Bernstein-Zelevinsky data and the crystal basis of U_q^- in type $A_{n-1}^{(1)}$

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Aim : Construct "Bernstein-Zelevinsky (BZ for short) data" in affine type A.

Plan

- What is BZ data? (Geometric background)
- Review on type A_n (Prototype)
- Toward BZ data in affine type A : a combinatorial approach (work in progress)

§ **What is BZ data? (MV polytopes and BZ data)**

Mirković-Vilonen (1997∼)

- MV cycles : Algebraic cycles in affine Grassmaniann associated to
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- Moment map image ⇒ MV polytopes : Polytopes in $\mathfrak{h}_{\mathbb{R}}(R^{\vee})$.

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That is, the Kamnitzer's result tells us

a realization of $B(\infty)$ in terms of MV polytopes.

○ **MV polytopes of type** A_n

$\mathbf{k} \subset [1, n+1]$: a subset (called a *Maya diagram of rank* n)
$\mathcal{M}_n$: the set of all Maya diagrams of rank n
$\mathcal{M}_n^{\times} := \mathcal{M}_n \setminus \{\phi, [1, n+1]\}$

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$\Rightarrow$ We can identify $\mathcal{M}_n^{\times}$ with $\Gamma_n := \bigsqcup_{w \in W,\ i \in I} W\Lambda_i$ via

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- For $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_n^{\times}}$, consider a polytope in $\mathfrak{h}_{\mathbb{R}}$

$$P(\mathbf{M}) := \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle h, \mathbf{k} \rangle \geq M_{\mathbf{k}}\ (\forall \mathbf{k} \in \mathcal{M}_n^{\times})\}.$$

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(BZ-1) *Edge inequalities* :
for every two indices $i \neq j$ and every $\mathbf{k} \in \mathcal{M}_n$ with $\mathbf{k} \cap \{i, j\} = \phi$,

$$M_{\mathbf{k}i} + M_{\mathbf{k}j} \leq M_{\mathbf{k}ij} + M_{\mathbf{k}}.$$

Here we denote $\mathbf{k}ij = \mathbf{k} \cup \{i, j\}$ *etc.*, and we set $M_{\phi} = M_{[1,n+1]} = 0$.

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• A polytope $P(\mathbf{M})$ is a *MV polytope* if $\mathbf{M} = (M_{\mathbf{k}})$ is a BZ datum.

Remark

$P(\mathbf{M})$: a MV polytope

$\Rightarrow$ $P(\mathbf{M})$ is the convex hull of $\mu_\bullet := (\mu_w)_{w\in W} \subset \mathfrak{h}_\mathbb{R}$ (GGMS datum) where

$$\mu_w := \sum_{i=1}^{n} M_{w[1,i]} w\alpha_i^\vee \in \mathfrak{h}_\mathbb{R} \qquad (w \in W).$$

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$$P(\mathbf{M}) \quad \leftrightarrow \quad \mathbf{M} = (M_\mathbf{k})_{\mathbf{k} \in \mathcal{M}_n^\times}.$$

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• We denote by $\mathcal{BZ}_n^{w_0}$ the set of all BZ data which satisfy the following normalization condition:

(BZ-0) $M_{[i+1,n+1]} = 0$ for any $1 \leq i \leq n$.

An element of $\mathcal{BZ}_n^{w_0}$ is called an <u>w_0-*BZ datum*</u>.

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• $\mathcal{BZ}_n^{w_0}$ has a crystal structure which is isomorphic to $B(\infty)$ (Kamnitzer).

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In stead of (BZ-0), we consider another normalization condition:

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For an e-BZ datum $\mathbf{M}$, define a new collection $\Theta(\mathbf{M})$ by

$$\Big(\Theta(\mathbf{M})\Big)_{\mathbf{k}} := (\mathbf{M})_{\mathbf{k}^c},$$

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- $\mathcal{BZ}_n^e$ has the induced crystal structure which is isomorphic to $B(\infty)$:

$$\widetilde{e}_i^* := \Theta \circ \widetilde{e}_i \circ \Theta, \quad \widetilde{f}_i^* := \Theta \circ \widetilde{f}_i \circ \Theta, \ \textit{etc.}$$

§ **Review on type A_n (Prototype)**

In type A, explicit relationships on the following three realizations of $B(\infty)$ are known.

- (limits of) semi-standard tableaux : $\mathcal{B}_n$
- irreducible Lagrangians : $\displaystyle\bigsqcup_{\nu \in Q_+} \mathrm{Irr}\Lambda(\nu)$
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$\mathcal{BZ}_n^e$: e-BZ data $\longleftrightarrow$ $\mathcal{B}_n$: "limits" of SST

◦ Notations

$U_q = U_q(\mathfrak{sl}_{n+1}) = \langle e_i,\ f_i,\ t_i^{\pm 1}\ (i \in I)\rangle.$

$B(\infty)$: the crystal basis of U_q^-

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$* : U_q \to U_q$: a $\mathbb{Q}(q)$-algebra anti-automorphism

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Set

$$\varepsilon_i^*(b) := \varepsilon_i(b^*), \quad \varphi_i^*(b) := \varphi_i(b^*), \quad \widetilde{e}_i^* := * \circ \widetilde{e}_i \circ *, \quad \widetilde{f}_i^* := * \circ \widetilde{f}_i \circ *.$$

$\Rightarrow B(\infty)$ endowed with maps wt, ε_i^*, φ_i^*, $\widetilde{e}_i^*$, $\widetilde{f}_i^*$ is a crystal.
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That is, $B(\infty)$ has two crystal structures :

$$(B(\infty)\ ;\ \mathrm{wt},\ \varepsilon_i,\ \varphi_i,\ \tilde{e}_i,\ \tilde{f}_i),$$

$$(B(\infty)^* = B(\infty)\ ;\ \mathrm{wt},\ \varepsilon_i^*,\ \varphi_i^*,\ \tilde{e}_i^*,\ \tilde{f}_i^*).$$

∘ Realization in terms of limits of semi-standard tableaux

$\lambda \in P_+$: dominant integral weight

$V(\lambda)$: irreducible U_q-module with h.w. λ

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Theorem (Kashiwara-Nakashima)

$$B(\lambda) \cong SST(\lambda).$$

Here $SST(\lambda)$ is the set of semistandard tableaux of shape λ.

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Theorem (Kashiwara-Nakashima)

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Here $SST(\lambda)$ is the set of semistandard tableaux of shape λ.

- Take $\lambda \to \infty$ (w.r.t. $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in Q_+$)

$$\begin{array}{ccc} B(\lambda) & \cong & SST(\lambda) \\ \downarrow & & \downarrow \\ B(\infty) & \cong & \mathcal{B}_n \end{array}$$

$\mathcal{B}_n$: The set of all $n(n+1)/2$ tuples of non-negative integers
$\mathbf{a} = (a_{i,j})_{1 \leq i < j \leq n+1}$ (which is called a *Lusztig datum*).

We regard $\mathcal{B}_n$ as "$SST(\infty)$" via

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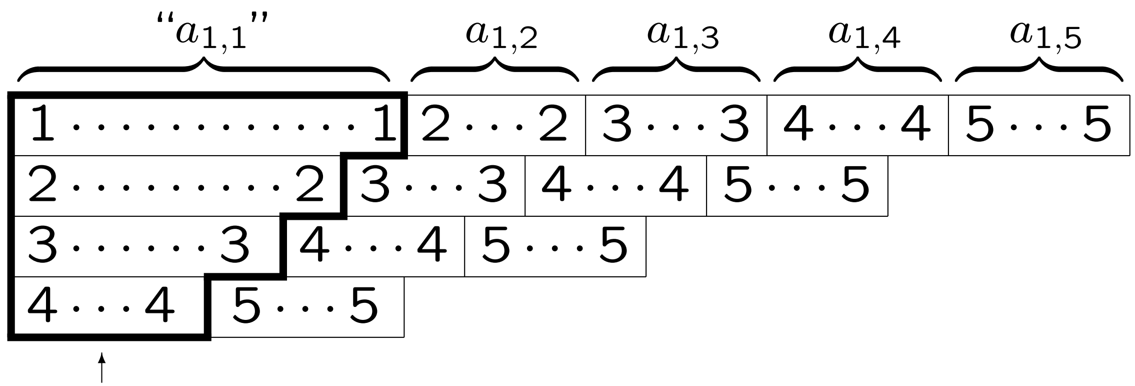
Example. A_4 case:

$$\begin{matrix} a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ & a_{2,3} & a_{2,4} & a_{2,5} \\ & & a_{3,4} & a_{3,5} \\ & & & a_{4,5} \end{matrix} \quad \longleftrightarrow$$



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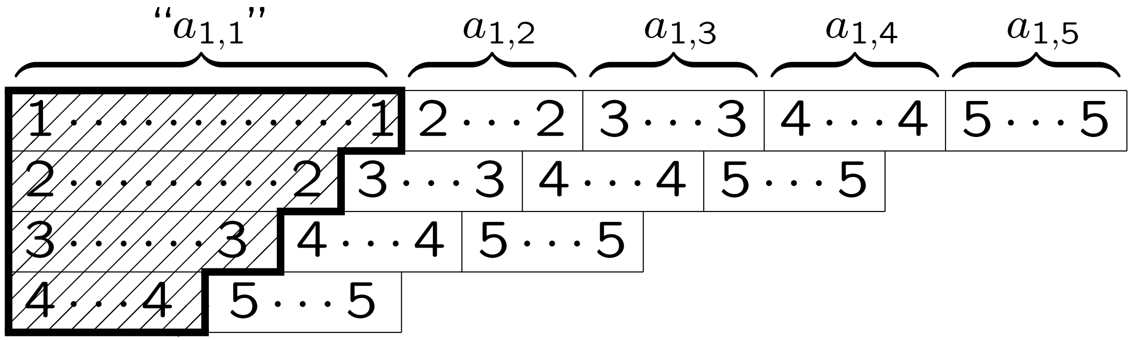

