumption on L, we see that $\tilde{L}_{T'}$ also satisfies Assumption 3.1 with the same function F. Therefore, we can do the same argument as $[0, T']$ for small interval. By iterating this procedure finite time, say, N-times, we obtain a controlled path (Z_t, Z'_t) $(0 \le t \le T)$. This is a solution to (3.7) and (3.8). Clearly,

$$N - 1 \le K(1 + \widetilde{\|\mathbf{X}\|_\beta})^\kappa \left(1 + F(K\widetilde{\|\mathbf{X}\|_\beta})\right)^\kappa \omega(0, T) \tag{4.16}$$

We need to show $(Z, \Phi) \in \mathcal{W}_{T,\beta,\beta,1,\xi,\eta}$ and its estimate with respect to the norm $\|\cdot\|_\beta$. We give the estimate of the solution on $[0, T']$. The solution (Z, Z') which we obtained satisfies

$$\|Z'\|_{\alpha,[0,T']} + \|R^Z\|_{\alpha,[0,T']} + \|\Phi\|_{q-var,\tilde{\alpha},[0,T']} \le 1. \tag{4.17}$$

Let $0 \le u \le v \le T'$. From (4.17), (3.30) and (3.11), we have

$$\|Z\|_{\beta,[u,v]} \le K\widetilde{\|\mathbf{X}\|_\beta}. \tag{4.18}$$

Second, by (3.4), (3.30) and (3.12), we have

$$\|L(Z)\|_{1-var,[u,v]} \le F(K\widetilde{\|\mathbf{X}\|_\beta})K\widetilde{\|\mathbf{X}\|_\beta}\omega(u, v)^\beta. \tag{4.19}$$

Therefore Z and $L(Z)$ are (ω, β)-Hölder continuous paths. Thus, by combining these estimates and (3.33) (we need to replace $\tilde{\alpha}$ by β and this is possible), we obtain for $0 \le u \le v \le T'$,

$$|R^Z_{u,v}| = |Z_v - Z_u - \sigma(Y_u)X_{u,v}| \le K\widetilde{\|\mathbf{X}\|_\beta}\omega(u, v)^{2\beta}. \tag{4.20}$$

These local estimates hold on other small intervals. By the estimate (4.16), we obtain the desired estimate. □

5 Reflected rough differential equations

Let D be a connected domain. The boundary need not be smooth. As in [24, 19], we consider the following conditions (A), (B) on the boundary.

Definition 5.1. *We write $B(z, r) = \{y \in \mathbb{R}^d \mid |y - z| < r\}$, where $z \in \mathbb{R}^d$, $r > 0$. The set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ is defined by*

$$\mathcal{N}_x = \cup_{r>0}\mathcal{N}_{x,r}, \tag{5.1}$$

$$\mathcal{N}_{x,r} = \left\{\boldsymbol{n} \in \mathbb{R}^d \mid |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset\right\}. \tag{5.2}$$

(A) *There exists a constant $r_0 > 0$ such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \ne \emptyset \quad \text{for any } x \in \partial D.$$

(B) *There exist constants $\delta > 0$ and $0 < \delta' \le 1$ satisfying:*
for any $x \in \partial D$ there exists a unit vector l_x such that

$$(l_x, \boldsymbol{n}) \ge \delta' \qquad \text{for any } \boldsymbol{n} \in \cup_{y \in B(x,\delta)\cap\partial D}\mathcal{N}_y.$$

Let us recall the Skorohod equation. The Skorohod equation associated with a continuous path $w \in C([0,\infty),\mathbb{R}^d)$ with $w_0 \in \bar{D}$ is given by

$$y_t = w_t + \phi_t, \quad y_t \in \bar{D} \qquad t \geq 0, \tag{5.3}$$

$$\phi_t = \int_0^t 1_{\partial D}(y_s)\boldsymbol{n}(s)d\|\phi\|_{1-var,[0,s]} \quad t \geq 0, \qquad \boldsymbol{n}(s) \in \mathcal{N}_{y_s} \text{ if } y_s \in \partial D \tag{5.4}$$

Under the assumptions (A) and (B) on D, the Skorohod equation is uniquely solved. This is due to Saisho [24]. We write $\Gamma(w)_t = y_t$ and $L(w)_t = \phi_t$. By the uniqueness, we have the following flow property.

Lemma 5.2. *Assume* (A) *and* (B). *For any continuous path w on $\mathbb{R}^d$ with $w_0 \in \bar{D}$, we have for all $\tau, s \geq 0$*

$$\Gamma(w)_{\tau+s} = \Gamma\left(y_s + \theta_s w\right)_\tau, \tag{5.5}$$

$$L(w)_{\tau+s} = L(w)_s + L\left(y_s + \theta_s w\right)_\tau, \tag{5.6}$$

where $(\theta_s w)(\tau) = w(\tau+s) - w(s)$.

The following lemmas can be found in [2, 4] which are essentially due to Saisho [24].

Lemma 5.3. *Assume conditions* (A) *and* (B) *hold. Let w_t be a continuous path of finite q-variation ($q \geq 1$). Then we have the following estimate.*

$$\|L(w)\|_{1-var,[s,t]} \leq \delta'^{-1}\left(\left\{\delta^{-1}G(\|w\|_{\infty-var,[s,t]}) + 1\right\}^q \|w\|^q_{q-var,[s,t]} + 1\right) \times \left(G(\|w\|_{\infty-var,[s,t]}) + 2\right)\|w\|_{\infty-var,[s,t]}, \tag{5.7}$$

where

$$G(x) = 4\left\{1 + \delta'^{-1}\exp\left\{\delta'^{-1}\left(2\delta + x\right)/(2r_0)\right\}\right\}\exp\left\{\delta'^{-1}\left(2\delta + x\right)/(2r_0)\right\}, \quad x \in \mathbb{R} \tag{5.8}$$

and δ, δ', r_0 are constants in conditions (A) *and* (B).

