Subfactors and Conformal Field Theory

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Abstract. We discuss relations between the combinatorial structure of subfactors, solvable lattice models, (rational) conformal field theory, and topological quantum field theory.

1. Introduction

Since the breakthrough by V. F. R. Jones [48, 49], the theory of subfactors has had unexpectedly deep connections with topology and mathematical physics. The combinatorial structure of subfactors plays a key role, and Ocneanu's paragroup theory [64] gives the most powerful machinery for understanding this at present. Ocneanu, however, has not published details of his theory, and a lack of understanding of the paragroup theory prevented detailed study of the relationship between the paragroup theory and the topology and physics — beyond noticing apparent similarities in the various fields. Recently, full details and several applications of the paragroup theory have appeared [11, 32, 33, 34, 42, 44, 45, 52, 53, 54, 66, 67, 83] and our aim here is to give a full account of the recent interactions.

A paragroup is a (discrete) quantization of a (finite) group, and it appeared as a combinatorial characterization of higher relative commutants of finite index and finite depth subfactors of the approximately finite dimensional (AFD) factor of type II_1. (See [38] for the basics of subfactor theory and [71] for analytic aspects of the classification of subfactors.) It has been realized that the combinatorial structure of paragroups has considerable similarity with several other structures in mathematics and physics. The following table shows correspondences between various notions in paragroup theory, exactly solvable lattice model theory, quantum $6j$-symbols, rational conformal field theory and topological quantum field theory. Further details on these will be presented in later sections. Some correspondences are more

exact than others — some merely express similarity.

Paragroups (path algebras)	Paragroups (bimodules)	Lattice models	Quantum $6j$-symbols	RCFT	TQFT
connection		Boltzmann weight	$6j$-symbol	braiding matrix	local data for tetrahedron
unitarity					invariance under moves II, III
commuting square	Frobenius reciprocity	crossing symmetry	S_4-symmetry	tetrahedral symmetry	
flatness	associativity	Yang-Baxter equation	pentagon relation	braiding-fusion relation	invariance under move I

See [52, 53, 65, 66, 74] for the path algebra approach to paragroups, and [33, 66, 67, 84] for the bimodule approach to paragroups.

## 2. Statistical Mechanics and Subfactors

We enter this maze of interactions involving operator algebras, conformal quantum field theory, and topological quantum field theory, with certain statistical mechanical models on a two dimensional infinite oriented lattice L (which for our purposes here will be $\mathbf{Z}^2$). The configuration space of the classical model will be (compatible) distributions of the edges of some oriented graph Γ (here taken to be finite) on the edges of L. If there is at most one edge between any pair of vertices of Γ, and if there is an edge from α to β then there is another edge from β to α, then this will be the same as distributing the vertices of Γ over L so that labels of two vertices in L are joined in Γ if the original vertices are joined in L. For the Dynkin diagram A_3 with vertices labeled as
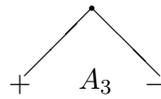
and square lattice L (with downward orientation)

$L = \mathbf{Z}^2$
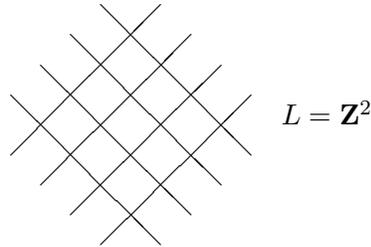
one obtains two copies of the configuration space of the Ising model by placing the frozen spin $\cdot$ on the even or odd sublattices of L.
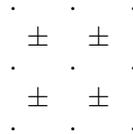
To complete the data for the model, we assign a complex number $W(\alpha, \beta, \gamma, \delta)$, the Boltzmann weight to each allowed configuration:
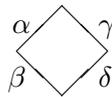
For the A_3 graph, we assign the appropriate Ising weights to the eight possible configurations:
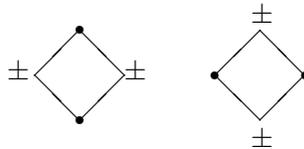
The Andrews-Baxter-Forrester models are associated with the Dynkin diagrams A_n [2]:

Thinking of this as an oriented graph with basic transitions •⟶•,
we can generalize to the Weyl alcove $A^{(n)}$ of the level k-integrable represen-
tations of the Kac-Moody algebra $\mathcal{A}_{N-1}^{(1)} = \widehat{SU(N)}$. For $SU(3)$ instead of
$SU(2)$, we replace

   by
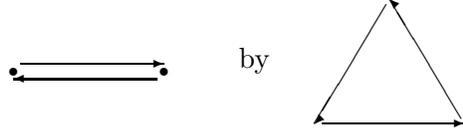
i.e., transitions generated by 3 vectors (instead of 2) which add to zero. Let

$$P_{++}^{(n)} = \{\lambda = \sum_{i=1}^{N-1} \lambda_i \Lambda_i; \lambda_i \geq 1, \sum_{i=1}^{N-1} \lambda_i \leq n - 1\},$$

where the Λ_i's are the $N-1$ weights of the fundamental representations and
$n = k + N$. The vertices of the graph $\mathcal{A}^{(n)}$ are the elements of $P_{++}^{(n)}$ and its
oriented edges are given by N vectors e_i defined by

$$\begin{aligned}
e_1 &= \Lambda_1, \\
e_i &= \Lambda_i - \Lambda_{i-1}, \qquad i = 1, 2, \ldots, N-1, \\
e_N &= -\Lambda_{N-1}.
\end{aligned}$$

These models were introduced and studied by Date, Jimbo, Miwa and Okado
[21].

The notion of an orbifold of such a model, dividing out by a symmetry
group G on Γ which preserves the Boltzmann weights was introduced by
[36, 37, 22]. The model based on Γ is replaced by one built on an orbifold
graph, denoted by Γ/G; i.e., we consider a model on the orbifold graph Γ/G
equipped with some Boltzmann weights. Fendley and Ginsparg considered
the $\mathbf{Z}_2$-orbifold of the A-series associated to $SU(2)$ and Di Francesco and
Zuber for the $\mathbf{Z}_3$-orbifolds of the corresponding $SU(3)$ orbifolds.
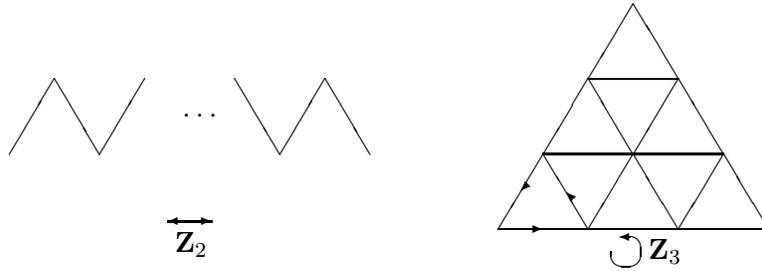
In particular, they expressed the Boltzmann weights for the original $\mathcal{A}^{(n)}$-model (for $N = 2, 3$ respectively) as

$$\left\langle\!\!\!\diagdown\ W'\ \diagup\!\!\!\right\rangle \quad = \sum \omega\ \left\langle\!\!\!\diagdown\ W\ \diagup\!\!\!\right\rangle \tag{1}$$

with complex coefficients $\omega^N = 1$, and could show by direct computation that the Yang-Baxter equation for the orbifold graph was inherited from the corresponding equation for the original graph. In this framework, the Yang-Baxter equation is expressed as equality of the partition functions

$$\text{(hexagon figure)} \quad = \quad \text{(hexagon figure)}$$

where both figures have the same configurations (edges of Γ) along the boundary and the sum (the product of 3 Boltzmann weights) over internal edges.