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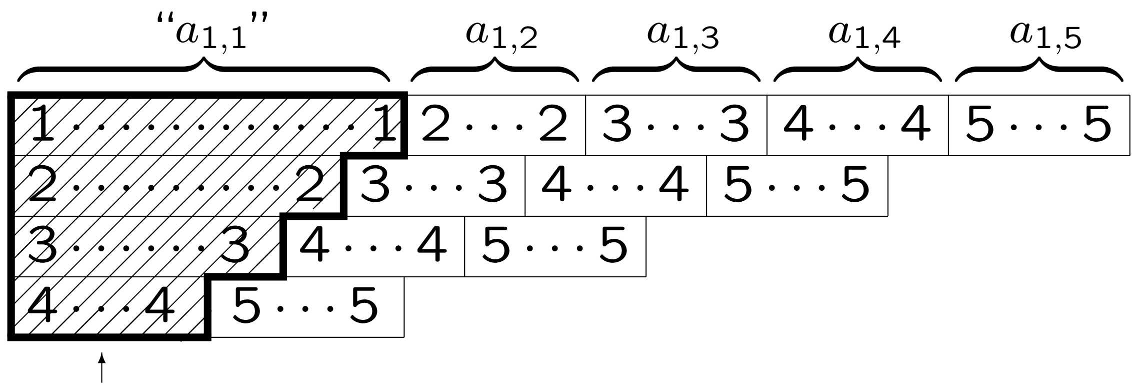

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Remark

(1) The explicit crystal structure of $\mathcal{B}_n$ can be determined.
(Here we use "the far-eastern reading" of a tableau.)

(2) Since $B(\infty)$ has the $*$-crystal structure, $\mathcal{B}_n$ also has the induced $*$-crystal structure.

(We omit to give them.)

∘ From $SST(\infty) \cong \mathcal{B}_n$ to $\mathcal{BZ}_n^e$

Definition *Let* $\mathbf{k} = \{k_1 < k_2 < \cdots < k_l\} \in \mathcal{M}_n^\times$ *be a Maya diagram. For such* $\mathbf{k}$*, we define a* $\mathbf{k}$*-tableau as an upper-triangular matrix* $C = (c_{p,q})_{1 \le p \le q \le l}$ *with integer entries satisfying*

$$c_{p,p} = k_p \qquad (1 \le p \le l),$$

and the usual monotonicity conditions for semi-standard tableaux:

$$c_{p,q} \le c_{p,q+1}, \qquad c_{p,q} < c_{p+1,q}.$$

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Example

$K = \{1, 3, 4\} \Rightarrow \mathbf{k}$-tableaux are :

$$\begin{pmatrix} 1 & 1 & 1 \\ & 3 & 3 \\ & & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ & 3 & 3 \\ & & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 2 \\ & 3 & 3 \\ & & 4 \end{pmatrix}.$$

For a giving $\mathbf{a} = (a_{i,j}) \in \mathcal{B}_n$, let $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k} \in \mathcal{M}_n^{\times}}$ be a collection of integers defined by

$$M_{\mathbf{k}}(\mathbf{a}) := -\sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} a_{i,k_j} + \min \left\{ \sum_{1 \le p < q \le l} a_{c_{p,q}, c_{p,q} + (q-p)} \;\middle|\; \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } \mathbf{k}\text{-tableau.} \end{array} \right\}$$

and denote the map $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ by Ψ_n.

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Proposition (Bernstein-Fomin-Zelevinsky)
For any $\mathbf{a} \in \mathcal{B}_n$, $\Psi_n(\mathbf{a}) = \mathbf{M}(\mathbf{a})$ *is an* e*-BZ datum. Moreover* $\Psi_n : \mathcal{B}_n \to \mathcal{BZ}_n^e$ *is a bijection.*

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Moreover we can prove

Theorem
The map $\Psi_n : \mathcal{B}_n \xrightarrow{\sim} \mathcal{BZ}_n^e$ is an isomorphism of $*$-crystals.

∘ On the realization by irreducible Lagrangians (Geometric side)

$\bigsqcup_{\nu} \mathrm{Irr}\,\Lambda(\nu)$: Irred. Lagrangians

↙↗ ↖↘

$\mathcal{BZ}_n^e$: e-BZ data ⟷ $\mathcal{B}_n$: Lusztig data (“limits” of SST)

◦ On the realization by irreducible Lagrangians (Geometric side)

$\bigsqcup_\nu \mathrm{Irr}\ \Lambda(\nu)$: Irred. Lagrangians

(b) ↙↗ ↖↘ (a)

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$\mathcal{BZ}_n^e$: e-BZ data $\longleftrightarrow$ $\mathcal{B}_n$: Lusztig data (“limits” of SST)

(a) Lusztig data $\leftrightarrow$ Irred. Lagrangians : well-known.

(b) BZ data $\leftrightarrow$ Irred. Lagrangians : Recently studied.

- Geiss-Leclerc-Schröer
- Kamnitzer et. al.
- S

§ Toward BZ data in affine type A : a combinatorial approach

Aim : Consider an affine analogue of

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$$\swarrow\!\!\nearrow \qquad\qquad \nwarrow\!\!\searrow$$

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Today : $\mathcal{BZ}_n^e$: e-BZ data $\longleftrightarrow$ $\mathcal{B}_n$: "limits" of SST.

This part can be generalized in combinatorial way.

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Today : $\boxed{\mathcal{BZ}_n^e : e\text{-BZ data} \longleftrightarrow \mathcal{B}_n : \text{"limits" of SST.}}$

This part can be generalized in combinatorial way.

Idea :

(1) Construct BZ data for "$\mathfrak{gl}(\infty)$".
(Replace $\{1, \cdots, n\}$ to $\mathbb{Z}$.)

(2) Reduction modulo $l \Rightarrow$ BZ data of type $A_{l-1}^{(1)}$.

∘ Index set of BZ data for "$\mathfrak{g}l(\infty)$"

$K \subset \mathbb{Z}$: a *Maya diagram of charge* i,
i.e. $\mathbf{k} = \{k_j \mid j \in \mathbb{Z}_{\le i}\}$ is a sequence of integers indexed by $\mathbb{Z}_{\le i}$ such that

$$k_{j-1} < k_j \ (j \le i), \quad k_j = j \ (j \ll i).$$

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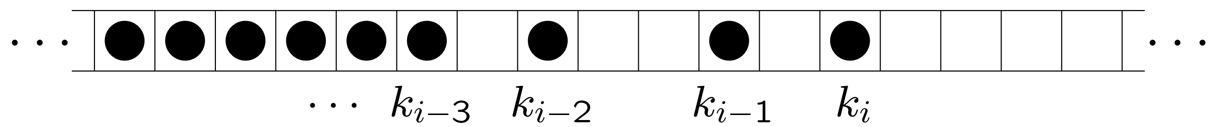
The following description of a Maya diagram is very useful:

$$\mathbf{k} = \{\cdots < k_{i-3} < k_{i-2} < k_{i-1} < k_i\}$$

$\longleftrightarrow$



○ **Index set of BZ data for "$\mathfrak{gl}(\infty)$"**

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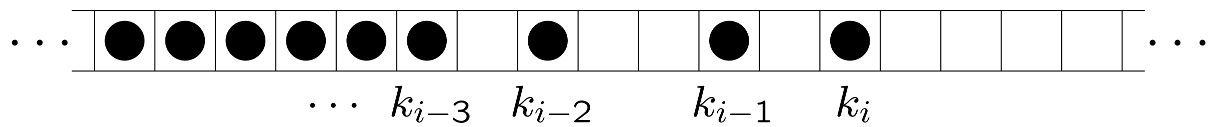

$\mathcal{M}_{\mathbb{Z}}^{(i)}$: the set of all Maya diagrams of charge i

$\mathcal{M}_{\mathbb{Z}} := \bigcup_{i \in \mathbb{Z}} \mathcal{M}_{\mathbb{Z}}^{(i)}$.

$\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$: a collection of integers indexed by $\mathcal{M}_{\mathbb{Z}}$.