Lemma 5.4. *Assume* (A) *and* (B). *Consider two Skorohod equations $y = w + \phi$, $y' = w' + \phi'$. Then*

$$|y_t - y'_t|^2 \leq \left\{|w_t - w'_t|^2 + 4\left(\|\phi\|_{1-var,[0,t]} + \|\phi'\|_{1-var,[0,t]}\right)\max_{0\leq s\leq t}|w(s) - w'(s)|\right\} \exp\left\{\left(\|\phi\|_{1-var,[0,t]} + \|\phi'\|_{1-var,[0,t]}\right)/r_0\right\}. \tag{5.9}$$

Lemma 5.3 shows that if w is a (ω,θ)-Hölder continuous path, $L(w) \in \mathcal{V}_{1,\theta}$ holds true. Actually, $\|L(w)\|_{1-var,[s,t]}$ can be estimated by the modulus of continuity of w and $\|w\|_{\infty-var,[s,t]}$. For example, see [24] and the proof of Lemma 2.3 in [4]. Hence, we see that L is a 1/2-Hölder continuous map on $C([0,\infty),\mathbb{R}^d)$.

By applying main theorem, we obtain the following result. In this theorem, $\mathbf{X}$ is a $1/\beta$-rough path on $\mathbb{R}^n$ as in Theorem 3.3.

Theorem 5.5. *Assume D satisfies conditions* (A) *and* (B). *Let $\sigma \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ and $\xi \in \bar{D}$. Then there exists a controlled path $(Z, Z') \in \mathscr{D}_X^{2\beta}(\mathbb{R}^d)$ and a bounded variation path $\Phi \in \mathcal{V}_{1,\beta}(\mathbb{R}^d)$ with $\Phi_0 = 0$ such that*

$$Y_t = Z_t + \Phi_t, \quad Z'_t = \sigma(Y_t), \tag{5.10}$$

$$Y_t = \xi + \int_0^t \sigma(Y_s) d\mathbf{X}_s + \Phi_t, \tag{5.11}$$

$$\Phi_t = L\left(\xi + \int_0^{\cdot} \sigma(Y_s) d\mathbf{X}_s\right)_t. \tag{5.12}$$

Further this solution satisfy the following estimate

$$\|Z\|_\beta + \|R^Z\|_{2\beta} + \|\Phi\|_{1,\beta} \le C_1 e^{C_2 \widetilde{\|\mathbf{X}\|_\beta}} (1 + \omega(0,T)) \widetilde{\|\mathbf{X}\|}_\beta, \tag{5.13}$$

where C_1, C_2 are constants which depend only on $\sigma, \beta, \gamma, p, \delta, \delta', r_0$.

Proof. We consider the following path-dependent equation:

$$Z_t = \xi + \int_0^t \sigma(Z_s + L(Z)_s) d\mathbf{X}_s, \quad Z'_t = \sigma(Z_t + L(Z_t)). \tag{5.14}$$

By applying Theorem 3.3, we complete the proof. □

Remark 5.6. Assume that condition (A) holds and there exists a positive constant C_D such that for any w, it holds that

$$\|L(w)\|_{1-var,[s,t]} \le C_D \|w\|_{\infty-var,[s,t]}. \tag{5.15}$$

We called this assumption (H1) in [2]. This condition holds if D is convex and there exists a unit vector $l \in \mathbb{R}^d$ such that $\inf\{(l, \boldsymbol{n}(x)) \mid \boldsymbol{n}(x) \in \mathcal{N}_x,\, x \in \partial D\} > 0$. In [2], we proved the existence of solutions and gave estimates for them for rough path with finite $1/\beta$- variation. That is, under the assumption $\sigma \in C_b^3$ and the above assumptions on D, we proved

$$\|Y\|_{\beta,[s,t]} \le C(1 + \omega(0,T))^3 \widetilde{\|\mathbf{X}\|}_\beta, \tag{5.16}$$

for certain positive C. We also note that one can prove the existence of solutions under the assumption $\sigma \in \mathrm{Lip}^{\gamma-1}$ and (H1) by modifying the proof in [2]. In Theorem 3.3 in this paper, we only assume (A) and (B) and we get worse exponential term. However, under the assumption (5.15), it is easy to obtain better bound as in (5.16) by the same proof as in this paper.

The solution Y_t we obtained in [2] is a 1st level path of a rough path which is a solution to the rough differential equation,

$$dY_t = \hat{\sigma}(Y_t) d\hat{X}_t \quad 0 \le t \le T, \quad Y_0 = \xi, \tag{5.17}$$

where $\hat{\sigma}(x)$ is a linear mapping from $\mathbb{R}^n \oplus \mathbb{R}^d$ to $\mathbb{R}^d$ defined by $\hat{\sigma}(x)(v_1, v_2) = \sigma(x)v_1 + v_2$ and the driving rough path $\hat{\mathbf{X}}$ is the rough path on $\mathbb{R}^n \oplus \mathbb{R}^d$ given by

$$\hat{X}^1_{s,t} = (X_{s,t}, \Phi_{s,t}) \tag{5.18}$$

$$\hat{\mathbb{X}}^2_{s,t} = \Big(\mathbb{X}_{s,t}, \int_s^t X_{s,u} \otimes d\Phi_u, \int_s^t \Phi_{s,u} \otimes dX_u, \int_s^t \Phi_{s,u} \otimes d\Phi_u\Big). \tag{5.19}$$

This is proved by showing that there exists $\varepsilon > 0$ such that

$$\left|Y_t - Y_s - \Big(\sigma(Y_s)X_{s,t} + (D\sigma)(Y_s)(\sigma(Y_s)\mathbb{X}_{s,t}) + (D\sigma)(Y_s)\int_s^t \Phi_{s,r}\otimes dX_r + \Phi_t - \Phi_s\Big)\right| \\ \le C\omega(s,t)^{\gamma/p} \quad \text{for all } 0\le s\le t\le T. \tag{5.20}$$

By the definition of the integral in (5.11) and Lemma 3.6 (2), Y_t and Φ_t in Theorem 3.3 also satisfies this estimate. Therefore, the solution obtained in this paper is also a 1st level path of a solution driven by $\hat{\mathbf{X}}$. Hence Theorem 5.5 is an extension of the main theorem in [2].