The orbifold procedure is best understood in the path algebras associated to graphs. The transfer matrix method transforms the two dimensional classical model to a one-dimensional quantum system [70, 60, 35, 3, 13, 58, 30]. The classical observables are $C(P)$, the commutative C^*-algebra of all continuous functions on the compact Hausdorff space of configurations, and at each inverse temperature β we may be interested in the simplex K_β of equilibrium states. The transfer matrix formalism transfers this setting to that of a one dimensional quantum model represented by a non-commutative C^*-algebra A, a path algebra on the graph Γ or the algebra generated by the local transfer matrices. For each inverse temperature β, one looks for a map from equilibrium states in K_β to linear functionals on A and a map $F \mapsto F_\beta$ from (local) classical observables in $C(P)$ to the quantum algebra A such that one recover the classical expectation or correlation functions from a knowledge of the quantum ones alone: $\mu(F) = \varphi_\mu(F_\beta)$.

The non-commutative path algebra $A(\Gamma)$ associated to the graph Γ is defined as follows. First take the space of all one-sided paths in Γ (beginning in some specified subset of vertices of Γ). Drawing the paths vertically, beginning with the initial vertices at level zero, we go down one level at a time, using the graph Γ, so that at each level we have a family of vertices. Consider the finite dimensional algebra $A[m, n]$ between the two levels m and n, generated by matrix units or partial isometries $|\xi\rangle\langle\eta| \equiv (\xi, \eta)$, where ξ, η are paths from level m to n, with $(\xi, \eta) = 0$ unless ξ, η have the same initial and same terminal vertices. Let $A(\Gamma) = \varinjlim A[m, n]$ be the completion, i.e., the

C^*-algebra generated by such pairs of paths (ξ, η), with $A[m, n]$ embedded into $A[m', n']$, $(m' \leq m \leq n \leq n')$ by

$$(\xi, \eta) \quad \rightarrow \quad \sum_{\alpha, \beta} (\alpha \xi \beta, \alpha \eta \beta),$$
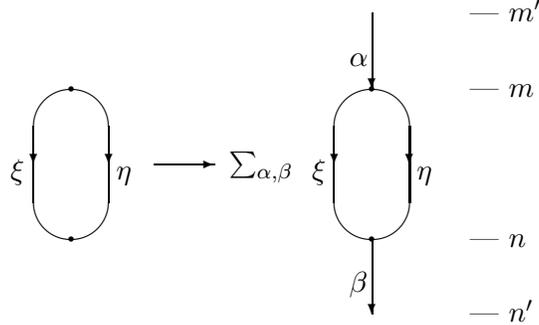
where the summation is over paths α from level m' to level m, β from level n to n'.

Writing

$$W = \sum \genfrac{}{}{0pt}{}{\alpha \quad \gamma}{\beta \quad \delta} (\alpha \beta, \gamma \delta)$$

we see that being given a (translation invariant) Boltzmann weight is the same as begin given an element W in the path algebra $A[n-1, n+1]$, for some n. Then for any n, letting W be the corresponding element of the path algebra, the Yang-Baxter equation corresponds to

$$W_n W_{n+1} W_n = W_{n+1} W_n W_{n+1}. \tag{2}$$

In general, the $\{W_n; n \in \mathbf{N}\}$ will not generate the path algebra, but they will in the case of the $\mathcal{A}^{(n)}$-models of [21], when W_n gives a representation of the Hecke algebra of type A. If the group G acts on Γ, then (under reasonable hypotheses) the fixed point algebra $A(\Gamma)^G$ is isomorphic to $A(\Gamma/G)$, the path algebra of the orbifold graph; with this isomorphisms explicitly leading to (2), as a Boltzmann weight invariant under the group action gives an invariant element of the path algebra $A(\Gamma)$. This gives a natural meaning for the construction of the orbifold graph. The fixed points of the graph $\mathcal{A}^{(n)}$ are replaced by N vertices representing the orbit and, the N vertices which are moves cyclically onto each other are replaced by singleton as in the following examples:

$$(3)$$
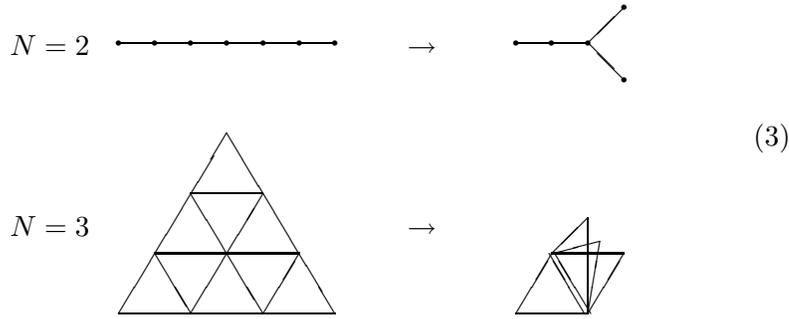
These models were first introduced by Pasquier [69] in the case of $N = 2$ by an ad hoc method of finding Boltzmann weights for the D-series which satisfies the Yang-Baxter equation, and later by [36, 37] for the case $N = 2$, $N = 3$, who used (1) to write down the Boltzmann weights and verified by hand that they satisfied the Yang-Baxter equation. Di Francesco and Zuber predicted similar models for general N would follow as a consequence of identities between theta functions. The methods of [32] shows that the best way to understand these models is via the path algebra picture outlined here. (Pasquier also used ad hoc methods to obtain a model at criticality for E_6; an embedding of A_{11} in E_6 via the Temperley-Lieb algebra.) This is a discrete version of orbifold theory in the structure theory of C^*-algebras, where the existence of fixed points in the symmetry group of some underlying structure tends to new phenomena. For example, consider the $\mathbf{Z}_2$-action of flipping the circle; $z \mapsto \bar{z}$. The crossed product of a pair of distinct points $\{z, \bar{z}\}$ moved by a symmetry $\mathbf{Z}_2$ is $C\{z, \bar{z}\} \times \mathbf{Z}_2 = M_2$, the 2×2-complex matrices generated by the diagonal $\mathbf{C}^2$, and the $\mathbf{Z}_2$-flip which interchanges the components. The crossed product of a singleton $\{z\}$ by $\mathbf{Z}_2$-action is $\mathbf{C}^2$ $(= C(\hat{\mathbf{Z}}_2))$. Gluing this together, the crossed product $C(\mathbf{T}) \times \mathbf{Z}_2$ is the M_2-valued functions on the interval (or a semicircle) diagonal at the endpoints, represented geometrically as
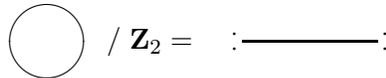
This example is the continuous analogue of (3) and is at the heart of recent progress in the structure theory of amenable C^*-algebras, where for example, one can show the existence of compact group actions on AF algebras whose fixed point algebras are not AF [5, 59, 6, 7, 8]. The analogous $\mathbf{Z}_2$-action on the non-commutative two torus A_θ leads to an AF algebra $A_\theta \times \mathbf{Z}_2$ [9, 10] (so that the dual action has again non-AF fixed point algebra, essentially A_θ);

the fixed point algebra $A_\theta^{\mathbf{Z}_2}$ is also AF, and this may help in understanding the Cantor spectra of discrete Schrödinger operators with almost periodic potentials.