∘ Restriction to subintervals

$I = \{m+1, m+2, \cdots, m+n\}$: a (finite) subinterval of $\mathbb{Z}$ ($m \in \mathbb{Z}$),
$\widetilde{I} := I \cup \{m+n+1\}$,
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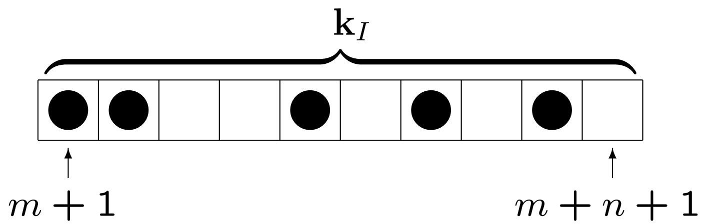

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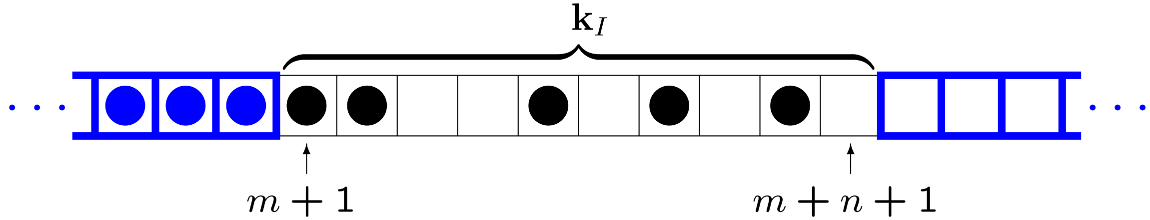

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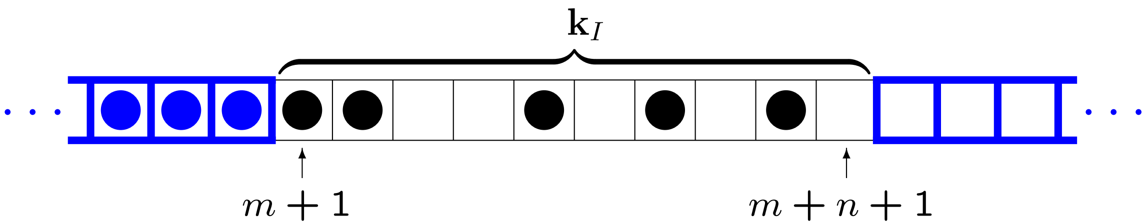

Conversely, for an element $\mathbf{k} = \left(\mathbb{Z}_{\le m} \cup \mathbf{k}_I\right) \in \mathcal{M}_\mathbb{Z}$ ($\mathbf{k}_I \in \mathcal{M}_I^\times$), we define

$$\mathrm{res}_I : \mathbf{k} \mapsto \mathbf{k}_I \quad \text{(remove blue parts).}$$

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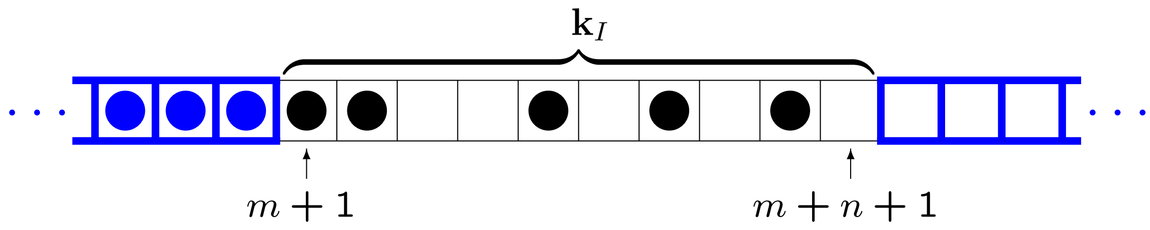

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For a collection of integers $\mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$, we define a new collection $\mathbf{M}_I = \left((M_I)_{\mathbf{k}_I}\right)_{\mathbf{k}_I \in \mathcal{M}_I^{\times}}$ by

$$(M_I)_{\mathbf{k}_I} := M_{\mathbb{Z}_{\leq m} \cup \mathbf{k}_I}.$$

○ BZ data for "$\mathfrak{gl}(\infty)$"

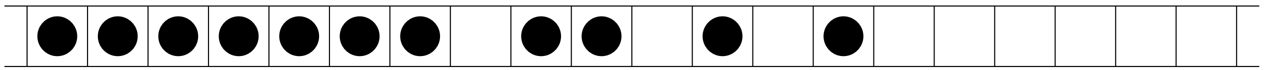
Definition (Naito-Sagaki-(S))
A collection of integers $\mathrm{M} = (M_{\mathbf{k}})_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ is called a BZ datum of type A_∞, if it satisfies the following conditions:
(a) *For any finite interval K, M_K is a e-BZ datum associated to K.*
(b) *For each Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, there exist an interval I such that*
 (i) $\mathbf{k} = \left(\mathbb{Z}_{\le m} \cup \mathbf{k}_I\right)$ *with* $\mathbf{k}_I \in \mathcal{M}_I^{\times}$,
 (ii) *for any finite interval $J \supset I$,*

$$(\mathrm{M}_J)_{\widetilde{J}\setminus res_J(\mathbf{k})} = (\mathrm{M}_I)_{\widetilde{I}\setminus res_I(\mathbf{k})}.$$

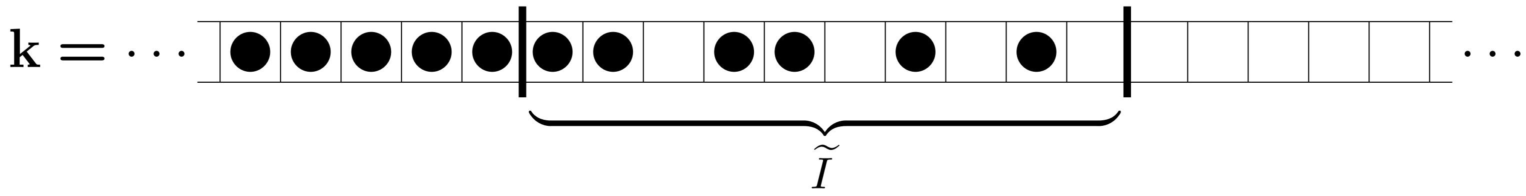
Let us denote by $\mathcal{BZ}_{\mathbb{Z}}$ the set of all BZ data of type A_∞.

Remark
(1) $J = \{p+1, p+q\} \supset I \;\Rightarrow \mathbf{k} = \left(\mathbb{Z}_{\le p} \cup \mathbf{k}_J\right)$ with $\mathbf{k}_J \in \mathcal{M}_J^{\times}$ and $\mathbf{k}_J \supset \mathbf{k}_I$.
(2) By definition we have $(\mathrm{M}_I)_{\widetilde{I}\setminus \mathrm{res}_I(\mathbf{k})} = M_{\mathbb{Z}_{\le m}\cup\left(\widetilde{I}\setminus \mathrm{res}_I(\mathbf{k})\right)}$.

Remark (1)

$\mathbf{k} = \cdots$  $\cdots$

Remark (1)



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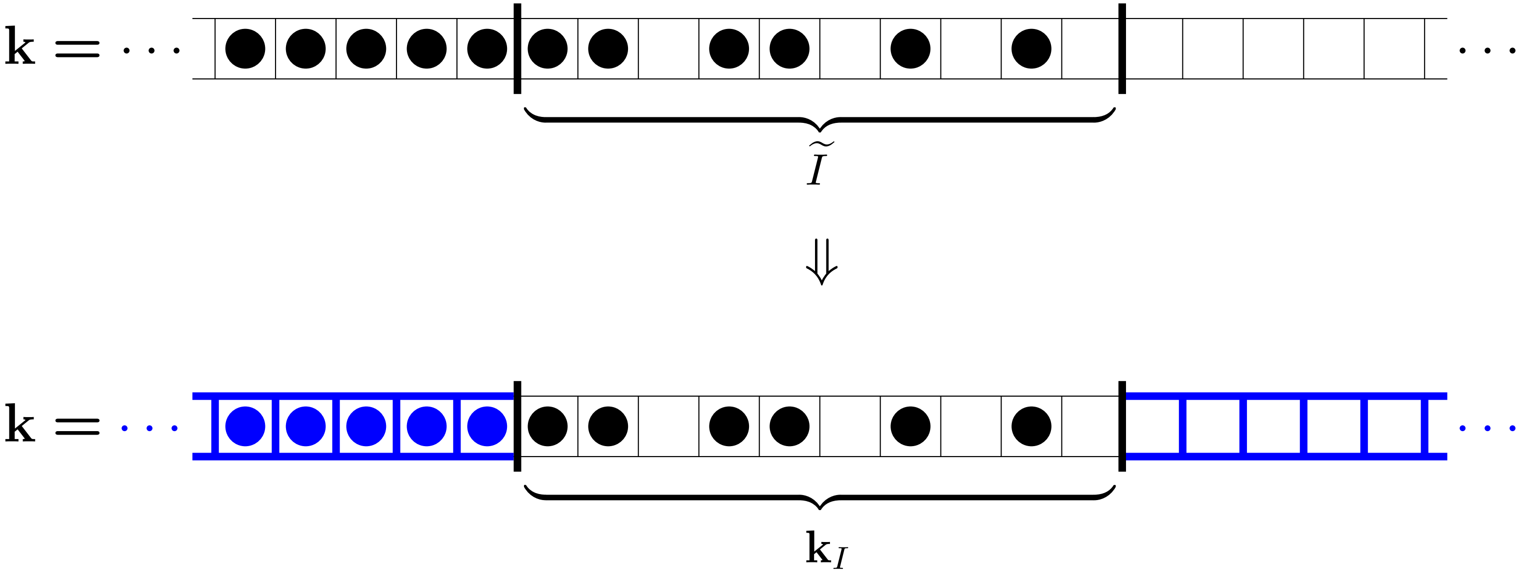

