Weak Poincaré inequalities on domains defined by Brownian rough paths

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§1. Introduction

(1) What is weak Poincaré inequality(=WPI)?

$(X, \mathfrak{B}, m)$: Probability space

$(\mathcal{E}, \mathrm{D}(\mathcal{E}))$: local symmetric Dirichlet form

(WPI): $\exists$ non-increasing function $\xi : (0, \infty) \to (0, \infty)$ s.t.

$$\int_X (f - \langle f \rangle_m)^2 dm \le \xi(\delta)\mathcal{E}(f, f) + \delta \|f\|_\infty \quad \text{for all } \delta > 0,\ f \in \mathrm{D}(\mathcal{E}) \cap L^\infty(X). \tag{1}$$

(Poincaré inequality=PI): $\exists\, C > 0$ s.t.

$$\int_X (f - \langle f \rangle_m)^2 dm \le C\mathcal{E}(f, f) \quad \text{for all } f \in \mathrm{D}(\mathcal{E})(\cap L^\infty(X)). \tag{2}$$

(Irreducibility of $(\mathcal{E}, \mathrm{D}(\mathcal{E}))$):

$\mathcal{E}(f, f) = 0 \Longrightarrow f = \text{const. a.s.}$

Remark:

(PI)$\Longrightarrow$(WPI)$\Longrightarrow$(Irreducibility)

Example

d-dimensional Wiener space: $(\mathrm{W}^d, \mathrm{H}^d, \mu)$

H-open set : $U \subset \mathrm{W}^d$

$$\mathcal{E}_U(f,f) := \int_U |Df(w)|_H^2 d\mu(w)$$

Theorem 1 (Feyel-Üstünel) *If U is an H-convex set, then LSI holds for $\mathcal{E}_U$.*

Theorem 2 (Kusuoka) *If U is an H-connected set + some conditions, then $\mathcal{E}_U$ satisfies WPI.*

Theorem 3 *If U is an H-connected set, then $\mathcal{E}_U$ is irreducible.*

By using Kusuoka's theorem, we can prove that

Theorem 4 ***Let M be a connected and simply connected compact Riemannian manifold. Let***

$$L_x(M) = C([0,1] \to M \mid \gamma(0) = \gamma(1) = x),$$

where $x \in M$ and consider the pinned Brownian motion measure. We fix a torsion skew-symmetric connection and define an H-derivative by the connection. Let $\mathcal{E}_x$ be the Dirichlet form which is defined by the H-derivative. Then $\mathcal{E}_x$ satisfies WPI.

The aim of this talk is to present a proof of Theorem 2 in the case where U is an inverse image of an open set by a continuous function in the sense of rough path analysis and prove Theorem 4 based on the results.

(2) A strategy to prove WPI

We present a proof of the WPI for $\mathcal{E}_U$ in the case where $U \subset \mathrm{W}^d$ is an open connected set.

Proposition 5 *Let us consider a general setting. We assume $\mathcal{E}$ has the square field operator such that*

$$\mathcal{E}(f,f) = \int_X \Gamma(f,f)dm.$$

For $U_i \subset X$, set

$$\mathcal{E}_i(f,f) := \int_{U_i} \Gamma(f,f)dm_i, \quad f \in \mathrm{D}(\mathcal{E}), \quad (3)$$

where $dm_i = dm/m(U_i)$. For $U := \cup_{i=1}^{\infty} U_i$, set

$$\mathcal{E}_U(f,f) = \int_U \Gamma(f,f)dm_U, \quad f \in \mathrm{D}(\mathcal{E}), \quad (4)$$

where $dm_U = dm/m(U)$. Assume the following (A1) *and* (A2).

(A1) *WPI holds for each $(\mathcal{E}_i, \mathrm{D}(\mathcal{E}))$ $(i \in \mathbb{N})$.*

(A2) *For any $n \in \mathbb{N}$,*

$m\left((\cup_{i=1}^n U_i) \cap U_{n+1}\right) > 0.$

Then WPI holds for $(\mathcal{E}_U, \mathrm{D}(\mathcal{E}))$.

Remark : **Assume that $U = \cup_{i=1}^{N} U_i$(finite union) and PI holds for each $\mathcal{E}_i$. Then PI holds for $\mathcal{E}_U$.**

The following is a consequence of Feyel and Üstünel's theorem.