Returning to our central framework, note that the graph A_3 is self-dual
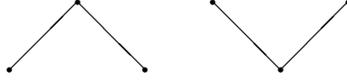
In this case, the process taking the orbifold model, is a non-trivial identification of the Ising model with itself; indeed it is the high temperature-low temperature duality
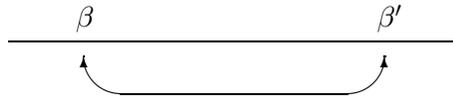
of Kramers and Wannier [19, 35]. Note that again the existence of a fixed point under this transformation is what leads to the interesting phenomena; it is the critical temperature of Onsagar β_c.

We now use the path formulation of AF algebras [28, 29, 64, 65, 75] to construct subfactors, with the main interest centered on those constructed from the solvable lattice models. Consider the following

$$(4)$$
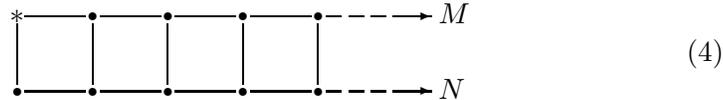
Here each line represents a graph, and each node represents a sequence of vertices say perpendicular to the plane of the paper, where at the top left hand corner we have one distinguished vertex $*$. Taking the Bratteli diagrams
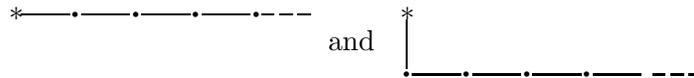
we obtain two AF algebras, completing with respect to some traces we obtain von Neumann algebras N and M; the vertical graphs will allow us to embed N in M — for the time being we will not be precise about this. We want to extend this picture to obtain a double complex and a sequence of factors $N \subset M \subset M_1 \subset \cdots$:

$$\begin{array}{c}\text{(diagram)}\end{array} \tag{5}$$
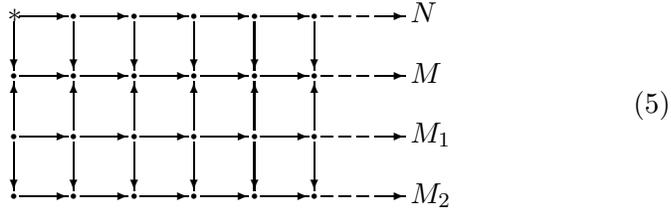
We do not wish to generate a random sequence of factors, but the Jones tower associated to the first inclusion $N \subset M$. The Jones basic extension from $N \subset M$ to $M \subset M_1$ can be thought of as a reflection of N through M; this can be achieved by reflecting the vertical graphs as indicated by alternating the orientations. This using the graphs $\mathcal{A}^{(n)}$, where the vertices are N-colorable, we obtain the following picture of colors [32]:

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow N-1 \longrightarrow 0 \longrightarrow 1 \longrightarrow$$

$$\begin{array}{c} 1 \\ \\ 0 \\ \\ 1 \end{array} \tag{6}$$
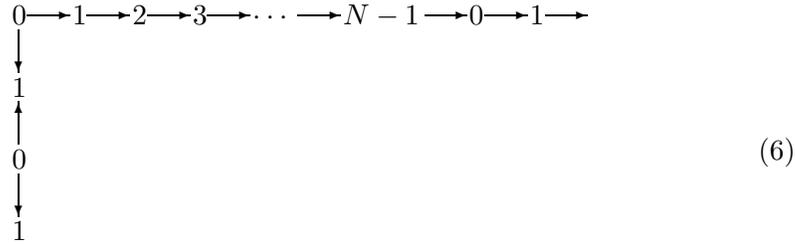
The Jones tower of relative commutants $N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \cdots$ is an inclusion of finite dimensional C^*-algebras in the case of finite index; its Bratteli diagram being the iteration of a single graph called the principal graph Γ (which may bear no relation in general to the graphs used in (4) to construct the subfactor). The inclusion $N \subset M$ is said to be of finite depth when the principal graph is finite, in which case the norm of its incidence matrix is the (square root) of the Jones index of the subfactor. Our task is to identify the principal graph in certain examples.

To compute the relative commutant $N' \cap M_k$
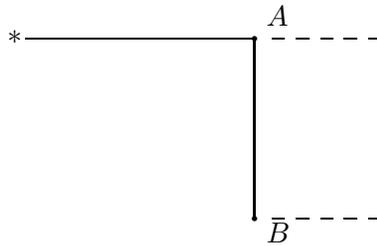
first consider at the finite dimension level the commutant $A' \cap B$. Straightening this out,
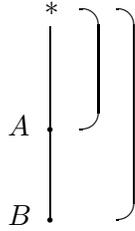
we need the commutant of a path algebra $A[0, n]$ in $A[0, m]$ which is easily
seen to be $A[n, m]$, the vertical path algebra
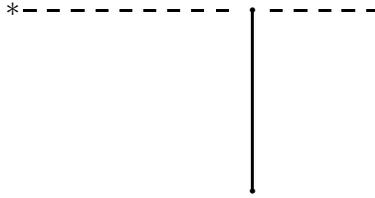
Since we need the relative commutant at the infinite stage, it is reasonable
to expect (and it is the case when the four basic graphs in (4) are finite) that
what we should take is that part of the vertical algebra which is transformed
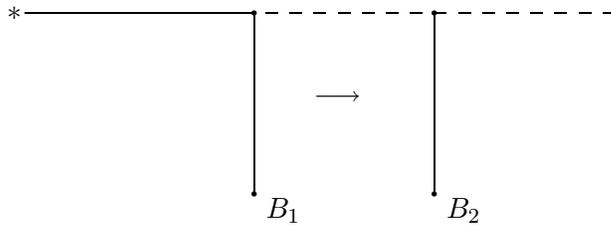
to another vertical path algebra when we embed B_1 in B_2. This is the notion
of parallel transport of Ocneanu [64, 66]; the principal graph is general being
the flat part (of a given complex (5)). The simplest case to describe is when
everything vertical transforms in a parallel fashion to a vertical algebra; in
which case the principal graph can be identified with the graph appearing
in the left most vertical part of the complex (5).

So far, we have not been specific about the vertical inclusions. If we take
trivial diagonal embeddings

then we would be in the worst possible situation where something vertical
becomes horizontal. We need to make a non-diagonal identification, and

this is where the Boltzmann weights become useful. They provide us, with suitable specialization of the spectral parameter, with a unitary to identify the path algebra ⌐ with ⌐ and when normalized, they give rise to a unitary to identify paths ⌐ with ⌐. This is a connection in the sense of Ocneanu [64, 65, 66]; the unitarity condition corresponds to the first inversion relation in the solvable lattice model theory, the biunitarity (unitarity of ⌐ under renormalization) or commuting square condition corresponds to the second inversion relation. It is the Yang-Baxter equation which is useful in helping to compute the flat part of a graph. By symmetry, flatness is the same as vertical parallel transport of the horizontal path algebras. Thus taking a face operator in the horizontal path algebra we have [73]:

using the connection

by Yang-Baxter equation

by unitarity
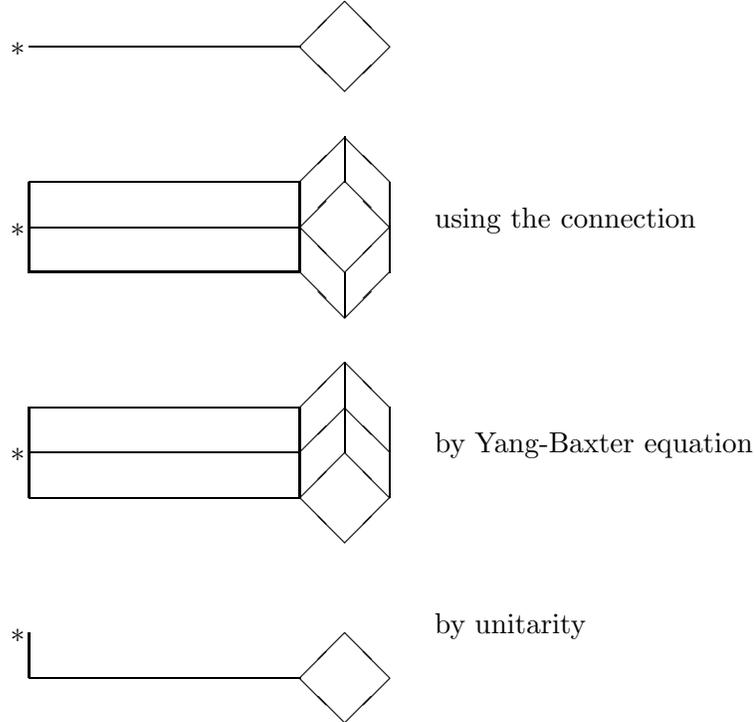
Thus if face operators generate the (first) horizontal algebra in (2), then we certainly have flatness. This is the case for the $\mathcal{A}^{(n)}$ models with connection given by the Boltzmann weights [46] so that the principal graphs of these subfactors (first constructed by Wenzl of [80] who computed their index) are their 0-1 parts [32] (c.f. (3)). Similarly embedding A_{11} in E_6 (since a Temperley-Lieb algebra can be embedded in the path algebra of a bipartite graph [38]), the principal graph is the intertwining graph (computed in [69, 31]) and so we recover the result of Okamoto [68]. Note that here we are using

the intertwining Yang-Baxter equation between the (critical) Boltzmann weights on A_{11} and E_6 (see [22]).

3. Modular Invariants and Subfactors

The A-D-E classification of (level k) $SU(2)$ Wess-Zumino-Witten models is as follows. The $SU(2)_k$ Kac-Moody algebras has $k+1$ integrable representations $[\varphi_j]$, $2j = 0, \ldots, k$, with $SU(2)$ isospin j and affine character $\chi_\lambda(\tau) = \text{Tr}_{\mathcal{H}_j} e^{2\pi\tau i(L_0 - c/24)}$, $\lambda = 2j + 1$. A one loop partition function on the torus should be expressed as $Z(\tau, \tau') = \sum N_{\lambda,\bar\lambda}\chi_\lambda(\tau)\overline{\chi_{\bar\lambda}}(\tau')$. (a continuum version of the transfer matrix formalism outlined above). The classification problems to find all such partition functions of the above form, where $N_{\lambda,\lambda'}$ are non-negative integers (arising from a decomposition of the underlying Hilbert space as $\oplus N_{\lambda,\bar\lambda}\mathcal{H}_\lambda \otimes \mathcal{H}_{\bar\lambda}$), and where Z is modular invariant (due to invariance under reparametrizations of the torus). Such modular invariant sesqui-linear forms were shown [12, 50] to be of type A-D-E.

　　The identification of a modular invariant with a graph is through noting that the diagonal part (writing $Z = \sum_{\lambda\in\mathcal{E}} |\chi_\lambda|^2 +$remainder) of a modular invariant is described by a set ($\mathcal{E}$) of exponents of an A-D-E Dynkin diagram. Thus for $k = 16$, we have three invariants corresponding to A_{17}, D_{10}, E_7 as follows:

modular invariant	exponents	graph
$\|\chi_1\|^2 + \|\chi_2\|^2 + \cdots + \|\chi_{17}\|^2$	$1, 2, 3, \ldots, 17$	A_{17}
$\|\chi_1 + \chi_{17}\|^2 + \|\chi_3 + \chi_{15}\|^2 + \|\chi_5 + \chi_{13}\|^2 + \|\chi_7 + \chi_{11}\|^2 + 2\|\chi_9\|^2$	$1, 3, 5, 7, 9, 11, 13, 15, 17$	D_{10}
$\|\chi_1 + \chi_{17}\|^2 + \|\chi_5 + \chi_{13}\|^2 + \|\chi_7 + \chi_{11}\|^2 + \|\chi_9\|^2 + (\chi_3 + \chi_{15})\chi_9^* + \chi_9(\chi_3 + \chi_{15})^*$	$1, 5, 7, 9, 11, 13, 17$	E_7

　　In the subfactor theory, only A-D-E Dynkin diagrams, with D_{even} and E_{even} appear as principal graphs of subfactors with index less than four. In the rational conformal field theory of $SU(2)$ Kac-Moody WZW models, described by A-D-E Dynkin diagrams, there is a degeneracy so that only D_{even} and E_{even} need be counted. For example in the case of $k = 16$, the modular invariant for E_7 reduces to that of D_{10} under the simple interchange of blocks χ_9 and $\chi_3 + \chi_{15}$. Note that there are two kinds of modular invariants here

$$\sum |\chi_i|^2 \qquad \text{type I}$$
$$\sum \chi_i \bar\chi_{\sigma(i)} \qquad \text{type II}$$

where σ is a permutation of the extended fusion rules. In this case, the flip on the extended blocks identifies the two modular invariants for D_{10} and E_7,

and led Zuber to conjecture that the subfactor built from the E_7 connection

$$
\begin{array}{c}
E_7 \\
E_7 \ \square \ E_7 \\
E_7
\end{array}
$$
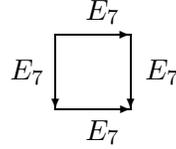
is the D_{10} subfactor (i.e., the flat part of E_7 is D_{10}). This conjecture was verified in [34]; indeed using a commuting square or connection

$$
\begin{array}{c}
D_{10} \\
D_{10} \ \square \ E_7 \\
E_7
\end{array}
$$
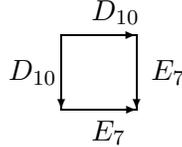
where the two D_{10}'s are glued together by a non-trivial identification corresponding exactly to the interchange blocks $\chi_9 \leftrightarrow \chi_3 + \chi_{15}$.
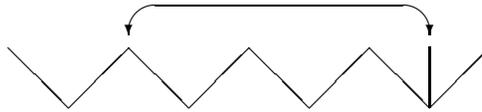
This is related to an observation in [24].

Note that the flat parts of the other Dynkin diagrams had been known; A_n, D_{2n}, E_6, E_8 are flat and D_{2n+1}, E_7 are non-flat as announced by Ocneanu [64] and proved by [4, 43, 44, 52, 76]. It is easy to see that the flat part of D_{2n+1} is A_{4n-1} and again corresponds to a degeneracy of a type II modular invariant for D_{2n+1} to a type I invariant for A_{4n-1}. See also [42] for an approach to flatness based on ideas from quantum groups.

4. Rational Conformal Field Theory (RCFT) and Subfactors

The combinatorial approach to rational conformal field theory such as [63, 82] has much in common with paragroup theory, and was explicitly worked out in [11] for the first time. We will explain this work [11] briefly and then show how F. Xu [83] combined this method and the orbifold construction of [32, 52]. The operator algebraic meaning of this construction will be clarified in Section 6.