Remark 5.7. When $D = \mathbb{R}^d$ and $\sigma \in \mathrm{Lip}^\gamma$, the mapping $\mathcal{M}$ is contraction mapping for small T' and we can apply contraction mapping theorem. Gubinelli [17] showed that the existence can be checked by applying Schauder's fixed point theorem when $D = \mathbb{R}^d$ even if $\sigma \in \mathrm{Lip}^{\gamma-1}$. When $\partial D \neq \emptyset$, as in Lemma 5.4, the Skorohod mapping Γ is 1/2-Hölder continuous in the uniform norm under the condition (A) and (B). If D is a half space of $\mathbb{R}^d$, Γ can be written down explicitly and it is easy to check that Γ is Lipschitz in the uniform norm. However, in general, the Lipschitz continuity does not hold any more. We refer the reader to [11] for this.

6 An approximate continuity property and support theorems

In this section, we continue the study of reflected rough differential equations on a domain $D \subset \mathbb{R}^d$ which satisfies the conditions (A) and (B). Let h be a Lipschitz path on $\mathbb{R}^n$ starting at 0. If σ is Lipschitz continuous, there exists a unique solution $(Y(h,\xi)_t, \Phi(h,\xi)_t)$ to the reflected ODE in usual sense,

$$Y_t = \xi + \int_0^t \sigma(Y_s)dh_s + \Phi_t, \quad \xi \in \bar{D}, \quad 0 \le t \le T. \tag{6.1}$$

We may omit denoting h, ξ. Moreover, $Z(h)_t = \xi + \int_0^t \sigma(Y_s(h))dh_s$, $Z_t(h)' = \sigma(Y_t(h))$ and $\Phi(h)_t$ are a unique pair of solution to the equation in Theorem 5.5 for the smooth rough path $\mathbf{h}_{s,t} = (h_{s,t}, \bar{h}^2_{s,t})$ defined by h, where

$$\bar{h}^2_{s,t} = \int_s^t (h_u - h_s) \otimes dh_u.$$

Hence the solution $(Z(h), R^{Z(h)}, \Phi(h))$ satisfies the estimate (5.13) with the same constant C_1, C_2.

It may not be clear to see the unique existence for the driving Lipschitz path h when we try to prove it via the (transformed) path-dependent ODE because the functional $w \mapsto L(w)$ is not Lipschitz in $C([0,T])$ with the uniform norm in general. However, by using some idea due to [25, 19, 24], the unique existence naturally follows from the view point of original reflected ODEs. We use the uniqueness in the argument below.

In general path-dependent setting, the trick for reflected case cannot be applied to prove the uniqueness of the solution. Of course, if we assume the Lipschitz continuity of L with respect to the uniform norm, we have the uniqueness of the solution although such an assumption does not hold for L associated with general reflected RDEs. By finding suitable assumptions on L, especially the uniqueness, the results in this section probably may be extended to more general path-dependent setting.

We consider the case where $\omega(s,t) = |t-s|$. This means we consider Hölder rough paths. Let us denote the set of β-Hölder geometric rough path $(1/3 < \beta \le 1/2)$ by $\mathscr{C}_g^\beta(\mathbb{R}^n)$ which is the closure of the set of smooth rough paths in the topology of $\mathscr{C}^\beta(\mathbb{R}^n)$. In this paper, smooth rough path means the rough path defined by a Lipschitz path and its iterated integral.

When the driving rough path is a geometric rough path, we can give another proof of the existence of solutions as in [3]. Below we use the notation $\mathcal{V}_{\theta-} = \cap_{\varepsilon>0}\mathcal{V}_{\theta-\varepsilon}, \mathcal{V}_{1+,\theta-} = \cap_{q>1}\mathcal{V}_{q,\theta/q}$. Clearly, these spaces are Fréchet spaces with naturally defined semi-norms.

Lemma 6.1. *Assume D satisfies conditions* (A) *and* (B) *and $\sigma \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n,\mathbb{R}^d))$.*

(1) *Let $\mathbf{X} \in \mathscr{C}_g^\beta(\mathbb{R}^n)$. Let $\{\mathbf{h}_n\}$ be a sequence of smooth rough paths associated with Lipschitz paths h_n such that $\lim_{n\to\infty}|||\mathbf{X}-\mathbf{h}_n|||_\beta = 0$. Let $((Z(h_n),\sigma(Y(h_n)),\Phi(h_n)))$ be the unique solution to* (6.1). *There exists a subsequence $\{(Z(h_{n_k}),\sigma(Y(h_{n_k})),\Phi(h_{n_k})\}$ which converges in $\mathcal{V}_{\beta-}(\mathbb{R}^d)\times\mathcal{V}_{\beta-}(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d))\times\mathcal{V}_{1+,\beta-}(\mathbb{R}^d)$.*

(2) *We denote by $Sol_{\beta-}(\mathbf{X})$ the subset of $\mathcal{V}_{\beta-}(\mathbb{R}^d)\times\mathcal{V}_{\beta-}(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d))\times\mathcal{V}_{1+,\beta-}(\mathbb{R}^d)$ consisting of all limit points obtained in* (1). *Then $Sol_{\beta-}(\mathbf{X})$ is a closed subset of $\mathscr{D}_X^{2\beta}(\mathbb{R}^d)\times\mathcal{V}_{1,\beta}(\mathbb{R}^d)$. Moreover $((Z,\sigma(Y)),\Phi) \in Sol_{\beta-}(\mathbf{X})$ $(Y_t = Z_t+\Phi_t)$ is a solution to* (5.11) *and satisfies the estimate* (5.13) *with the same constant.*