Lemma 6 ***Let us consider the Wiener space*** (W^d, H^d, μ)***. Let*** $\| \ \|$ ***be the norm of*** W^d***. (for example, sup norm, Hölder norm, etc). Let***

$$B_r(\mathrm{h}) = \left\{ \mathrm{w} \in \mathrm{W}^d \mid \|\mathrm{w} - \mathrm{h}\| < r \right\},$$

where $\mathrm{h} \in H^d$***. LSI holds for***

$$\mathcal{E}_{B_r(\mathrm{h})}(f, f) = \int_{B_r(\mathrm{h})} |Df(\mathrm{w})|^2 d\mu(\mathrm{w}) \qquad f \in \mathfrak{F}\mathfrak{C}_b^{\infty}(\mathrm{W}^d), \tag{5}$$

where $\mathfrak{F}\mathfrak{C}_b^{\infty}(\mathrm{W}^d)$ ***denotes a set of smooth cylindrical functions.***

Corollary 7 *Let $U \subset \mathrm{W}^d$ be a connected open set. Then WPI holds for $\mathcal{E}_U$.*

Proof There exist $B_{r_i}(\mathrm{h}_i)$ such that

$$U = \cup_{i=1}^{\infty} B_{r_i}(\mathrm{h}_i)$$

and for any n

$$\mu\left((\cup_{i=1}^{n} B_{r_i}(\mathrm{h}_i) \cap B_{r_{n+1}}(\mathrm{h}_{n+1}))\right) > 0.$$

■

But a typical set U which appears in Malliavin calculus is not an open set in the topology of W^d. Let M be a compact Riemannian manifold isometrically embedded in $\mathbb{R}^d$. Let $P(x) : \mathbb{R}^d \to T_x M$ be the projection operator. $X(t, x, \mathrm{w})$ be the solution to the following SDE:

$$\begin{aligned} dX(t, x, \mathrm{w}) &= P(X(t, x, \mathrm{w})) \circ d\mathrm{w}(t) \\ X(0, x, \mathrm{w}) &= x \in M. \end{aligned} \tag{6}$$

Let $V_\varepsilon = \{y \in M \mid d(x, y) < \varepsilon\}$ and set

$$U_{V_\varepsilon} = \left\{\mathrm{w} \in \mathrm{W}^d \mid X(1, x, \mathrm{w}) \in V_\varepsilon\right\}. \tag{7}$$

Then, in general, $X(t, x, \mathrm{w})$ is not a continuous function of w in the topology of W^d and so Corollary 7 cannot be applied. But $X(t, x, \mathrm{w})$ is a continuous function of w in the sense of rough path. Our main idea is

1. To prove a WPI for a ball like set in the sense of rough path analysis

2. To apply the proof of Corollay 7 and Proposition 5 to prove WPI

§2. Lyons' continuity theorem

(1) Notaion, p-variation norm

Let $\Delta = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le t \le 1\}$. Take $q > 1$. For $\psi : \Delta \to \mathbb{R}$, $\|\psi\|_q$ is defined by

$$\|\psi\|_q = \sup_D \left\{ \sum_{i=0}^{n-1} |\psi(t_i, t_{i+1})|^q \right\}^{1/q}, \qquad (8)$$

where $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ runs all partitions of $[0,1]$.

Let $T_2(\mathbb{R}^d) = \mathbb{R}^d \bigoplus (\mathbb{R}^d \bigotimes \mathbb{R}^d)$. Let $C(\Delta, T_2(\mathbb{R}^d))$ be the space of continuous functions. Let $e_i = {}^t(0, \ldots, \overset{i}{1}, \ldots, 0)$.

For $\eta = (\eta(\cdot,\cdot)_1, \eta(\cdot,\cdot)_2) \in C(\Delta, T_2(\mathbb{R}^d))$, set

$$\begin{aligned} \eta_{1,i}(s,t) &= (\eta(s,t)_1, e_i) \\ \eta_{2,k,l}(s,t) &= (\eta(s,t)_2, e_k \otimes e_l) \end{aligned}$$

and define

$$\|\eta(\cdot,\cdot)_1\|_q = \max_{1 \le i \le d} \|\eta(\cdot,\cdot)_{1,i}\|_q, \qquad (9)$$

$$\|\eta(\cdot,\cdot)_2\|_q = \max_{1 \le k,l \le d} \|\eta(\cdot,\cdot)_{2,k,l}\|_q. \qquad (10)$$

Let p be a positive number such that $2 < p < 3$. For $\eta(\cdot,\cdot) \in C(\Delta, T_2(\mathbb{R}^d))$, define

$$\|\eta(\cdot,\cdot)\|_{C^p} = \max\left\{\|\eta_1\|_p, \|\eta_2\|_{p/2}\right\}. \qquad (11)$$

Remark

For $\mathrm{w} \in \mathrm{W}^d$, let $\overline{\mathrm{w}}_1(s,t) := \mathrm{w}(t) - \mathrm{w}(s)$. Then it holds that $\|\overline{\mathrm{w}}_1\|_p < \infty$ for μ-a.e. w in the case where $2 < p < 3$.