Moore and Seiberg [63] showed that the braiding and fusion matrices and the modular matrix satisfying certain axioms contain the duality data of a conformal field theory. If we start from a connected, simply connected,

compact and simple Lie group G and an integer k called a level, one gets such combinatorial data and can compute an expectation value of knotted graphs as in [82]. In [11], de Boer and Goeree showed that this kind of combinatorial data produce a paragroup from an RCFT and a choice of a primary field. The biunitary connection is directly obtained from entries of the braiding matrices, and the only non-trivial axiom we have to verify is flatness. Their observation is that if we have a partition function diagram with $*$ at all the four corners, the computation of the value is reduced to that of the expectation value of the following knotted diagram.
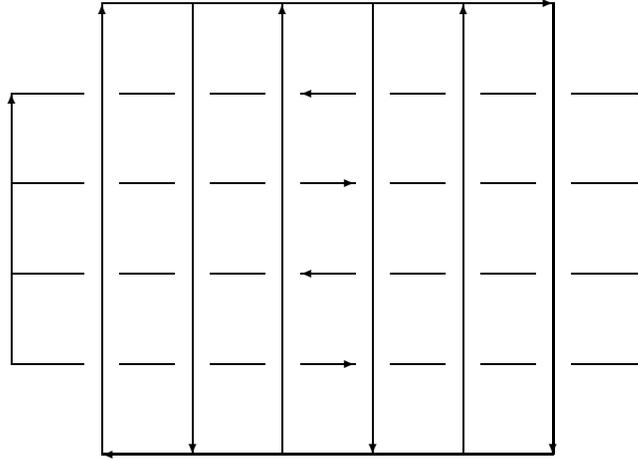
This diagram can be split into two pieces because the vertical arrows are over the horizontal arrows at all the crossings. Then it is very easy to compute the expectation value and we get flatness of the connection.

Next we explain the orbifold construction of subfactors from this viewpoint. The orbifold construction was first used in subfactor setting in [52] to prove classification of subfactors corresponding to the Dynkin diagram D_n which was announced by Ocneanu without a proof in [64], and established as a general method in [32]. The main idea is that if a paragroup has a symmetry of finite order, we can produce a candidate for a new paragroup which is the quotient of the original paragroup by the symmetry. This is interesting especially when the symmetry moves the distinguished vertex $*$ of the paragroup. In such a case, the symmetry cannot act on the higher relative commutants naturally, but we realize the subfactor in non-canonical way so that the symmetry can act. That is, the path algebra starts not just from $*$ but from all the images of $*$ under the symmetry. Then we can take fixed point algebras of the double sequence of the path algebras and get a natural biunitary connection on the double sequence of the fixed point algebras, then the only remaining problem is to verify flatness. The paragroups

corresponding to the Dynkin diagrams A_{2n-3} naturally have $\mathbf{Z}_2$-symmetries, and the fixed point algebras under the $\mathbf{Z}_2$-action on the double sequence of the path algebras starting from the two endpoints of A_{2n-3} give the connection on D_n. Then the flatness problem of D_n is reduced to the following identity of the partition functions as in [52].

$$
\begin{array}{ccc}
* & \longrightarrow & *' \\
\downarrow & & \downarrow \\
*' & \longrightarrow & *
\end{array} \quad = 1
$$

where $*'$ is the other endpoint than $*$, and the size of the diagram is $(2n - 4) \times (2n - 4)$. Then direct computation based on induction as in [52] shows that the value of the left hand side is $(-1)^n$, and this proves that the Dynkin diagrams D_{2n} are realized as paragroups (in unique way) and the diagrams D_{2n+1} are impossible. This gave the first complete proof of this statement in the literature. Impossibility of D_{2n+1} has been independently proved by [43] and [76] based on fusion rules, the original proof of realization of D_{2n} of Ocneanu is now known [53, appendix], and Izumi's method for E_8 also gives a simpler proof of realization of D_{2n}, but this orbifold method still gives the most natural conceptual understanding of the phenomena as seen below.

In [32], we first computed the higher relative commutants of the Hecke algebra subfactors of type A of Wenzl [80] based on solutions to the Yang-Baxter equation by Jimbo-Miwa-Okado [46] as explained in section 1, and then applied the orbifold construction to these series. The sequence of subfactors of Wenzl with indices converging to N^2 corresponds to $SU(N)$. We proved in [32] that if $N = 3$, the orbifold construction always gives a flat connection. Thus $SU(2)$ and $SU(3)$ behave quite differently in this question of flatness. We further noted with a little bit more work that the same result holds for odd prime N, and conjectured that the parity of N causes this difference. F. Xu [83] proved that this is indeed the case. That is, if N is odd, the orbifold construction always gives a flat connection, and if N is even, the sequence of the resulting connections contains a flat connection and a non-flat connection alternately. Xu's method is more general and applicable to any subfactor arising from RCFT corresponding to G_k Wess-Zumino-Witten models as explained as above, if we work at the symmetry of the paragroup arising from the center Z of the Lie group G.

By [32] and [83], the problem of flatness is reduced to verifying the identity

$$
\begin{array}{ccc}
* & \xrightarrow{\ \eta\ } & \rho^j(*) \\
\xi \downarrow & & \downarrow \rho^j(\xi) \\
\rho^k(*) & \xrightarrow[\ \rho^k(\eta)\ ]{} & \rho^{j+k}(*)
\end{array}
\quad = 1,
$$

where ρ denote the automorphism of the graph arising from the symmetry. Xu further noticed that it is enough to assume $k = -j$. By the same argument as in [11], this computation is reduced to that of the expectation value of the following knotted graph.
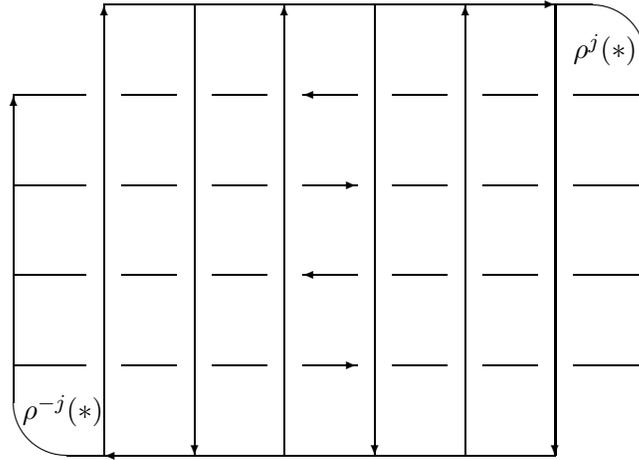
We again want to simplify the picture, but this expectation value is invariant only under regular isotopy, and this time we have a writhe for a curve at the upper right corner of the original graph. If we remove this writhe with the first Reidemeister move, we have a coefficient $\exp(\pm 2\pi i h_j)$, where h_j is the conformal dimension of $\rho^j(*)$, as in [82, page 639]. Thus we get flatness if and only if h_j's are integers. The conformal dimensions for $SU(N)$ have been known and we can decide when it is an integer.

Thus the difference between the Dynkin diagrams D_{2n} and D_{2n+1}, first observed by Ocneanu in paragroup settings, is naturally understood as a

very special case of general phenomena caused by conformal dimensions and the orbifold construction.

Xu further noticed that the condition on the Lie group G and level k equivalent to flatness of the resulting connections in the orbifold construction is the same as the one equivalent to existence of Chern-Simons gauge theory based on G/Z at level k.

We conclude this section by an important remark. Although the orbifold construction has its origin in conformal field theory [26] and works well in paragroup setting as seen above, it does not produce the combinatorial data necessary for rational conformal field theory in the following sense. The simplest case is the series D_{2n} obtained as Z_2-orbifolds of the series A_{4n-3} which comes from rational conformal field theory corresponding to $SU(2)$. Although the Dynkin diagrams D_{2n} are realized as paragroups, we cannot have a corresponding rational conformal field theory as in [11] since we do not have symmetric S-matrices.