Proof. For simplicity, we write $Z(h_n) = Z_n, \Phi(h_n) = \Phi_n, Y_n = Z_n + \Phi_n$. By Theorem 5.5,

$$\|Z_n\|_\beta + \|R^{Z_n}\|_{2\beta} + \|\Phi_n\|_{1,\beta} \le C_1 e^{C_2\widetilde{|||\mathbf{h_n}|||_\beta}}\widetilde{|||\mathbf{h_n}|||}_\beta. \tag{6.2}$$

By

$$\|h_n\|_\beta \le \|h_n - X\|_\beta + \|X\|_\beta, \quad \|\bar{h}_n^2\|_{2\beta} \le \|\bar{h}_n^2 - \mathbb{X}\|_{2\beta} + \|\mathbb{X}\|_{2\beta}, \tag{6.3}$$

we have

$$\limsup_{n\to\infty}|||\mathbf{h}_n|||_\beta \le |||\mathbf{X}|||_\beta. \tag{6.4}$$

This implies there exists a subsequence $\{Z_{n_k}\}$ and $\{\Phi_{n_k}\}$ which converges in $\mathcal{V}_{\beta-}$ and $\mathcal{V}_{1+,\beta-}$ respectively and the limit point (Z,Φ) also belongs to $\mathcal{V}_\beta\times\mathcal{V}_{1,\beta}$. Moreover,

$$\begin{aligned}|Z_t - Z_s - \sigma(Y_s)X_{s,t}| &= \lim_{n\to\infty}\left|Z_n(t) - Z_n(s) - \sigma(Y_n(s))X_|s,t\right| \\ &= \lim_{k\to\infty}|R^{Z_{n_k}}_{s,t}|.\end{aligned} \tag{6.5}$$

Therefore, letting $R^Z_{s,t} = Z_t - Z_s - \sigma(Y_s)Z_{s,t}$, we see that Z, R^Z, Φ satisfies the estimate (5.13). Hence $(Z,\sigma(Y)) \in \mathscr{D}_X(\mathbb{R}^d)$. By (3.29) in Lemma 3.6, there exists $C>0$ such that for any n,

$$\begin{aligned}&\Big|Z_n(t) - Z_n(s) - \Big\{\sigma(Y_n(s))X_{s,t} + (D\sigma)(Y_n(s))\left(\sigma(Y_n(s))\mathbb{X}_{s,t}\right) \\ &\qquad + (D\sigma)(Y_n(s))\left(\int_s^t(\Phi_n)_{s,t}\otimes dX_r\right)\Big\}\Big| \le C|t-s|^{\gamma\beta}.\end{aligned} \tag{6.6}$$

Thus, we get

$$\begin{aligned}&\left|Z_t - Z_s - \left\{\sigma(Y_s)X_{s,t} + (D\sigma)(Y_s)\left(\sigma(Y_s)\mathbb{X}_{s,t}\right) + (D\sigma)(Y_s)\left(\int_s^t\Phi_{s,t}\otimes dX_r\right)\right\}\right| \\ &\qquad \le C|t-s|^{\gamma\beta}\end{aligned} \tag{6.7}$$

which implies that $Y_t = Z_t + \Phi_t$ and Φ_t is a solution to (5.10) and (5.11). We need to prove (5.12). It holds that

$$Y_n(t) = \xi + \int_0^t \sigma(Y_n(s))dh_n(s) + \Phi_n(t), \tag{6.8}$$

$$\Phi_n(t) = L\left(\xi + \int_0^{\cdot} \sigma(Y_n(s))dh_n(s)\right)_t. \tag{6.9}$$

Since Y_{n_k} and Φ_{n_k} converges uniformly to Y and Φ respectively, $\int_0^{\cdot} \sigma(Y_{n_k}(s))dh_{n_k}(s)$ also converges to $\int_0^{\cdot} \sigma(Y_s)d\mathbf{X}_s$ uniformly. Hence the continuity of L implies that (5.12) holds. This completes the proof. □

By a similar argument to the proof of Theorem 4.9 in [2], we can prove the existence of universally measurable selection mapping of solutions as follows.

Proposition 6.2. *Assume the same conditions on D and σ in Lemma 6.1. Then there exists a universally measurable mapping*

$$\mathcal{I} : \mathscr{C}_g^{\beta}(\mathbb{R}^n) \ni \mathbf{X} \mapsto \Big(\Big(Z(\mathbf{X},\xi), \sigma(Y(\mathbf{X},\xi)\Big), \Phi(\mathbf{X},\xi)\Big) \in \mathcal{V}_{\beta-} \times \mathcal{V}_{\beta-} \times \mathcal{V}_{1+,\beta-},$$
$$\text{where } Y_t(\mathbf{X},\xi) = Z_t(\mathbf{X},\xi) + \Phi_t(\mathbf{X},\xi)$$

satisfying the following.

(1) $(Z(\mathbf{X},\xi), \sigma(Y(\mathbf{X},\xi)) \in \mathscr{D}_X^{2\beta}(\mathbb{R}^d)$ *and* $\Big((Z(\mathbf{X},\xi), \sigma(Y(\mathbf{X},\xi))\,, \Phi(\mathbf{X},\xi)\Big)$ *is a solution in Theorem* 3.3 *and satisfies the estimate in* (5.13)*.*

(2) *There exists a sequence of Lipschitz paths h_n such that $\|\|\mathbf{X} - \mathbf{h}_n\|\|_\beta \to 0$ and $\mathcal{I}(\mathbf{h}_n, \xi)$ converges to $\mathcal{I}(\mathbf{X},\xi)$ in $\mathcal{V}_{\beta-}(\mathbb{R}^d) \times \mathcal{V}_{\beta-}(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d)) \times \mathcal{V}_{1+,\beta-}(\mathbb{R}^n)$.*