(2) For $\mathrm{h} \in \mathrm{H}^d$ define a smooth rough path $\overline{\mathrm{h}} \in C(\Delta \to T_2(\mathbb{R}^d))$:

$$\overline{\mathrm{h}}(s,t) := (\overline{\mathrm{h}}(s,t)_1, \overline{\mathrm{h}}(s,t)_2) \tag{12}$$

$$\overline{\mathrm{h}}(s,t)_1 := \mathrm{h}(t) - \mathrm{h}(s) \tag{13}$$

$$\overline{\mathrm{h}}(s,t)_2 := \int_s^t (\mathrm{h}(u) - \mathrm{h}(s)) \otimes d\mathrm{h}(u). \tag{14}$$

Theorem 8 (T.Lyons) ***Let us consider an ODE which is driven by*** **$\mathrm{h} \in \mathrm{H}^d$*:***

$$\dot{\xi}(t,x,\mathrm{h}) = P\left(\xi(t,x,\mathrm{h})\right)\dot{\mathrm{h}}(t)$$

$$\xi(0,x,\mathrm{h}) = x \in M.$$

Then **$\forall\ R > 0$, $\exists C(R) > 0$** ***such that*** **$\forall \mathrm{h}, \forall \mathrm{h}' \in \mathrm{H}^d$** ***with*** **$\|\overline{\mathrm{h}}\|_{C^p} \le R, \|\bar{\mathrm{h}}'\|_{C^p} \le R$*,*** ***we have***

$$\|\bar{\xi}(\cdot,x,\mathrm{h}) - \bar{\xi}(\cdot,x,\mathrm{h}')\|_{C^p} \le C(R)\|\overline{\mathrm{h}} - \bar{\mathrm{h}}'\|_{C^p}$$

(3) A realization of Brownian rough path:

Theorem 8 itself has nothing to do with SDE and Brownian motion. In order to relate it with Brownian path and the solution of SDE, we need an approximation theorem and a realization of Brownian path as a limit of smooth rough path.

For $\mathrm{w} \in \mathrm{W}^d$, set

$$(P_n\mathrm{w})(t) = \mathrm{w}\left(t_k^n\right) + 2^n\left(\mathrm{w}\left(t_{k+1}^n\right) - \mathrm{w}\left(t_k^n\right)\right)\left(t - t_k^n\right) \quad (t_k^n \le t \le t_{k+1}^n),$$

where $t_k^n = \frac{k}{2^n}$ $(0 \le k \le 2^n)$. Since $P_n\mathrm{w} \in \mathrm{H}^d$, $\overline{P_n\mathrm{w}} \in C(\Delta \to T_2(\mathbb{R}^d))$.

Theorem 9 (Hambly,Ledoux,Lyons,Qian)

Let $\mathrm{Y}^d \subset \mathrm{W}^d$ be the set which consists of w such that $\overline{P_n\mathrm{w}}$ is a Cauchy sequence in the topology of C^p.Then $\mu(\mathrm{Y}^d) = 1$. We denote the limit by $\overline{\mathrm{w}}$ for $\mathrm{w} \in \mathrm{Y}^d$. Also it holds that $\lim_{n\to\infty} E[\|\overline{P_n\mathrm{w}} - \overline{\mathrm{w}}\|_{C^p}] = 0$.

The limit $\overline{\mathrm{w}}$ is called a Brownian rough path.

Remark If $\mathrm{w} \in \mathrm{Y}^d$, then $\mathrm{w} + \mathrm{h} \in \mathrm{Y}^d$ for all $\mathrm{h} \in \mathrm{H}^d$.

By the method similar to the above, we can prove the following facts which we need.