5. Topological Quantum Field Theory (TQFT) and Subfactors

Turaev-Viro [78] introduced a topological invariant of 3-dimensional closed manifolds based on triangulations using the quantum $6j$-symbols of Kirillov-Reshetikhin [56]. They define the invariant as a state sum for a triangulation; topological invariance of this state sum is based on certain algebraic relations of the $6j$-symbols. To achieve this, they proved that classical Alexander moves [1] can be replaced simpler local moves of tetrahedra. (For a simple proof of this claim in 3 and 4 dimensions, see [39].) A. Ocneanu [67] claimed without a full proof that this type of invariants can be generalized to a subfactor setting and that algebraic axioms necessary for topological invariance are equivalent to the paragroup axioms. Although he has not published the exact statement of the theorem or a complete proof, a complete proof was given in [33] and the full machinery was worked out there. We discuss this aspect of paragroup theory here.

The basic idea of the construction in [78] is as follows. We have a set of finitely many colors with multiplication rules and each color has a certain weight. For a fixed triangulation of a given 3-dimensional compact manifold, we consider all the admissible configurations of colors assigned to edges appearing in the triangulation. (Admissibility is determined by the multiplication rule.) Each tetrahedron with six colors on six edges has a complex value called a $6j$-symbol. For a given configuration, we take the product of all the $6j$-symbol values, and then take a weighted sum of all the products over all configurations, which is called the state sum. In order to show that this complex number is really a topological invariant, we have to prove that the $6j$-symbol has a certain symmetry and that the state sum does not depend on a choice of triangulation. Turaev-Viro proved that

these two properties follow from three axioms on $6j$-symbols and presented the Kirillov-Reshetikhin quantum $6j$-symbol for $U_q(sl_2)$ [56] satisfied these axioms. The symmetry of $6j$-symbols we need means that the value given to a tetrahedron does not depend on the way we project a tetrahedron on paper, and is called tetrahedral symmetry. The other two axioms in [78] are unitarity and the pentagon relation. In order to prove that this construction gives a complex number independent on triangulation, we have to compare two triangulations of the same manifold. Alexander [1] already showed that such two triangulations can be transformed to each other by finite steps of local moves, called Alexander moves. These moves, however, are infinitely many, and are not easy to handle, so Turaev-Viro replaced the Alexander moves in 3-dimensions by three local moves. The unitarity and the pentagon relation are axioms equivalent to invariance of the state sum under these three moves. Dijkgraaf-Witten proposed a similar invariant arising from 3-cocycles of finite groups based on physical ideas. A rigorous presentation of this invariant based on Alexander's theorem is given by [79].

To generalize this machinery to a general subfactor setting, we first need an appropriate $6j$-symbol. This can be done as the generalization of a connection as in [67]. In paragroup theory, we look at four bimodules, and composition of four intertwiners among the tensor products of these four bimodules and $_N L^2(M)_M$ and $_M L^2(M)_N$. Extending this setting, we look at *six* bimodules and composition of four intertwiners among their tensor products. We also have a fusion algebra consisting of four kind of bimodules: N-N, N-M, M-N, and M-M. Each bimodule has a weight given by the corresponding Jones index. In this way, we have a generalized $6j$-symbols with tetrahedral symmetry, unitarity, and flatness. The tetrahedral symmetry (after a certain normalization) comes from the Frobenius reciprocity of bimodules and intertwiners [67, 84] and unitarity is just as usual. So all we have to prove is the pentagon relation.

First write down the following identity as 3×3-flatness. Then split the square into two pieces as follows.

It is not difficult to see that the Cauchy-Schwarz inequality implies that these two pieces have the same partition function value. (This kind of use of the Cauchy-Schwarz inequality was first made in [32, §5].) It is easy to see that all the four corner square of the picture have value 1 up to some normalization constant. Thus we have non-trivial values for two squares for the left picture and three square for the right picture. The pentagon relation is obtained immediately by writing down this identity among five squares.

Now we turn to the problem of the construction in the converse direction, constructing a subfactor from a topological invariant. Of course, we assume that the topological invariant is of Turaev-Viro type in the sense that it is a state sum based on triangulation. We also need a finitely generated fusion algebra with positive weights on the generators. Such conditions are often called rationality and unitarity respectively. By reversing the above argument, we get a biunitary connection and its 3×3-flatness in the sense that partition functions of size 3×3 have value 1. All we need is (ordinary) flatness now. This can be done as follows. First we split the 3×3-picture in the following way.
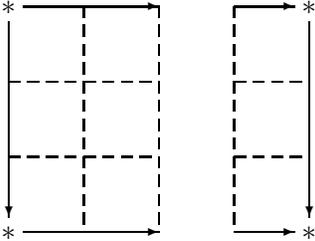
Then again by the Cauchy-Schwarz argument, we can reduce the size $3 \times n$ of a large diagram to 3×3 as follows.
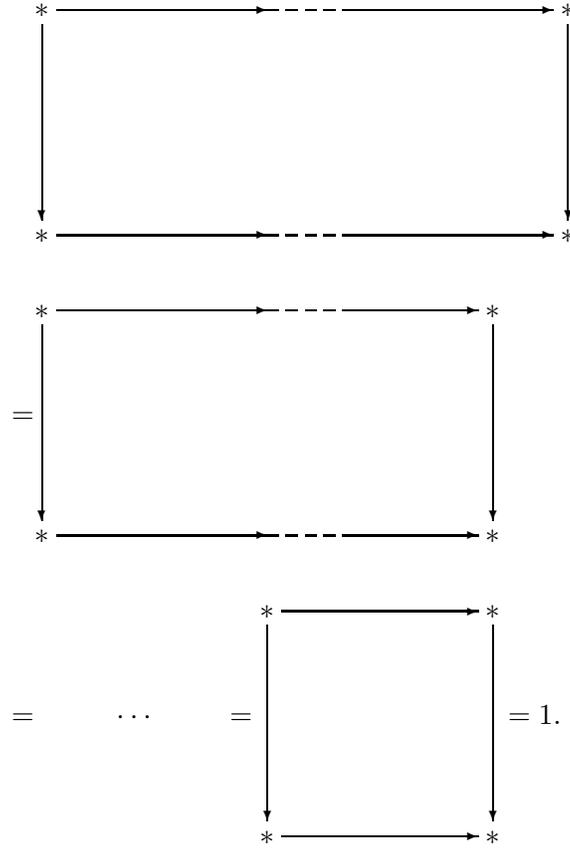
Then by the same kind of argument, we can reduce the size $m \times n$ of a large diagram to $3 \times n$. In this way, we can prove ordinary flatness from 3×3-flatness. This is an analogue of the argument of de Boer-Goeree [11] mentioned in Section 4 in the sense that we get a global identity from local identities.