Proof. Below, we omit writing ξ. We consider the product space,

$$E = \mathscr{C}_g^{\beta}(\mathbb{R}^n) \times \mathcal{V}_{\beta-}(\mathbb{R}^d) \times \mathcal{V}_{\beta-}(\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d)) \times \mathcal{V}_{1+,\beta-}(\mathbb{R}^n) \tag{6.10}$$

and its subset

$$E_0 = \Big\{\Big(\mathbf{h}, Z(\mathbf{h}), \sigma(Y(\mathbf{h})), \Phi(\mathbf{h})\Big) \in E \mid \mathbf{h} \text{ is a smooth rough path}\Big\} \tag{6.11}$$

We consider the closure $\bar{E}_0$ in E. Then $\bar{E}_0$ is a separable closed subset of E. The separability follows from the continuity of the mapping $h \mapsto (Z(h), \sigma(Y(h)), \Phi(h))$. See Lemma 6.4. Note that $Sol_{\beta-}(\mathbf{X})$ coincides with the projection of the subset of $\bar{E}_0$ whose first component is $\mathbf{X}$. Hence by the measurable selection theorem (See 13.2.7. Theorem in [10]), there exists a universally measurable mapping $\mathcal{I} : \mathscr{C}_g^{\beta}(\mathbb{R}^n) \to E$ such that $\mathcal{I}(\mathbf{X}) \in \{\mathbf{X}\} \times Sol_{\beta-}(\mathbf{X})$. By Lemma 6.1, this mapping satisfies the desired properties in (1) and (2). □

Let $\mathbf{X}_{s,t}^{-h}$ be the translated rough path of $\mathbf{X} \in \mathscr{C}^{\beta}(\mathbb{R}^n)$ by h. That is, the 1st level path and the second level path are given by,

$$X_{s,t}^{-h} = X_{s,t} - h_{s,t} \tag{6.12}$$

$$\mathbb{X}_{s,t}^{-h} = \mathbb{X}_{s,t} - \bar{h}_{s,t}^2 - \int_s^t X_{s,u}^{-h} \otimes dh_u - \int_s^t h_{s,u} \otimes dX_{s,u}^{-h}. \tag{6.13}$$

Hence

$$\|X^{-h}\|_\beta \le \|X-h\|_\beta, \tag{6.14}$$

$$\|\mathbb{X}^{-h}\|_{2\beta} \le \|\mathbb{X}-\bar{h}^2\|_{2\beta} + \left(1+\frac{2}{1+\beta}\right) T^{1-\beta}\|X-h\|_\beta\|h\|_1. \tag{6.15}$$

By the definition of controlled paths, we immediately obtain the following.

Lemma 6.3. *Let h be a Lipschitz path. If $(Z,Z')\in\mathscr{D}_X^{2\beta}$, then $(Z,Z')\in\mathscr{D}_{X-h}^{2\beta}$. In fact,*

$$\left|Z_{s,t}-Z'_sX^{-h}_{s,t}\right| \le \left(\|R^Z\|_{2\beta}+(|Z'_0|+\|Z'\|_\beta s^\beta)\|h\|_1(t-s)^{1-2\beta}\right)(t-s)^{2\beta}. \tag{6.16}$$

Let $(Z,Z')\in\mathscr{D}_X^{2\alpha}(\mathbb{R}^d)$ and $\Phi\in\mathcal{V}_{\tilde\alpha,q}(\mathbb{R}^d)$ with $\Phi_0=0$ and $q,\alpha,\tilde\alpha$ satisfy the assumptions in Lemma 3.5. By the above lemma, we can define the integral $\int_s^t\sigma(Y_u)d\mathbf{X}^{-h}_u$ and the estimates in Lemma 3.6 hold for this integral. Here $Y_u=Z_u+\Phi_u$. Moreover, $\Xi_{s,t}$ in (3.19) which is defined by $\mathbf{X}^{-h}_{s,t}$ reads

$$\Xi_{s,t}=\sigma(Y_s)X^{-h}_{s,t}+(D\sigma)(Y_s)Z'_s\mathbb{X}^{-h}_{s,t}+(D\sigma)(Y_s)\int_s^t\Phi_{s,u}\otimes dX^{-h}_u \tag{6.17}$$

$$=\sigma(Y_s)X_{s,t}+(D\sigma)(Y_s)Z'_s\mathbb{X}_{s,t}+(D\sigma)(Y_s)\int_s^t\Phi_{s,u}\otimes dX_u-\sigma(Y_s)h_{s,t}+\tilde{\Xi}_{s,t}, \tag{6.18}$$

where

$$\tilde{\Xi}_{s,t}=-(D\sigma)(Y_s)Z'_s\left(\bar{h}^2_{s,t}+\int_s^tX^{-h}_{s,u}\otimes dh_u+\int_s^th_{s,u}\otimes dX^{-h}_{s,u}\right)+(D\sigma)(Y_s)\int_s^t\Phi_{s,u}\otimes dh_u. \tag{6.19}$$

Since $|\tilde{\Xi}_{s,t}|\le C(t-s)^{1+\tilde\alpha}$, the sum of these terms converges to 0. Thus we obtain

$$\int_s^t\sigma(Y_u)d\mathbf{X}^{-h}_u=\int_s^t\sigma(Y_u)d\mathbf{X}_u-\int_s^t\sigma(Y_u)dh_u. \tag{6.20}$$

Using this simple relation, we prove that the solution mapping is continuous at any smooth rough paths in the following sense.