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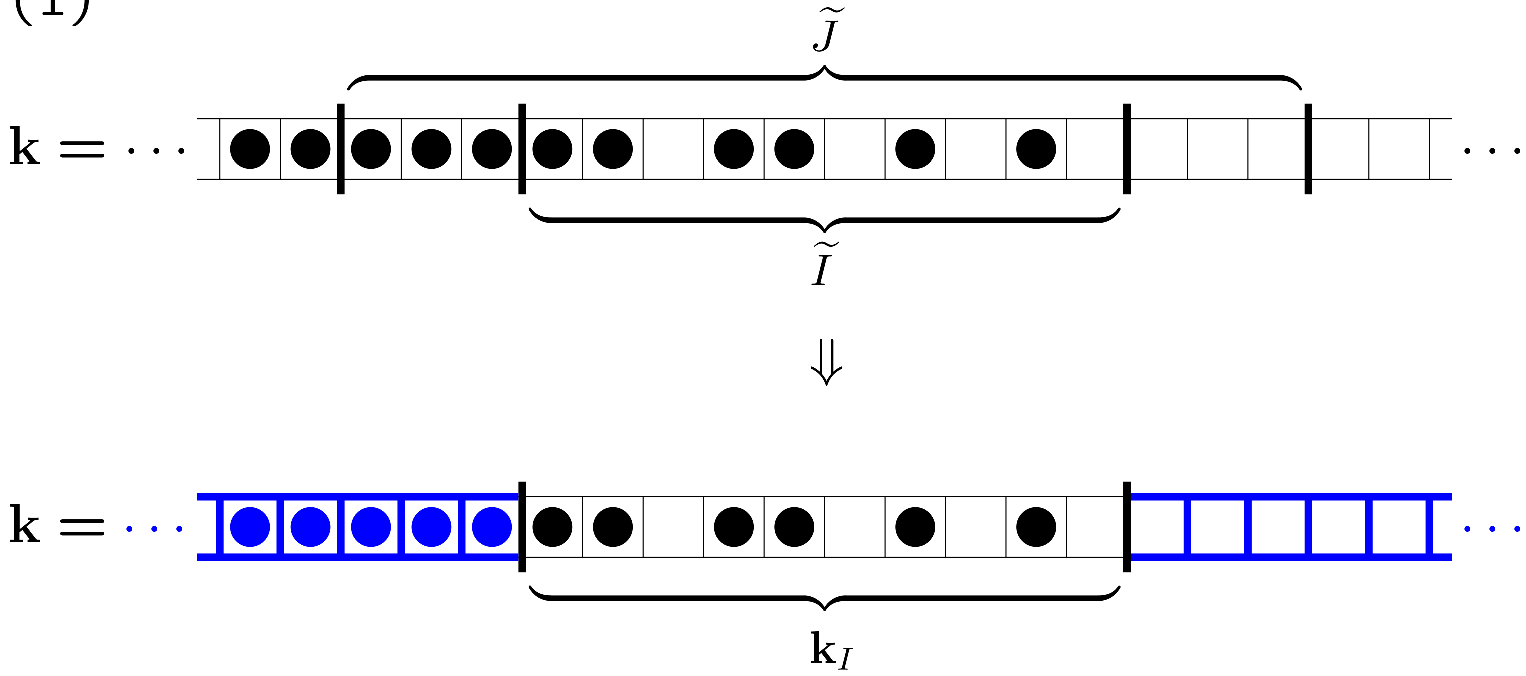
$\mathbf{k} = \cdots$
$\cdots$
$\widetilde{J}$
$\widetilde{I}$
$\Downarrow$
$\mathbf{k} = \cdots$
$\cdots$
$\mathbf{k}_I$

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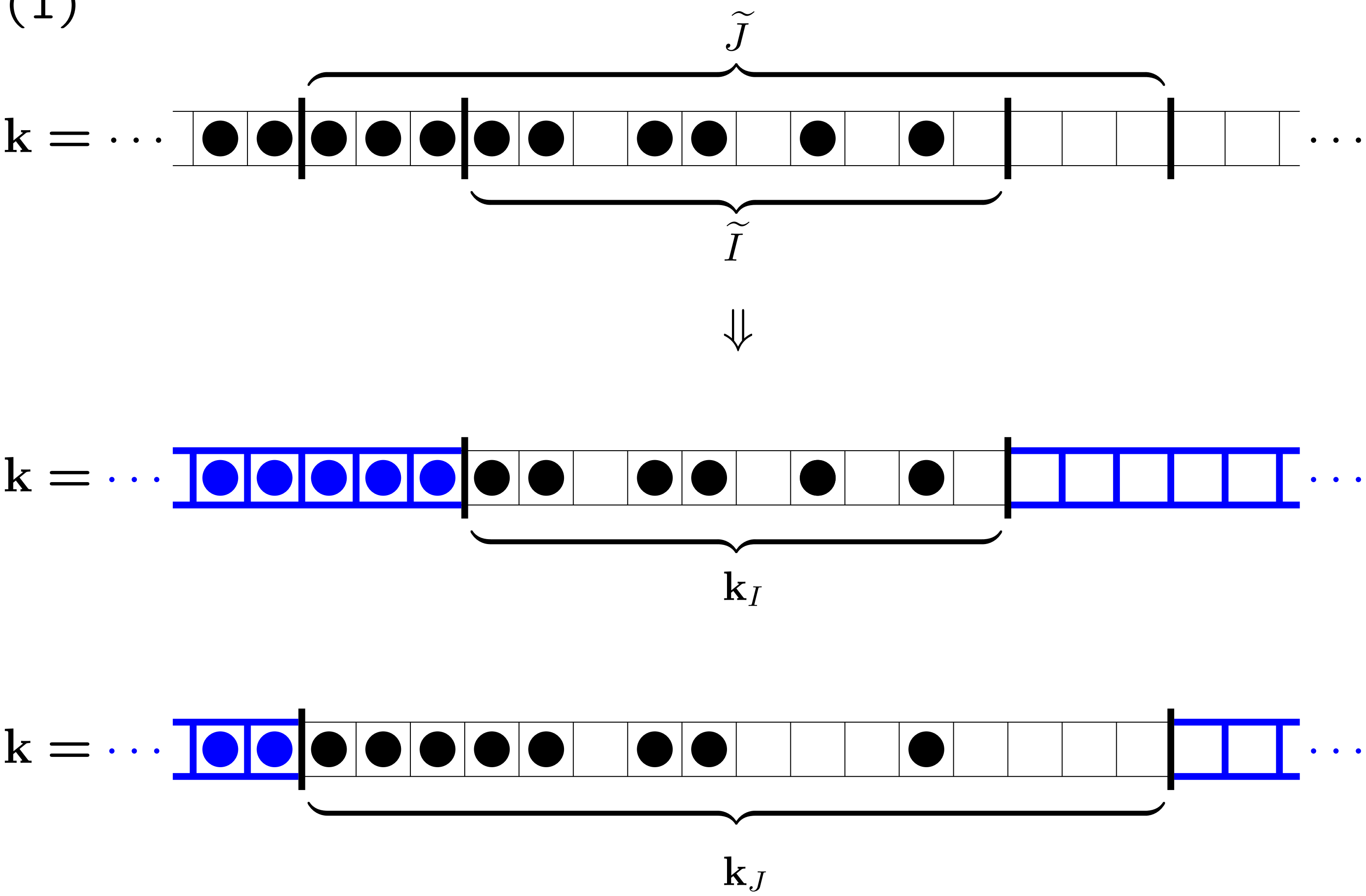

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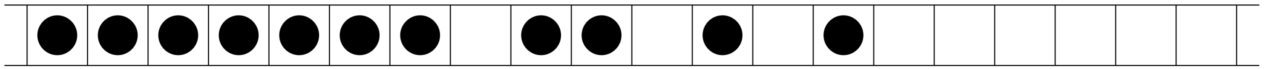
$\mathbf{k} = \cdots$
$\cdots$
$\mathbf{k}_I$
$\mathbf{k} = \cdots$
$\cdots$
$\mathbf{k}_J$

Remark (2)

$\mathbf{k} = \cdots$  $\cdots$

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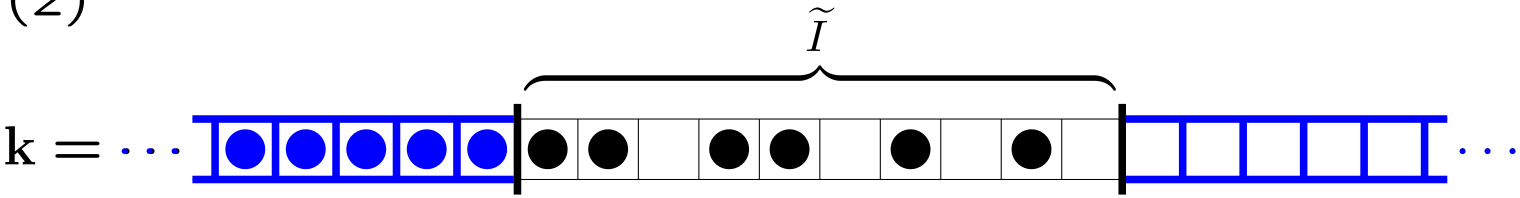