Lemma 10 *Let* $\mathrm{X}^d \subset \mathrm{Y}^d$ *be the set which consists of* w *such that* $\|C_{P_n\mathrm{w}-\mathrm{w},P_n\mathrm{w}}\|_{p/2} \to 0$ *and* $\|\overline{P_n\mathrm{w} - \mathrm{w}}\|_{C^p} \to 0$. *Then* $\mu(\mathrm{X}^d) = 1$.

Here

$$C_{\mathrm{w},\mathrm{h}}(s,t) = \int_s^t (\mathrm{w}(u) - \mathrm{w}(s)) \otimes d\mathrm{h}(u).$$

Remark If $\mathrm{w} \in \mathrm{X}^d$, then $\mathrm{w} + \mathrm{h} \in \mathrm{X}^d$ for all $\mathrm{h} \in \mathrm{H}^d$.

Theorem 11 (Approximation theorem)

Let $X(t,x,\mathrm{w})$ *and* $\xi(t,x,\mathrm{h})$ *be the solutions which were defined already. Then for all* $t \geq 0$

$$X(t,x,\mathrm{w}) = \lim_{n\to\infty} \xi(t,x,P_n\mathrm{w}) \qquad \mu - \text{a.e. w.} \tag{15}$$

This approximation theorem and Theorem 8 and Theorem 9 imply that

$$\mathbf{w}(\in \mathbf{Y}^{\boldsymbol{d}}) \to \boldsymbol{X}(\cdot, \boldsymbol{x}, \mathbf{w})$$

has a continuous modification of $\overline{\mathbf{w}}$ in the topology of $\| \ \|_{C^p}$.

§3. Main theorem

Now we introduce ball like set in the sense of rough path: For $r > 0$ and $\mathrm{h} \in \mathrm{H}^d$:

$$U_{r,\mathrm{h}} = \Big\{ \mathrm{w} \in \mathrm{X}^d \;\Big|\; \|\overline{\mathrm{w}}\|_{C^p} < r, \|C_{\mathrm{w},\mathrm{h}}\|_{p/2} < r, \|C_{\mathrm{h},\mathrm{w}}\|_{p/2} < r \Big\}. \tag{16}$$

$$\begin{aligned} B_{r,\mathrm{h}} &:= U_{r,\mathrm{h}} + \mathrm{h} \qquad (\mathrm{h} \in \mathrm{H}^d) \\ &= \Big\{ \mathrm{w} \in \mathrm{X}^d \;\Big|\; \|\overline{(\mathrm{w}-\mathrm{h})}\|_{C^p} < r, \|C_{\mathrm{w}-\mathrm{h},\mathrm{h}}\|_{p/2} < r, \|C_{\mathrm{h},\mathrm{w}-\mathrm{h}}\|_{p/2} < r \Big\}. \end{aligned}$$

The following is a key result in our analysis.

Lemma 12 *$\mu(U_{r,\mathrm{h}}) > 0, \mu(B_{r,\mathrm{h}}) > 0$ hold and WPI hold for $U_{r,\mathrm{h}}$ and $B_{r,\mathrm{h}}$.*

We use the induction on the latter half of the proof. In the first step, $d = 1$, we use Feyel-Üstünel's result.

Our main theorem is as follows:

Theorem 13 ***Assume that*** $F : \mathrm{H}^d \to \mathbb{R}$ ***satisfies the following continuity condition:***
$\forall\, R > 0$, $\exists C(R) > 0$ ***such that*** $\forall \mathrm{h}, \forall \mathrm{h}' \in \mathrm{H}^d$ ***with*** $\|\bar{\mathrm{h}}\|_{C^p} \le R, \|\bar{\mathrm{h}}'\|_{C^p} \le R$, ***it holds that***

$$|F(\mathrm{h}) - F(\mathrm{h}')| \le C(R)\|\bar{\mathrm{h}} - \bar{\mathrm{h}}'\|_{C^p}. \qquad (17)$$

Then
(1) $\lim_{n\to\infty} F(P_n \mathrm{w})$ ***exists for all*** $\mathrm{w} \in \mathrm{X}^d$. ***We denote the limit by*** $\tilde{F}(\mathrm{w})$.
(2) ***Let***

$$U_F := \left\{\mathrm{h} \in \mathrm{H}^d \mid F(\mathrm{h}) > 0\right\}$$
$$U_{\tilde{F}} := \left\{\mathrm{w} \in \mathrm{X}^d \mid \tilde{F}(\mathrm{w}) > 0\right\}$$

$U_F \neq \emptyset$ ***is equivalent to*** $\mu(U_{\tilde{F}}) > 0$.
(3) ***If*** $U_F (\neq \emptyset)$ ***is a connected set in*** H^d, ***then WPI holds for*** $\mathcal{E}_{U_{\tilde{F}}}$.