Lemma 6.4. *Assume that D satisfies the condition* (A), (B) *and $\sigma\in\mathrm{Lip}^{\gamma-1}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^n,\mathbb{R}^d))$. For each $\mathbf{X}\in\mathscr{C}_g^\beta(\mathbb{R}^n)$, let us choose $(Z_t(\mathbf{X},\xi),\Phi_t(\mathbf{X},\xi))\in Sol_{\beta-}(\mathbf{X})$. Let h be a Lipschitz path and $Y(h,\xi)_t$ be the solution to* (6.1). *Then we have the following results.*

(1) *$Y_t(\mathbf{X},\xi)$ satisfies the following equation,*

$$Y_t(\mathbf{X},\xi)=\xi+\int_0^t\sigma\left(Y_s(\mathbf{X},\xi)\right)d\mathbf{X}^{-h}_s+\int_0^t\sigma\left(Y_s(\mathbf{X},\xi)\right)dh_s+\Phi_t(\mathbf{X},\xi). \tag{6.21}$$

(2) *If $\|\!|\mathbf{h}-\mathbf{X}\|\!|_\beta\le\varepsilon\le1$, then $\|\!|\mathbf{X}^{-h}\|\!|_\beta\le\left(1+\sqrt{\left(1+\frac{2}{1+\beta}\right)T^{1-\beta}\|h\|_1}\right)\varepsilon$.*

(3) *For any $\varepsilon>0$, there exists $\delta>0$ such that if $\|\!|\mathbf{h}-\mathbf{X}\|\!|_\beta\le\delta$, then $\|Y(\mathbf{X},\xi)-Y(h,\xi)\|_{\beta-}\le\varepsilon$ and $\|\Phi(\mathbf{X},\xi)-\Phi(h,\xi)\|_{1+,\beta-}\le\varepsilon$.*

Proof. We proved (1). The statement (2) easily follows (6.14) and (6.15). We prove (3) by contradiction. Suppose that there exists a positive number $\varepsilon > 0$ and a sequence of Lipschitz paths $\mathbf{X}_n \in \mathscr{C}_g^\beta(\mathbb{R}^n)$ such that $\lim_{n\to\infty} |||\mathbf{h} - \mathbf{X}_n|||_\beta = 0$ and

$$\|Y(\mathbf{X}_n, \xi) - Y(h, \xi)\|_{\beta-} \geq \varepsilon \quad \text{or} \quad \|\Phi(\mathbf{X}_n, \xi) - \Phi(h, \xi)\|_{1+,\beta-} \geq \varepsilon. \tag{6.22}$$

Since $\sup_n |||\mathbf{X}_n|||_\beta < \infty$, there exists a subsequence $Y(\mathbf{X}_{n_k}, \xi)$ and $\Phi(\mathbf{X}_{n_k}, \xi)$ converges to $\tilde{Y}$ and Φ uniformly on $[0, T]$. Moreover, by Lemma 6.3, Lemma 3.6, and Theorem 5.5,

$$\int_0^\cdot \sigma\left(Y_s(\mathbf{X}_n, \xi)\right) d(\mathbf{X}_n^{-h})_s \text{ converges to 0 uniformly.} \tag{6.23}$$

From (6.21), we have

$$\tilde{Y}_t = \xi + \int_0^t \sigma(\tilde{Y}_s) dh_s + \tilde{\Phi}_t \quad 0 \leq t \leq T. \tag{6.24}$$

By the continuity property of the Skorohod mapping, $L\left(\xi + \int_0^\cdot \sigma(\tilde{Y}_s) dh_s\right)_t = \tilde{\Phi}_t$ $(0 \leq t \leq T)$ holds. By the uniqueness of the reflected ODE driven by h, it holds that $\tilde{Y} = Y(h)$ and $\tilde{\Phi} = \Phi(h)$. Since $\sup_n \left(\|Y(\mathbf{X}_n, \xi)\|_\beta + \|\Phi(\mathbf{X}_n, \xi)\|_{1,\beta}\right) < \infty$, this contradicts (6.22). □

We now consider the following condition on the boundary.
Condition (C): There exists a C^γ function f on $\mathbb{R}^d$ and a positive constant k such that for any $x \in \partial D$, $y \in \bar{D}$, $\mathbf{n} \in \mathcal{N}_x$ it holds that

$$(y - x, \mathbf{n}) + \frac{1}{k}\left((Df)(x), \mathbf{n}\right)|y - x|^2 \geq 0. \tag{6.25}$$

Usually, the function f is assume to be C_b^2 in the condition (C). Here, we assume $f \in C^\gamma$ to make use of estimates in Lemma 3.6.

Under additional condition (C), we can prove the following explicit modulus of continuity.

Lemma 6.5. *Assume that D satisfies the conditions* (A), (B), (C) *and $\sigma \in \mathrm{Lip}^{\gamma-1}$. Let $Y_t(\mathbf{X}, \xi), Z_t(\mathbf{X}, \xi), \Phi_t(\mathbf{X}, \xi), Y_t(h, \zeta), \Phi_t(h, \zeta)$ be a solution as in Lemma* 6.4. *Assume $|||\mathbf{h} - \mathbf{X}|||_\beta \leq \varepsilon \leq 1$. Then there exists a positive constant C which depends only on σ, f, k such that*

$$\sup_{0 \leq t \leq T} |Y_t(\mathbf{X}, \xi) - Y(h, \zeta)_t| \leq C(1 + \|h\|_1)^{3/2} e^{C\|h\|_1}(|\xi - \zeta| + \varepsilon). \tag{6.26}$$

Proof. We write $Y_t = Y_t(\mathbf{X}, \xi)$, $\Phi(\mathbf{X}, \xi)_t = \Phi_t$ and $\tilde{Y}_t = Y(h, \zeta)_t$, $\tilde{\Phi}_t = \Phi(h, \zeta)_t$ for simplicity. Let $Z_t = e^{-\frac{2}{k}\left(f(Y_t) + f(\tilde{Y}_t)\right)}|Y_t - \tilde{Y}_t|^2$. We have