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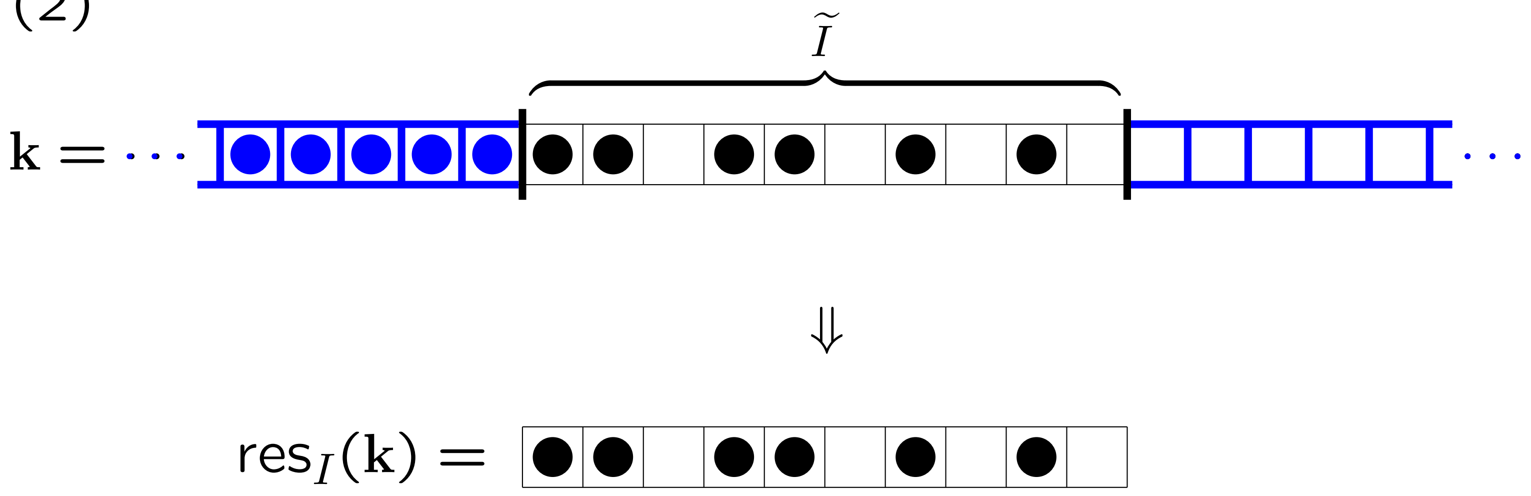
$\widetilde{\widetilde{I}}$
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$\cdots$
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$\mathrm{res}_I(\mathbf{k}) =$

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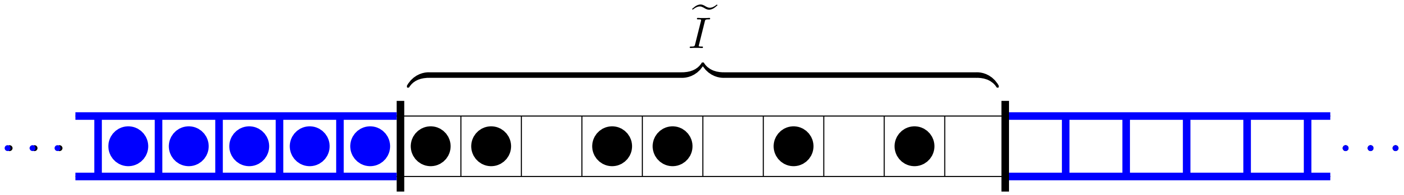
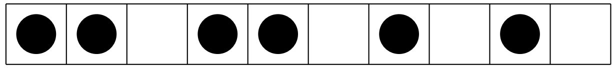
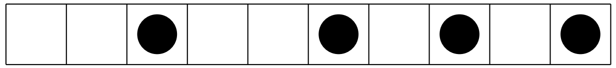
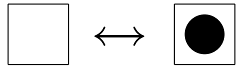
$\mathbf{k} =$ 

$$\Downarrow$$

$\mathrm{res}_I(\mathbf{k}) =$ 

$\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k}) =$ 

Replace 

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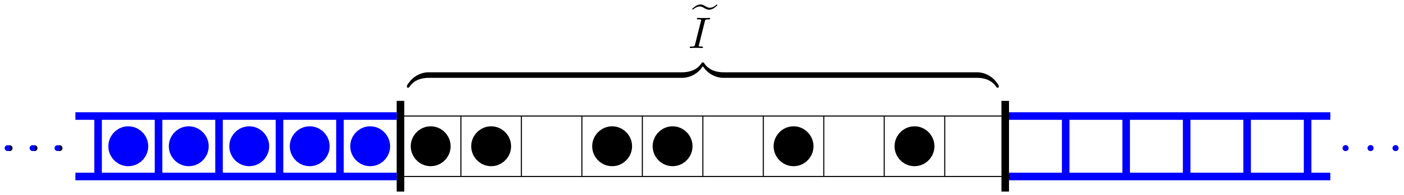
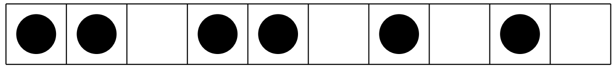
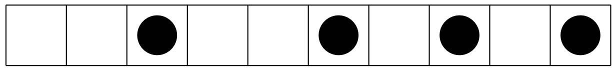
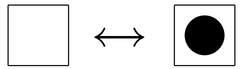
$\mathbf{k} =$ 

$\Downarrow$

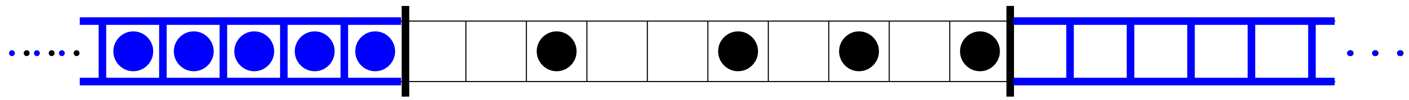
$\mathrm{res}_I(\mathbf{k}) =$ 

$\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k}) =$ 

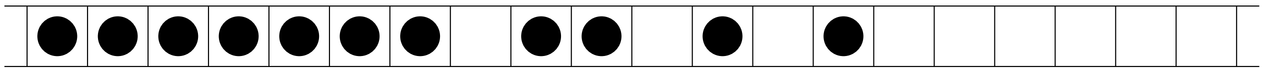
Replace 

$\Downarrow$

$\mathbb{Z}_{\leq m} \cup \left(\widetilde{I} \setminus \mathrm{res}_I(\mathbf{k})\right)$

$=$ 

Remark (2)

$\mathbf{k} = \cdots$  $\cdots$

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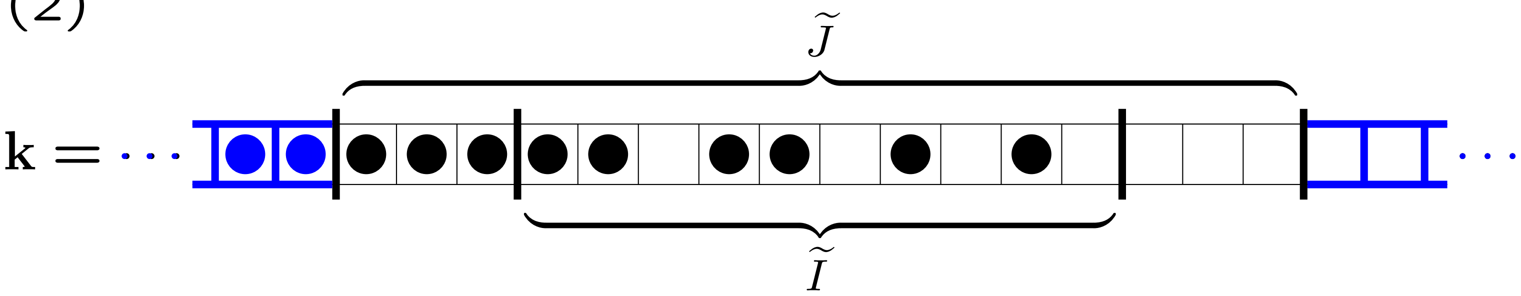

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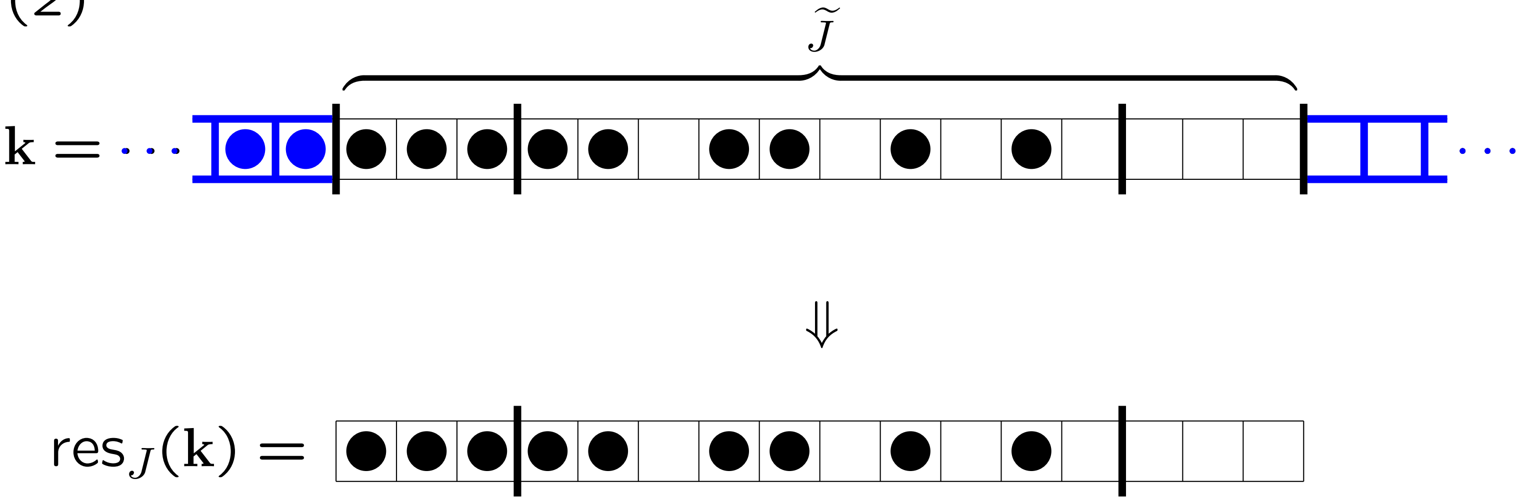