Proof of (3):

We can prove that there exists a countable set $\{\mathrm{h}_i\}_{i=1}^{\infty} \subset U_F$ **and positive numbers** r_i **such that**

(i) $U_{\tilde{F}} = \cup_{i=1}^{\infty} B_{r_i,\mathrm{h}_i}$

(ii) For any $n \in \mathbb{N}$,

$$\mu\left(\cup_{i=1}^{n} B_{r_i,\mathrm{h}_i} \cap B_{r_{n+1},\mathrm{h}_{n+1}}\right) > 0.$$

Hence Proposition 5 and Lemma 12 imply the conclusion.■

To prove WPI on loop spaces, we need the following results:

Lemma 14 *Let us consider two probability spaces* $(X_i, \mathfrak{B}_i, m_i)$ *and two pre-Dirichlet forms* $(\mathcal{E}_i, \mathfrak{F}_i)$ *on them. Assume that there exists a measurable map* $I : X_1 \to X_2$ *such that*

(A1) *There exists a positive constant* C *such that for all* $f \in \mathfrak{F}_2$, $f \circ I \in \mathfrak{F}_1$ *hold and*

$$\mathcal{E}_1(f \circ I, f \circ I) \le C \mathcal{E}_2(f, f).$$

(A2) *The image measure* $I_* m_1$ *is equivalent to* m_2 *and the density function is a bounded function.*

Then if $(\mathcal{E}_1, \mathfrak{F}_1)$ *satisfies WPI, then* $(\mathcal{E}_2, \mathfrak{F}_2)$ *satisfies WPI.*

Lemma 15

Let us consider the case where $X_2 = L_x(M)$, m_2 = pinned measure and $\mathcal{E}_2$ is the Dirichlet form on it as explained in Theorem 4. We can construct a function F which is the continuous function in the sense of rough path in Theorem 13 and the assumption of Lemma 14 holds for

$$X_1 := U_{\tilde{F}} = \{\mathrm{w} \in \mathrm{X}^d \mid \tilde{F}(\mathrm{w}) > 0\} \quad (18)$$

and $\mathcal{E}_{U_{\tilde{F}}}$ and the above X_2.

To construct a function in Lemma 15, we use the following:

Proposition 16 ***Let*** $V_\xi : \mathrm{H}^d \to \mathrm{H}^d$ ***(***$\xi \in \mathbb{R}^N$***) be a family of vector fields on*** H^d***. Assume that there exista a positive function*** $C(\cdot)$ ***such that***

(A1) $\forall \mathrm{h}, \forall \mathrm{h}' \in \mathrm{H}^d$ ***with*** $\|\bar{\mathrm{h}}\|_{C^p} \leq R, \|\bar{\mathrm{h}'}\|_{C^p} \leq R$***, it holds that***

$$\sup_\xi |V_\xi(\mathrm{h}) - V_\xi(\mathrm{h}')| \leq C(R)\|\bar{\mathrm{h}} - \bar{\mathrm{h}'}\|_{C^p}. \quad (19)$$

(A2) ***For all*** $\mathrm{h} \in \mathrm{H}^d$ ***with*** $\|\bar{\mathrm{h}}\|_{C^p} \leq R$***,***

$$\sup_\xi \|DV_\xi(\mathrm{h})\|_{L(H,H)} \leq C(R).$$

(A3) ***For all multi-index*** α***,***

$$\sup_{\xi,\mathrm{h}} \|\partial_\xi^\alpha V_\xi(\mathrm{h})\| < \infty.$$

Let $\phi_t(\xi, \mathrm{h})$ ***be the solution to the following ODE in*** H^d***:***

$$\dot{\phi}_t(\xi, \mathrm{h}) = V_\xi(\mathrm{h} + \phi_t(\xi, \mathrm{h})) \quad (20)$$

$$\phi_0(\xi, \mathrm{h}) = 0 \quad (21)$$

Let $F(\mathrm{h})$ ***be a continuous function in Theorem 13. Let us fix*** $T > 0$***. Then*** $\{\phi_t(F(\mathrm{h}), \mathrm{h})\}_{0 \leq t \leq T}$ ***satisifes the continuity in Theorem 13.***