$$Z_t - e^{-\frac{2}{k}(f(\xi) + f(\zeta))}|\xi - \zeta|^2 \tag{6.27}$$

$$\begin{aligned}
&= \int_0^t 2e^{-\frac{2}{k}\left(f(Y_t) + f(\tilde{Y}_t)\right)}\left\{\left(Y_s - \tilde{Y}_s, \left(\sigma(Y_s) - \sigma(\tilde{Y}_s)\right) h_s'\right) ds + \left(Y_s - \tilde{Y}_s, \sigma(Y_s) dX_s^{-h}\right)\right\} \\
&\quad - \frac{2}{k}\int_0^t Z_s\left(\sigma(Y_s)^* Df(Y_s) + \sigma(\tilde{Y}_s)^* Df(\tilde{Y}_s), h_s'\right) ds - \frac{2}{k}\int_0^t Z_s\left(Df(Y_s), \sigma(Y_s) dX_s^{-h}\right) \\
&\quad - \int_0^t 2e^{-\frac{2}{k}\left(f(Y_s) + f(\tilde{Y}_s)\right)}\left\{\left(\tilde{Y}_s - Y_s, d\Phi_s - d\tilde{\Phi}_s\right) + \frac{1}{k}\left(Df(Y_s), d\Phi_s\right) + \frac{1}{k}\left(Df(\tilde{Y}_s), d\tilde{\Phi}_s\right)\right\}.
\end{aligned} \tag{6.28}$$

Condition (C) implies that the fourth integral on the right-hand side of the equation (6.28) is always negative. By the estimates of the solution $Y, \tilde{Y}, \Phi, \tilde{\Phi}$ in the main theorem and the estimates in Lemma 3.6 and the Gronwall inequality, we obtain the desired estimate. □

We now consider a probability measure μ on $\mathscr{C}_g^\beta(\mathbb{R}^n)$. Here we restrict the map $\mathcal{I}$ in Proposition 6.2 as $\mathcal{I} : \mathscr{C}_g^\beta(\mathbb{R}^n) \ni \mathbf{X} \mapsto Y(\mathbf{X}, \xi) \in \mathcal{V}_{\beta-}$. It is now immediate to determine the support of the image measure of μ by $\mathcal{I}$ under a certain assumption on μ.

Theorem 6.6. *Assume that the set of smooth rough path is included in the topological support of μ. Then*

$$\mathrm{Supp}\,(\mathcal{I}_*\mu) = \overline{\{\mathcal{I}(h) \mid h \in \mathcal{V}_1\}}^{\mathcal{V}_{\beta-}}, \tag{6.29}$$

where $\mathrm{Supp}\,(\mathcal{I}_*\mu)$ *denotes the topological support of* $\mathcal{I}_*\mu$.

Proof. The inclusion $\mathrm{Supp}\,(\mathcal{I}_*\mu) \subset \overline{\{\mathcal{I}(h) \mid h \in \mathcal{V}_1\}}^{\mathcal{V}_{\beta-}}$ follows from Proposition 6.2 (2). The converse inclusion follows from the continuity in Lemma 6.4 (3) and the assumption on μ. □

Finally, we consider the case of Brownian rough path. Let B_t be the standard Brownian motion on $\mathbb{R}^n$. Let B_t^N be the dyadic polygonal approximation, that is,

$$B^N(t) = B(t_{i-1}^N) + \frac{B(t_i^N) - B(t_{i-1}^N)}{2^{-N}T}(t - t_i^N), \quad t_{i-1}^N \le t \le t_i^N,\ t_i^N = \frac{iT}{2^N},\ 0 \le i \le 2^N. \tag{6.30}$$

Let $\mathbb{B}_{s,t}^N = \int_s^t B_{s,u}^N \otimes dB_u^N$. Let us define

$$\Omega_1 = \left\{B \in W^n \mid \text{the smooth rough path } (B_{s,t}^N, \mathbb{B}_{s,t}^N) \text{ converges in } \mathscr{C}^\beta(\mathbb{R}^n) \text{ for all } \beta < 1/2.\right\} \tag{6.31}$$

It is known that $\mu^W(\Omega_1) = 1$ and the limit point of $(B_{s,t}^N, \mathbb{B}_{s,t}^N)$ for $B \in \Omega_1$ is called the Brownian rough path. Here μ^W is the Wiener measure. We identify the element of $B \in \Omega_1$ and the associated Brownian rough path $\mathbf{B}$. For application to the support theorem of diffusion processes, it is important to obtain the support of $\mathbf{B}$. The following is due to Ledoux-Qian-Zhang [18]. More general results can be found in [16].

Theorem 6.7. *Let $\beta < 1/2$. The topological support of the probability measure of $\mathbf{B}$ on $\mathscr{C}^\beta(\mathbb{R}^n)$ is equal to the closure of the set of smooth rough paths in the topology of $\mathscr{C}^\beta(\mathbb{R}^n)$.*

Let $\sigma \in C_b^2(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$ and consider a Stratonovich reflected SDE,

$$dY_t = \sigma(Y_t) \circ dB_t + d\Phi_t, \quad Y_0 = \xi \in D. \tag{6.32}$$

The corresponding solution Y_t^N which is obtained by replacing B_t by B_t^N is called the Wong-Zakai approximation of Y_t. It is proved in [4, 26, 3] that the Wong-Zakai approximations of the solution to a reflected Stratonovich SDE under (A), (B) and (C) converge to the solution in the uniform convergence topology almost surely. Note that a similar statement under the conditions (A) and (B) is proved in [3]. By using the result in [4, 26], a support theorem for the reflected diffusion under the conditions (A), (B) and (C) is proved by Ren and Wu [23]. We now prove a support theorem for the reflected diffusion under (A) and (B) by using the estimates in rough path analysis in this paper and in [3, 4]. In [3, 4], it is proved that the following hold.

Theorem 6.8. *Assume $\sigma \in C_b^2(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$.*

(1) *Assume D satisfies condition* (A), (B), (C). *Let*

$$\Omega_2 = \left\{ B \in W^n \;\middle|\; \max_{0\le t\le T} |Y_t^N - Y_t| \to 0 \quad \text{as } N \to \infty \right\}. \tag{6.33}$$

Then $\mu^W(\Omega_2) = 1$.