$\widetilde{J}$
$\mathbf{k} = \cdots$
$\cdots$
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$\mathrm{res}_J(\mathbf{k}) =$

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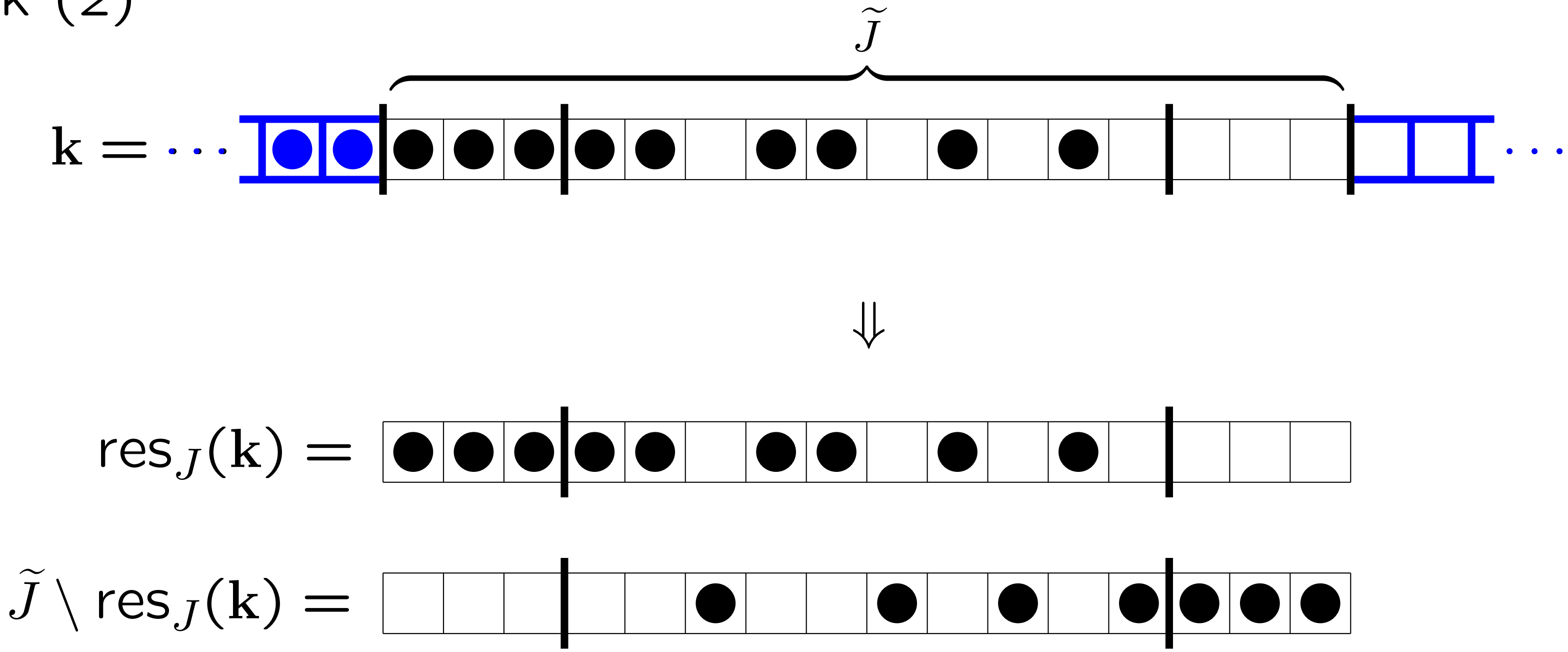
$\widetilde{J}$
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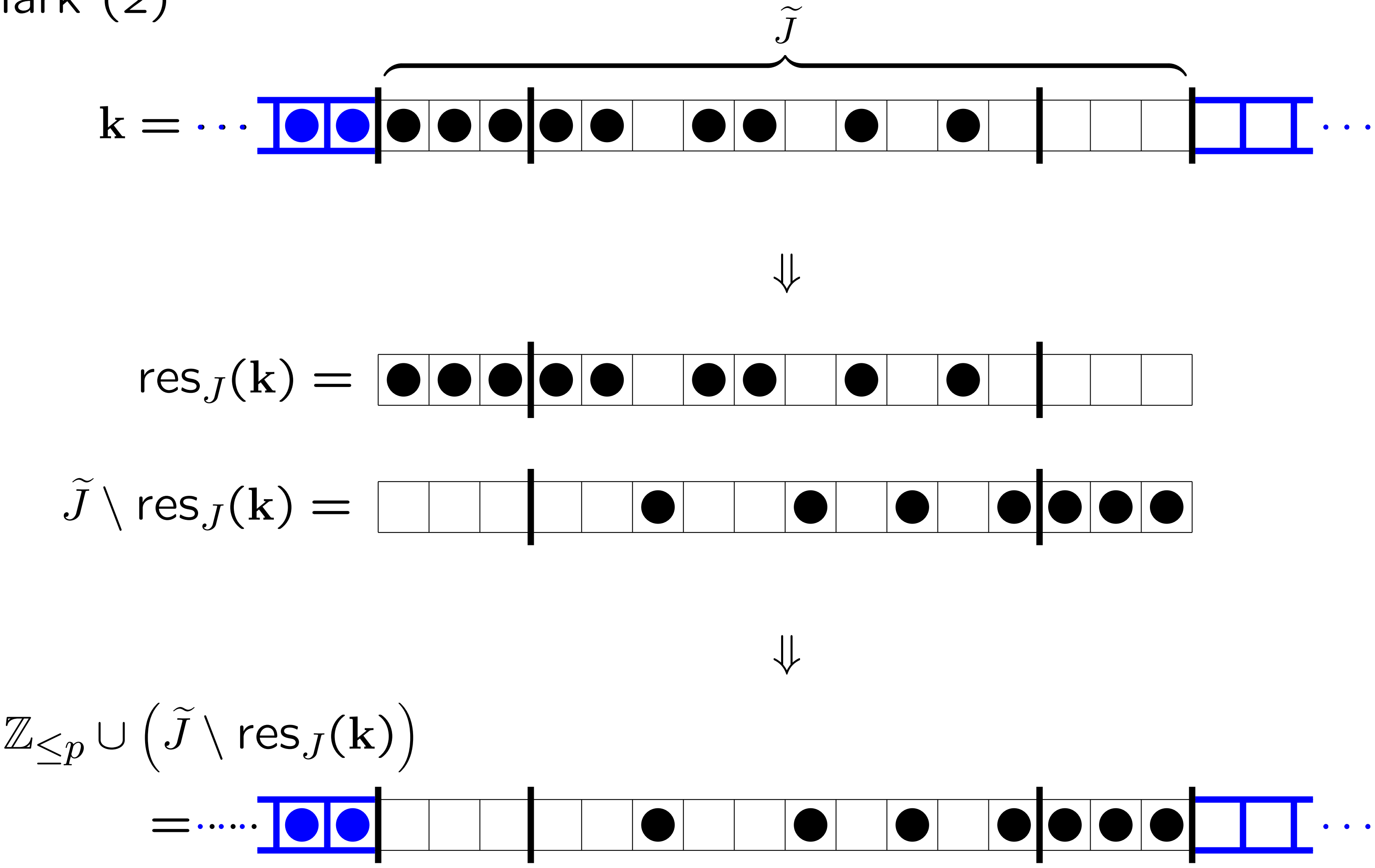
$\cdots$
$\Downarrow$
$\mathrm{res}_J(\mathbf{k}) =$
$\widetilde{J} \setminus \mathrm{res}_J(\mathbf{k}) =$

Remark (2)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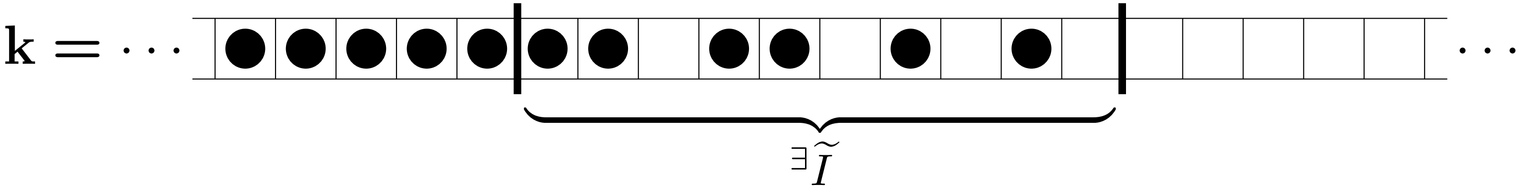
$= \cdots$