In this way, we can say that flatness is equivalent to the pentagon relation which means invariance of the state sum under the following move which we call move I.
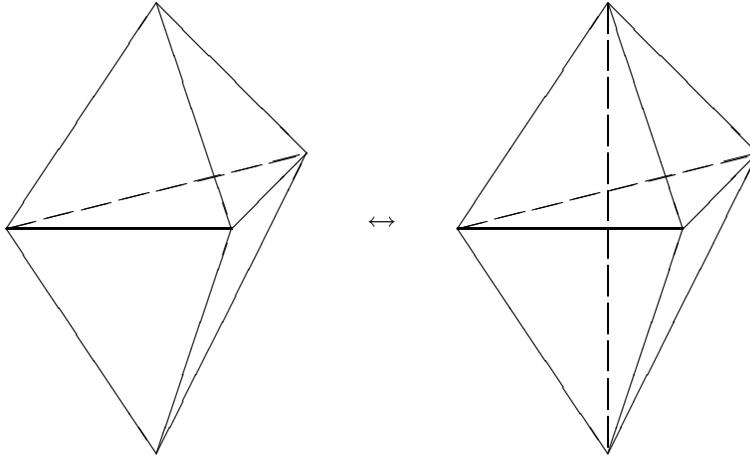
where the left picture has two tetrahedra and the right has three tetrahedra.

At the end of this section, we mention two subtle differences between the Turaev-Viro construction and topological invariants arising from subfactors. The first difference is unitarity, or positivity of the trace. If we start from a subfactor, each element of the fusion algebra has a positive weight given by the Jones index, and we need this positivity for the construction in the converse direction. Turaev-Viro machinery, however, does not require this positivity, and actually, the Kirillov-Reshetikhin quantum $6j$-symbol does not satisfy this positivity in general. At the end of [78], they compute the invariant for the case Dynkin diagram A_2 for two choices of roots of unity and get different values. One of the two choices does not satisfy positivity condition and does not come from a subfactor. In general, we have to choose a root of unity q to be $\exp(\pi i/k)$ to get this positivity. The other difference comes from $\mathbf{Z}_2$-grading of vertices of the principal graph. In the above state sum approach, each vertex has an assignment of M or N in our setting unlike in [78]. We do *not* have to take a sum over configurations of M or N on vertices since each fixed configuration on vertices gives the same value as proved in [33]. (The key argument for this is as follows. In one of the three local moves, one vertex disappears. The assignment of M or N to this vertex does not change the value we obtain.) In particular, we can use only N-N bimodules or M-M bimodules. In the Turaev-Viro case of A_n Dynkin diagrams, this means that we use only a fusion subalgebra generated by "colors" labeled by integers instead of the entire algebra generated by "colors" labeled by half-integers. This causes another difference between the Turaev-Viro invariant and the invariant coming from the Jones subfactor of type A_n.

6. Automorphisms of Subfactors

As seen above, the orbifold construction gives new series of interesting subfactors as simultaneous fixed point algebras of finite group actions on the original subfactors. This suggests that automorphisms appearing in this construction have interesting operator algebraic properties. The aim of this section is to show that this is indeed the case. Beginning with a series of papers by A. Connes, there has been extensive work on the automorphisms of operator algebras, and many results on the classification of automorphisms and group actions on injective factors have been obtained. In this work, the roles of two classes of automorphisms have been emphasized. One is a class of approximately inner automorphisms and the other is that of centrally trivial automorphisms. In the subfactor setting, Loi introduced analogues of these two classes for subfactors and we use the same names "approximately inner" and "centrally trivial" for these analogues. Let $N \subset M$ be a subfactor of type II$_1$. An automorphism $\alpha \in \mathrm{Aut}(M, N)$ is said to be approximately inner if it is of the form $\lim_n \mathrm{Ad}(u_n)$ with $u_n \in N$ and it is said to be centrally trivial if α acts trivially on $N^\omega \cap M'$, where ω is a free ultrafilter over $\mathbf{N}$.

In the following, let $N \subset M$ be a strongly amenable subfactor of type II$_1$ in the sense of S. Popa [71]. This condition is equivalent to the generating property in the sense that the higher relative commutants reproduce the subfactor for some choice of tunnel [71]. P. Loi studied the classification problem of $\alpha \in \mathrm{Aut}(M, N)$ to classify type III$_\lambda$ subfactors, $0 < \lambda < 1$, via subfactor version of Connes' discrete decomposition in [61], and introduced his invariant of automorphisms of subfactors. This invariant is given by the action of (the extension of) an automorphism on the higher relative commutants, and he proved that triviality of this invariant is equivalent to approximate innerness of the automorphism. He also proved that if any nontrivial power of $\alpha \in \mathrm{Aut}(M, N)$ is outer on M and $N \subset M$ has finite depth, then no non-trivial power of this α is centrally trivial. He further noticed that if $\alpha \in \mathrm{Aut}(M, N)$ is approximately inner and no non-trivial power of α is centrally trivial, a Connes type argument as in [17] uniquely determines α up to outer conjugacy.

Consider our orbifold action α of $\mathbf{Z}_2$ on subfactors of type A_{4n-3}. Loi's invariant is always trivial on subfactors of type A_n, so all automorphisms of these subfactors are approximately inner. Because the simultaneous fixed point algebra gives a different subfactor D_{2n}, this action α is different from the "standard" action of $\mathbf{Z}_2$, which is of the form $\mathrm{id} \otimes \sigma$ on $N \bar{\otimes} R \subset M \bar{\otimes} R$, where σ is a (unique) outer action of $\mathbf{Z}_2$ on the common hyperfinite II$_1$ factor R and $N \bar{\otimes} R \subset M \bar{\otimes} R$ is conjugate to $N \subset M$. By applying Loi's Connes type argument to $\mathbf{Z}_2$, we conclude that this action α is centrally trivial.

This method, however, is quite indirect, and does not work in the case of A_{4n-1}, where the simultaneous fixed point algebras are of the same type.

A more direct method using partition functions gives a better proof. Let α be an automorphism appearing in the orbifold construction for subfactors arising from RCFT as in section 4. Then computations based on topological moves of knotted graphs show the following identities for any k.

$$
\begin{array}{ccc}
* & \xrightarrow{\;\xi\;} & \rho^k(*) \\
\eta_+ \downarrow & & \downarrow \zeta_+ \\
& & \\
\eta_- \uparrow & & \uparrow \zeta_- \\
* & \xrightarrow[\;\xi\;]{} & \rho^k(*)
\end{array}
\quad = \delta_{\rho^k(\eta_+),\zeta_+}\delta_{\rho^k(\eta_-),\zeta_-},
$$

$$
\begin{array}{ccc}
* \xrightarrow{\eta_+} & \xleftarrow{\eta_-} & * \\
\xi \downarrow & & \downarrow \xi \\
\rho^k(*) \xrightarrow{\zeta_+} & \xleftarrow{\zeta_-} & \rho^k(*)
\end{array}
\quad = \delta_{\rho^k(\eta_+),\zeta_+}\delta_{\rho^k(\eta_-),\zeta_-},
$$

where ρ is the induced symmetry of the graph.

We can show that the first identity implies that α acts trivially on higher relative commutants, and the second implies that α acts trivially on central sequences. (See [32, Remark 5.9], [54, Lemma 4.3] respectively.) In this sense, approximate innerness and central triviality appear in a symmetric way, and we get that $\alpha \in \mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)$. Note that we get this regardless of whether the orbifold construction produces a flat connection.

In a single factor case, Connes [16] introduced an invariant $\chi(M) = (\mathrm{Ct}(M) \cap \overline{\mathrm{Int}}(M))/\mathrm{Int}(M)$ and proved several interesting results. (Also see [47].) The above observation means that the orbifold construction produces

non-trivial elements in the subfactor version of $\chi(M)$. We define $\chi(M, N) = (\mathrm{Ct}(M, N) \cap \overline{\mathrm{Int}}(M, N))/\mathrm{Int}(M, N)$, where $\mathrm{Ct}(M, N), \overline{\mathrm{Int}}(M, N), \mathrm{Int}(M, N)$ mean the class of centrally trivial automorphisms, the approximately inner automorphisms, and inner automorphisms given by unitaries in N, respectively. This is an abelian group as in the single factor case, and we have an analogue [54, Theorem 1.1] of the exact sequence in [16].