(2) *Assume* (A), (B) *hold. Then there exists an increasing subsequence of natural numbers N_k such that $\mu^W(\Omega_3) = 1$ holds where,*

$$\Omega_3 = \left\{ B \in W^n \;\middle|\; \max_{0\le t\le T} |Y_t^{N_k} - Y_t| \to 0 \quad \text{as } N_k \to \infty \right\}. \tag{6.34}$$

It is proved in [3] that $\max_{0\le t\le T} |Y_t^N - Y_t|$ converges to 0 in probability under (A) and (B). This and the moment estimate in [4] implies the almost sure convergence in Theorem 6.8 (2).

The following is a support theorem for reflected diffusion Y_t.

Theorem 6.9. *Assume D satisfies* (A) *and* (B) *and $\sigma \in C_b^2$. Let P^Y be the law of Y. Let $0 < \beta < 1/2$. The law of P^Y is a probability measure on $\mathcal{V}_\beta(\mathbb{R}^d; Y_0 = \xi)$ and*

$$\mathrm{Supp}\,(P^Y) = \overline{\{Y(h) \mid h \in \mathcal{V}_1(\mathbb{R}^n)\}}^{\mathcal{V}_\beta}. \tag{6.35}$$

Proof. Take $1/2 > \beta' > \beta$. For $B \in \Omega_1 \cap \Omega_3$, define $Y_t(B) = \lim_{k\to\infty} Y_t^{N_k}(B)$. Then for $\mathbf{B} \in \mathscr{C}_g^{\beta'}(\mathbb{R}^n)$, $Y(\mathbf{B}) \in Sol_{\beta'-}(\mathbf{B})$ and the mapping $\Omega_1 \cap \Omega_2 \ni \mathbf{B} \mapsto Y(\mathbf{B}) \in \mathcal{V}_{\beta'-}$ is μ^W-almost surely defined Borel measurable mapping. By the estimate in Theorem 5.5, the convergence $Y^{N_k}(B) \to Y(B)$ takes place in the topology of $\mathcal{V}_{\beta'-}$. Hence $\mathrm{Supp}\,(P^Y) \subset \overline{\{Y(h) \mid h \in \mathcal{V}_1(\mathbb{R}^n)\}}^{\mathcal{V}_{\beta'-}}$ holds. The converse inclusion follows from Lemma 6.4 and Theorem 6.7. □

7 Path-dependent RDE with drift

We consider path-dependent rough differential equations with drift term. It is necessary to study such kind of equations for the study of diffusion operators with drift. In the case of n-dimensional Brownian motion $B_t = (B_t^1, \ldots, B_t^n))$, one possible approach to include the drift term is to consider the geometric rough path defined by $\tilde{B}_t = (B_t, t) \in \mathbb{R}^{n+1}$. By considering the geometric rough path which is naturally defined by Brownian rough path $\mathbf{B}_{s,t}$, we may extend all results in previous sections to the corresponding results for the solutions to the equation,

$$Z_t = \xi + \int_0^t \sigma(Z_s, L(Z)_s) d\mathbf{B}_s + \int_0^t b(Z_s, L(Z_s)) ds. \tag{7.1}$$

However, we need to assume $b \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ $(2 < \gamma \le 3)$ to do so and the assumption on b is too strong. Hence, we explain different approach to deal with the drift term. Let us consider β-Hölder rough path, that is, the case where $\omega(s,t) = |t-s|$. Let $b \in C_b^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ and consider the equation,

$$Z_t = \xi + \int_0^t \sigma(Z_s, \Phi_s) d\mathbf{X}_s + \int_0^t b(Z_s, \Phi_s) ds, \tag{7.2}$$

$$\Phi_t = L\left(\xi + \int_0^{\cdot} \sigma(Z_s, \Phi_s) d\mathbf{X}_s + \int_0^{\cdot} b(Z_s, \Phi_s) ds\right)_t. \tag{7.3}$$

The meaning of this equation is as follows. The controlled path (Z, Z') and Φ are elements as in the definition of $\Xi_{s,t}$ and $I(Z,\Phi)_{s,t}$. In the present case, we consider

$$\tilde{\Xi}_{s,t} := \sigma(Y_s)X_{s,t} + (D_1\sigma)(Y_s)Z'_s\mathbb{X}_{s,t} + (D_2\sigma)(Y_s)\int_s^t \Phi_{s,r} \otimes dX_r + b(Y_s)(t-s). \tag{7.4}$$

Then, $(\delta\tilde{\Xi})_{s,u,t} = (\delta\Xi)_{s,u,t} + (b(Y_s) - b(Y_u))\,(t-u)$ for $s < u < t$ and

$$\begin{aligned} &|\,(b(Y_s) - b(Y_u))\,(t-u)| \\ &\leq \|Db\|_\infty \left(\|Z'\|_\infty \|X\|_\beta (t-s)^\beta + \|R^Z\|_{2\beta}(t-s)^{2\beta} + \|\Phi\|_{q-var,\tilde{\alpha}}(t-s)^{\tilde{\alpha}}\right)(t-s). \end{aligned} \tag{7.5}$$

By using this, we define

$$I(Z,\Phi)_{s,t} := \int_s^t \sigma(Z_u,\Phi_u)d\mathbf{X}_u + \int_s^t b(Z_u,\Phi_u)du = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \tilde{\Xi}_{u,v}. \tag{7.6}$$

For this, $(I(Z,\Phi)_{0,t}, \sigma(Z_t,\Phi_t)) \in \mathscr{D}_X^{2\beta}$ and similar estimates to Lemma 3.6 holds. Moreover, Lemma 3.7 and Lemma 4.1 hold. Thus, Theorem 3.3 holds for suitable constants κ, C_1, C_2 which depend only on $\sigma, b, \beta, p, \gamma$. In the case of reflected rough differential equation, all statements can be extended to differential equations with drift term $b \in C_b^1$. In particular, this results include the support theorem for reflected diffusions which is proved in [23].

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