Condition (b)-(ii) in the definition



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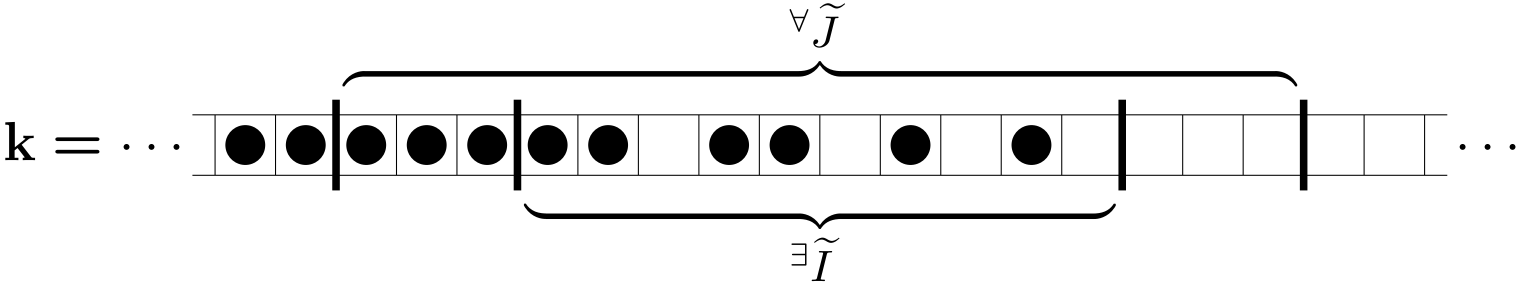

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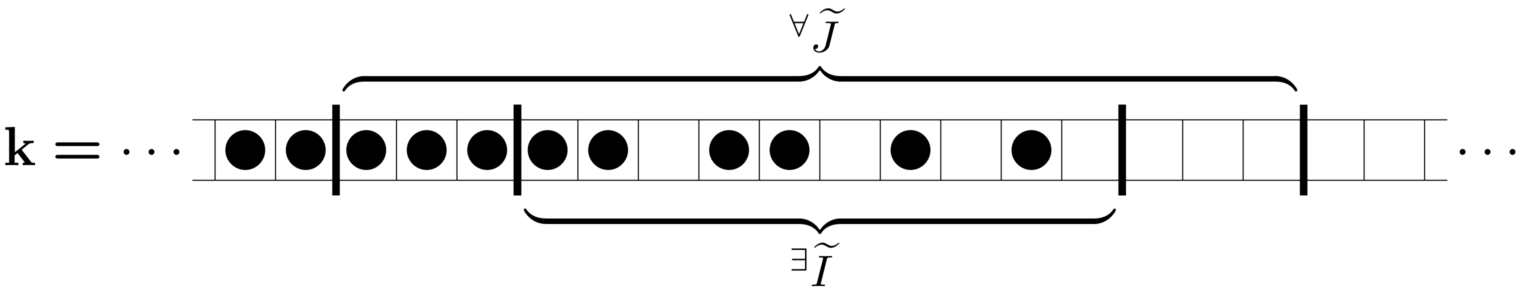
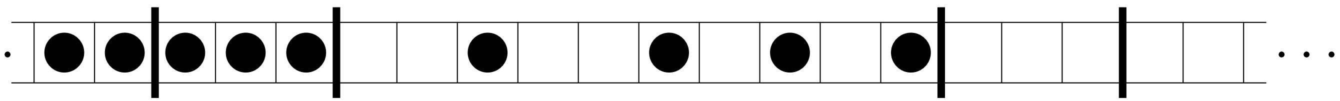


$$\Downarrow$$

$$\mathbb{Z}_{\leq m} \cup \left(\widetilde{I} \setminus \mathsf{res}_I(\mathbf{k})\right) = \cdots$$



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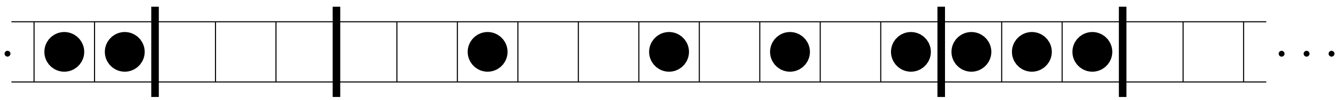
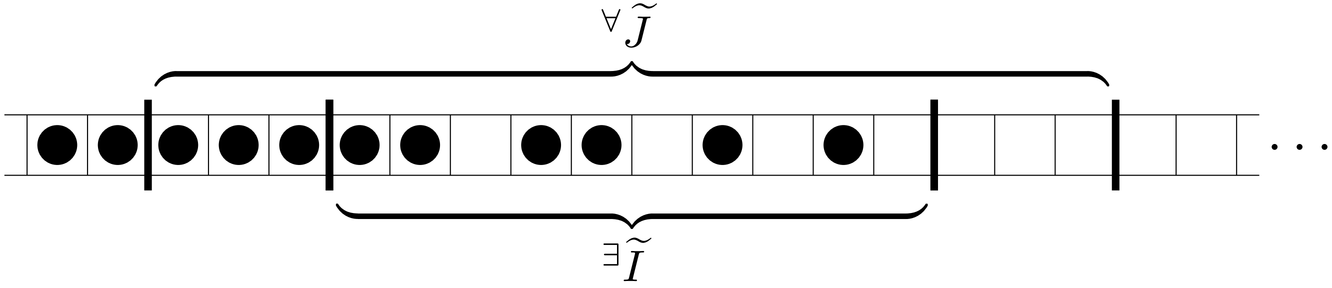


Condition (b)-(ii) in the definition

$\mathbf{k} = \cdots$



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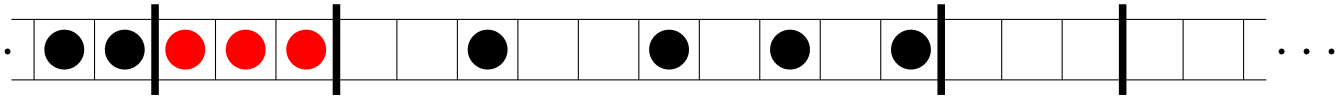

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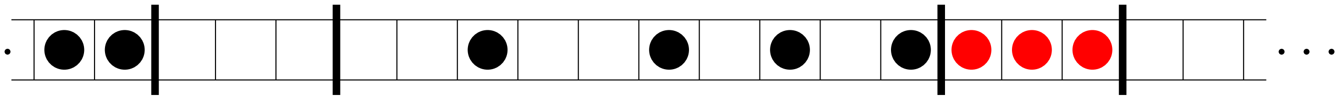

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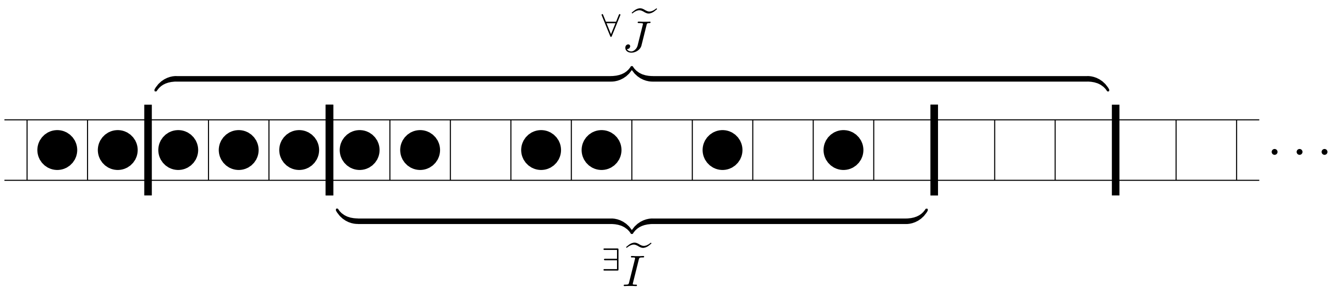

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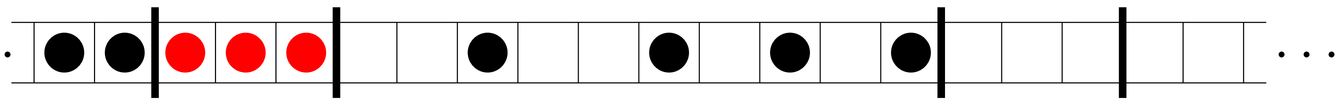

The corresponding components of a BZ datum coincide!

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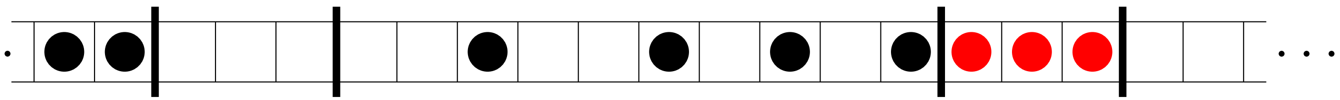

∘ BZ data of type $A_{l-1}^{(1)}$

$$\sigma : \mathbb{Z} \to \mathbb{Z} \quad \text{defined by} \quad i \mapsto i + l.$$

$$\mathcal{BZ}_{\mathbb{Z}}^{\sigma} := \left\{ \mathbf{M} = (M_{\mathbf{k}})_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}} \in \mathcal{BZ}_{\mathbb{Z}} \;\middle|\; M_{\sigma(\mathbf{k})} = M_{\mathbf{k}} \text{ for any } \mathbf{k} \in \mathcal{M}_{\mathbb{Z}} \right\}$$

(The set of all BZ data of type $A_{l-1}^{(1)}$).