Next we aim to show that the automorphisms appearing in the orbifold construction for subfactors arising from RCFT as in section 4 actually give the entire $\chi(M, N)$. For example, in the case of Hecke algebra subfactors of type A by Wenzl [80] with indices $\sin^2(N\pi/k)/\sin^2(\pi/k)$, we get $\chi(M, N) = \mathbf{Z}_d$ with $d = (k, N)$. (In [54, Proposition 4.5], this was proved only for the case where N is prime, but the same method based on the above computation of partition functions gives this.) This claim is proved by getting an upper bound for the size of the group $\mathrm{Ct}(M, N)$ as follows.

Popa and Choda-Kosaki independently introduced the same notion of outerness of an automorphism in different situations. Popa calls an $\alpha \in \mathrm{Aut}(M, N)$ *properly outer* if the identity $ax = \alpha(x)a$ for some $a \in M_k$ and all x implies $a = 0$. Choda-Kosaki calls the same condition *strongly outer*. Popa [72] proved that if a subfactor is strongly amenable, then not being properly outer is equivalent to central triviality. Choda-Kosaki [15, 57] proved that $\alpha \in \mathrm{Aut}(M, N)$ is not strongly outer if and only if α appears at even vertices of the dual principal graph in sector/bimodule approach. (If a subfactor is of type III, this condition means that α appears in irreducible decompositions of powers of Longo's canonical endomorphism.) Combining these two results, we know that the order of the group $\mathrm{Ct}(M, N)/\mathrm{Int}(M, N)$ is bounded by the number of even vertices of the dual principal graph with normalized Perron-Frobenius weights 1. In the case of subfactors arising from RCFT as in Section 4, this upper bound is attained and we get $\chi(M, N)$.

This upper bound was proved in the finite depth case first in [54] based on Ocneanu's approach with central sequence subfactors and asymptotic inclusion [64, 66]. (The statements, made without proof, by Ocneanu needed in this approach were proved in [54].) Although this approach is more complicated than combining results of Choda-Kosaki and Popa, it is expected to give more information on subfactors with finite depth. For example, Ocneanu has suggested that considerable information on TQFT of Section 5 can be obtained from central sequence subfactors and asymptotic inclusions.

Note that central triviality is easier to handle with partition functions than the property of not being properly outer is, although these two are equivalent in our situation. For example, in the orbifold construction for A_{2n-3}, it is easy to write down a non-zero $a \in M_k$ with $ax = \alpha(x)a$, $x \in N$, explicitly as in [52] if n is even, but it is not clear at all how to get such an a, if n is odd, while central triviality is easily established in both cases. With this advantage, we can prove the following: If a subfactor arises from

RCFT as in Section 4, automorphisms appearing on even vertices of the dual principal graph are the same as those appearing in the orbifold construction. That is, the orbifold construction gives a very concrete realization of these automorphisms appearing in a more abstract way in the sector/bimodule approach.

Note that a recent construction of a subfactor with finite depth and index $(5 + \sqrt{13})/2$ by Haagerup [40] shows that the above upper bound is not always attained. (This is related to a problem whether one can adjust a certain automorphism M so that it fixes N globally. See [57].)

Thus we have many examples of subfactors with non-trivial $\chi(M, N)$, and it is not difficult to realize all finite abelian groups as $\chi(M, N)$ for subfactors with finite depth using tensor products. In the case of single factor, Connes [16, 47] constructed several examples with non-trivial $\chi(M)$, but these are rather unusual II$_1$ factors in the sense that they are non-AFD with property Γ. Subfactors arising from RCFT such as Hecke algebras subfactors of Wenzl are very natural and basic objects in subfactor theory, and they should not regarded as analogues of such exotic single factors. The right analogues of these subfactors in the single factor case turn out to be (injective) type III factors.

Loi's invariant has considerable similarity with the Connes-Takesaki module [20], and paragroups can be regarded as discrete analogues of flows of weights [20], as pointed out in [51]. For example, an automorphism of a strongly amenable subfactors is approximately inner if and only if its Loi invariant is trivial [61, Theorem 5.4], while an automorphism of an injective type III factor is approximately inner if and only if its module is trivial, as announced by Connes [18, section 3.8] and proved by [55, Theorem 1].

As to the other important class of automorphisms, centrally trivial automorphisms, Connes [18] announced that an automorphism α of an injective type III factor is centrally trivial if and only if it is an extended modular automorphism up to inner perturbation, and this was proved in [55, Theorem 1]. From this viewpoint, we come to the idea that an orbifold action arising from RCFT is a (discrete) analogue of modular automorphisms. With this understanding, we can interpret the result of Choda-Kosaki [15, 57] naturally as an analogue of a result of Haagerup-Størmer [41, Proposition 5.4].

It is easy to see that the orbifold $\mathbf{Z}_2$ actions on A_{4n-3} and the $\mathbf{Z}_2$ actions flipping the two tails of D_{2n} are in Takesaki type duality. This duality is again a discrete analogue of that between the Connes-Takesaki module [20] and the Sutherland-Takesaki modular invariant [77].

Popa [72] obtained the classification result of discrete amenable group actions on strongly amenable subfactors in the sense that properly outer such actions are classified by the Loi invariant. Because such actions are centrally free, this is again an analogue of the classification of centrally

free actions of discrete amenable groups on injective type III factors by the Connes-Takesaki module.

Longo had already noticed in [62] that his canonical endomorphisms are similar to modular automorphism groups. From our viewpoints, automorphisms appearing in the powers of canonical endomorphisms have further similarity to modular automorphisms, and in this approach the analogy makes sense even in type II_1 settings.

This kind of central sequence approach is useful for the classification of group actions on subfactors. For example, there are two and only two outer actions of $\mathbf{Z}_2$ on subfactors of type A_{2n+1}, and one is given by the standard form and the other is given by the orbifold construction. There is also an analogue of modular obstruction. For more on this classification, see [54, §5].

A basic philosophy in the above type of analogy is that any phenomenon on type III (single) factors appears in type II_1 subfactor in a discrete form. A mixture of the both will appear in type III subfactors. Winsløw's simultaneous generalization of the Connes-Takesaki module and Loi invariant in [81] is the first example of such a mixture.

It is a very important problem to distinguish non-amenable subfactors with the same indices using $\chi(M, N)$ as an analogue of Connes' construction in [16] for single II_1 factors. Our exact sequence [54, Theorem 1.1] is expected to be useful for this purpose, but so far, no example has been computed in the non-amenable subfactor case.

Finally, we discuss the problem of finding non-commutative symmetries of paragroups. In Xu's construction [83], the finite group acting on subfactors arising from RCFT is given as the center of the Lie group with which we started. In particular, this finite group is always abelian. In general, there seems to be no reason why a symmetry group of a given paragroup is abelian and the problem to find an interesting non-abelian symmetry has caught attention of physicists. From the viewpoint of automorphisms, $\chi(M, N)$ is always abelian, but $Ct(M, N)/Int(M, N)$ is not necessary abelian. (A trivial example is a subfactor $R^G \subset R$ with G finite, non-commutative, and acting on R outerly.) We feel that "interesting" symmetries should give $\chi(M, N)$ and so are commutative by some deeper reason, but nothing is clear at present.

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