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(The set of all BZ data of type $A_{l-1}^{(1)}$).

• There exist an BZ datum whose $\mathbf{k}$-component is zero for any $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$. We denote it $\mathbf{O}$.

Theorem (Naito-Sagaki-(S))

(1) $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ *has a crystal structure which is induced from one of the set of all* e*-BZ data for a finite interval.*

(2) *Let* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ *be the connected component of (the crystal graph of) the crystal* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ *containing* $\mathbf{O}$. *Then* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$ *is isomorphic to* $B(\infty)$.

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Remark

(1) The total crystal structure of $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$ is not known.

(2) Recall the condition (b) in the definition of $\mathcal{BZ}_{\mathbb{Z}}$:

For each Maya diagram $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$, there exist an interval I such that

(i) $\mathbf{k} = \left(\mathbb{Z}_{\leq m} \cup \mathbf{k}_I\right)$ with $\mathbf{k}_I \in \mathcal{M}_I^{\times}$,

(ii) for any finite interval $J \supset I$,

$$(\mathbf{M}_J)_{\widetilde{J}\setminus \mathrm{res}_J(\mathbf{k})} = (\mathbf{M}_I)_{\widetilde{I}\setminus \mathrm{res}_I(\mathbf{k})}.$$

To define the weight of a BZ datum, we need this condition.

∘ Lusztig data of type $A_{l-1}^{(1)}$

Recall

$$\mathbf{a} = (a_{i,j})_{1 \leq i < j \leq n+1} \in \mathcal{B}_n \cong \text{``}SST(\infty)\text{''} \quad \text{(Lusztig datum)}.$$

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Definition

(1) *Let* $\Delta^+ = \{(i,j) \mid i,j \in \mathbb{Z} \text{ with } i < j\}$*. A collection of non-negative integers* $\mathbf{a} = (a_{i,j})_{(i,j)\in\Delta^+}$ *is called a Lusztig datum of type “*$\mathfrak{gl}(\infty)$*” if there exist* $N > 0$ *such that*

$$a_{i,j} = 0 \quad \text{for} \quad j - i \geq N.$$

We denote by $\mathcal{B}_{\mathbb{Z}}$ *the set of all Lusztig data of type “*$\mathfrak{gl}(\infty)$*”.*

$\circ$ **Lusztig data of type $A^{(1)}_{l-1}$**

Recall

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We denote by $\mathcal{B}_{\mathbb{Z}}$ *the set of all Lusztig data of type “$\mathfrak{gl}(\infty)$”.*

(2) $$\mathcal{B}^{(1)}_{l-1} := \left\{ \mathbf{a} = (a_{i,j}) \in \mathcal{B}_{\mathbb{Z}} \;\middle|\; a_{i,j} = a_{i+l,j+l} \text{ for any } (i,j) \in \Delta^+ \right\}.$$

An element of $\mathcal{B}^{(1)}_{l-1}$ *will be called a Lusztig datum of type* $A^{(1)}_{l-1}$.

○ $\mathcal{B}_{l-1}^{(1)}$ v.s. multisegments in the LLTA theory

• A *segment* of length r is a sequence of r consecutive values in $\mathbb{Z}/l\mathbb{Z}$

x_1	x_2	$\cdots$	x_r

where $x_p = i + p - 1$ $(1 \leq p \leq r)$ for some $i \in \mathbb{Z}/l\mathbb{Z}$.

• A *multisegment* is a multiset of segments.

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- A *multisegment* is a multiset of segments.

- Then we have a bijection

$$\mathcal{B}^{(1)}_{l-1} \stackrel{\sim}{\longleftrightarrow} \text{the set of all multisegments}$$

via

$$\Delta^+ \ni (i,j) \quad \mapsto \quad \begin{array}{c}\text{the segment of length } r = j - i \\ \text{with } x_1 = i\end{array}.$$

Note that

$$a_{i,j} = \text{the multiplicity of the corresponding segment.}$$

Known facts.

(1) $\mathcal{B}^{(1)}_{l-1}$ *has a crystal structure of type* $A^{(1)}_{l-1}$ *which is a natural generalization of one of* $\mathcal{B}_n$*.*
(2) *The set of all multisegment also has a crystal structure of type* $A^{(1)}_{l-1}$*. Moreover, under the above identification, it coincides with one of* $\mathcal{B}^{(1)}_{l-1}$*.*

Known facts.

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• A Lusztig datum $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$ is called *aperiodic* if it satisfies the following condition:

for any $(i,j) \in \Delta^+$, there exist an element which is equal to 0 in the set

$$\{a_{i,j}, a_{i+1,j+1}, \cdots, a_{i+l-1,j+l-1}\}.$$

We denote by $\left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$ the set of all aperiodic Lusztig datum.

Let $\mathbf{0}$ be the Lusztig datum whose coefficient is equal to 0 for any $(i, j) \in \Delta^+$.

Known facts.

(3) $\left(\mathcal{B}^{(1)}_{l-1}\right)^{ap}$ *coincides with the connected component of the crystal* $\mathcal{B}^{(1)}_{l-1}$ *containing* $\mathbf{0}$*. In other words, "aperiodicity" characterizes that component.*

(4) $\left(\mathcal{B}^{(1)}_{l-1}\right)^{ap}$ *is isomorphic to* $B(\infty)$*.*

◦ **An explicit correspondence between** $\left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$ **and** $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}$

Let us return to finite interval cases.
$\mathcal{B}_I$: the set of all Lusztig data associated to a finite interval I
$\mathcal{BZ}_I^e$: the set of all e-BZ data associated to I

$$\Psi_I : \mathcal{B}_I \xrightarrow{\sim} \mathcal{BZ}_I^e, \quad \mathbf{a} \mapsto \mathbf{M}(\mathbf{a}) \quad \text{(isom. of crystals).}$$

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How to define $\mathbf{M}(\mathbf{a})$ for $\mathbf{a} \in \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$?

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How to define $\mathbf{M}(\mathbf{a})$ for $\mathbf{a} \in \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$?

Idea :

Consider $\mathrm{Res}_I : \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap} \to \mathcal{B}_I$ by

$$\left(\mathcal{B}_{l-1}^{(1)}\right)^{ap} \ni \mathbf{a} = (a_{i,j})_{\substack{i<j \\ i,j\in\mathbb{Z}}} \mapsto (a_{i,j})_{\substack{i<j \\ i,j\in\widetilde{I}}} \in \mathcal{B}_I.$$

Define $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k}\in\mathcal{M}_{\mathbb{Z}}}$ for $\mathbf{a} \in \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$ by

$$M_{\mathbf{k}}(\mathbf{a}) := \varprojlim_{I} M_{\mathrm{res}_I(\mathbf{k})}(\mathrm{Res}_I(\mathbf{a}))$$

Lemma

For a given $\mathbf{a} \in \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$ *and* $\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}$*, there exist an interval* I_0 *such that, for any* $I \supset I_0$,

$$M_{res_{I}(\mathbf{k})}(Res_{I}(\mathbf{a})) = M_{res_{I_0}(\mathbf{k})}(Res_{I_0}(\mathbf{a})).$$

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$\Rightarrow$ We can define $M_{\mathbf{k}}(\mathbf{a}) := M_{\mathrm{res}_{I_0}(\mathbf{k})}(\mathrm{Res}_{I_0}(\mathbf{a}))$.

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Theorem

Let $\mathbf{a} \in \left(\mathcal{B}_{l-1}^{(1)}\right)^{ap}$*. Then the collection of integers* $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))_{\mathbf{k} \in \mathcal{M}_{\mathbb{Z}}}$ *is an element of* $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$*. Moreover, the map* $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ *gives an isomorphism of crystals.*

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Remark
(1) We can define $\mathbf{M}(\mathbf{a}) = (M_{\mathbf{k}}(\mathbf{a}))$ for $\mathbf{a} \in \mathcal{B}_{\mathbb{Z}}$ by similar way.
(2) For $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$, the map $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ is *not* injective.
(3) For a general $\mathbf{a} \in \mathcal{B}_{l-1}^{(1)}$, $\mathbf{M}(\mathbf{a})$ is *not* a BZ datum of type $A_{l-1}^{(1)}$.

§ Conclusions

Affine A case :

$$\begin{array}{ccc} & \bigsqcup_\nu \mathrm{Irr}\,\Lambda(\nu) \text{ : Irred. Lagrangians} & \\ \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \text{ (a) known} \\ ??? & \longleftrightarrow & \mathcal{B}^{(1)}_{l-1}. \end{array}$$

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$\bigsqcup_\nu \mathrm{Irr}\ \Lambda(\nu)$: Irred. Lagrangians

(a) known

Today :

??? $\longleftrightarrow$ $\mathcal{B}^{(1)}_{l-1}$.

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§ Conclusions

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- We gave an answer for ???. $\Rightarrow$ It is $\mathcal{BZ}_{\mathbb{Z}}^{\sigma}(\mathbf{O})$.

§ Conclusions

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Today : $\mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O}) \longleftrightarrow \mathcal{B}^{(1)}_{l-1}$. (c)

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- We gave an answer for ???. ⇒ It is $\mathcal{BZ}^{\sigma}_{\mathbb{Z}}(\mathbf{O})$.
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- The correspondence (c) : described in explicit way.

- The correspondence (b) : work in progress.

Thank you!