Mirković-Vilonen polytopes and quiver construction of crystal basis in type A

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§ Introduction

Mirković-Vilonen (1997∼)

- MV cycles : Algebraic cycles in affine Grassmannian

Kamnitzer (2005∼)

- Moment map image ⇒ MV polytopes : Polytopes in $\mathfrak{h}_{\mathbb{R}}(R^{\vee})$
- $\mathcal{MV}$: the set of all MV polytopes has a crystal structure, and $\mathcal{MV} \cong B(\infty)$.
- Prove the Anderson-Mirković conjecture in type A.

  (It tells us the explicit action of $\widetilde{f}_i$ on $\mathcal{MV}$.)

That is, the Kamnitzer's result tells us a realization of $B(\infty)$ in terms of MV polytopes.

Today :

In type A, compare the Kamnitzer's realization with

- a realization of $B(\infty)$ in terms of Young tableaux,
- a realization of $B(\infty)$ in terms of irreducible Lagrangians

and, as an application, we will give

- a new proof of AM conjecture.

§ Notations

$U_q = U_q(\mathfrak{s}l_{n+1}) = \langle e_i,\ f_i,\ t_i^{\pm 1}\ (i \in I)\rangle$.

$B(\infty)$: the crystal basis of U_q^-

$* : U_q \to U_q$: a $\mathbb{Q}(q)$-algebra anti-automorphism

$$e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad t_i^{\pm} \mapsto t_i^{\mp}.$$

$\Rightarrow * : B(\infty) \to B(\infty)$.

Set

$$\varepsilon_i^*(b) := \varepsilon_i(b^*), \qquad \varphi_i^*(b) := \varphi_i(b^*),$$
$$\widetilde{e}_i^* := * \circ \widetilde{e}_i \circ *, \qquad \widetilde{f}_i^* := * \circ \widetilde{f}_i \circ *.$$

$\Rightarrow$ The set $B(\infty)$ endowed with maps wt, $\varepsilon_i^*,\ \varphi_i^*,\ \widetilde{e}_i^*,\ \widetilde{f}_i^*$ is a crystal.

("*the $*$-crystal structure*" on $B(\infty)$)

That is, $B(\infty)$ has two crystal structures :

$$(B(\infty)\ ;\ \mathrm{wt},\ \varepsilon_i,\ \varphi_i,\ \widetilde{e}_i,\ \widetilde{f}_i),$$
$$(B(\infty)^* = B(\infty)\ ;\ \mathrm{wt},\ \varepsilon_i^*,\ \varphi_i^*,\ \widetilde{e}_i^*,\ \widetilde{f}_i^*).$$

§ Realization I : Young tableaux

$\lambda \in P_+$: dominant integral weight

$V(\lambda)$: irreducible U_q-module with h.w. λ

$B(\lambda)$: crystal basis of $V(\lambda)$

Theorem (Kashiwara-Nakashima).

$$B(\lambda) \cong SST(\lambda).$$

Here $SST(\lambda)$ is the set of semistandard Young tableaux of shape λ.

- Take $\lambda \to \infty$ (w.r.t. $\lambda \geq \mu \Leftrightarrow \lambda - \mu \in Q_+$)

$$\begin{array}{ccc} B(\lambda) & \cong & SST(\lambda) \\ \downarrow & & \downarrow \\ B(\infty) & \cong & \mathcal{B} \end{array}$$

$\mathcal{B}$: The set of all $n(n+1)/2$ tuples of non-negative integers
$\mathbf{a} = (a_{i,j})_{1 \leq i < j \leq n+1}$

We regard $\mathcal{B}$ as "$SST(\infty)$" via
$a_{i,j}$="the number of $\boxed{j}$ in the i-th row of a tableau"

Remark .

(1) The explicit crystal structure of $\mathcal{B}$ can be determined.

(2) Since $B(\infty)$ has the $*$-crystal structure, $\mathcal{B}$ also has the induced $*$-crystal structure.

(We omit to give them.)

§ Realization II : Lagrangian construction

(I, H) : (double) quiver of type A_n

$(I = \{1, \cdots, n\}$: set of vertices, H : set of arrows)

$\Omega \subset H$: an orientation $\Rightarrow$ (I, Ω) : a quiver of type A_n.

$V = \oplus_{i \in I} V_i$: finite dimensional I-graded complex vector space

$$\begin{aligned} E_{V,\Omega} &:= \bigoplus_{\tau \in \Omega} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)}), \\ X_{V,\Omega} &:= \bigoplus_{\tau \in H} \mathrm{Hom}_{\mathbb{C}}(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)}) \\ &= E_{V,\Omega} \oplus E_{V,\overline{\Omega}} \\ &\cong T^* E_{V,\Omega}. \end{aligned}$$

$G_V := \prod_{i \in I} GL(V_i) \curvearrowright E_{V,\Omega}$ and X_V.

$\Rightarrow \mu : X_V \to (\mathrm{Lie} G_V)^* \cong \oplus_{i \in I} \mathrm{End}(V_i)$: moment map

$$\Lambda_V := \mu^{-1}(0) : \text{ a Lagrangian subvariety of } X_V.$$
$$\mathrm{Irr}\ \Lambda_V : \text{ the set of all irreducible components of } \Lambda_V.$$

Theorem (Kashiwara-S)**.**

(1) $\bigsqcup_V \mathrm{Irr}\ \Lambda_V \cong B(\infty)$.

(2) $\bigsqcup_V \mathrm{Irr}\ \Lambda_V$ *has a $*$-crystal structure induced from the map*

$$* : B \mapsto {}^t B \quad (B \in X_V),$$

and $\bigsqcup_V \mathrm{Irr}\ \Lambda_V \cong B(\infty)$ *as $*$-crystals.*

Remark . Since (I, Ω) is of type A_n, for $\Lambda \in \Lambda_V$, there is a unique G_V-orbit $\mathcal{O} \subset E_{V,\Omega}$ such that

$$\Lambda = \overline{T^*_{\mathcal{O}} E_{V,\Omega}}.$$

§ Comparison I & II

○ Preliminaries

- There is a natural one to one correspondence

$$G_V\text{-orbits in } E_{V,\Omega} \;\cong\; \begin{array}{c}\text{isomorphism classes of reps. of } (I,\Omega)\\ \text{with dimension vector} = \dim V.\end{array}$$

- There is a one to one correspondence

$$\Delta_+ \;\cong\; \begin{array}{c}\text{isomorphism classes of}\\ \text{indecomposable reps. of } (I,\Omega),\end{array}$$

where Δ_+ is the set of positive roots (Gabriel's theorem).

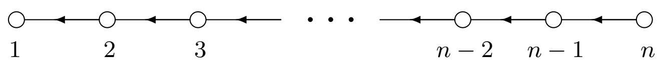
For a positive root $\alpha \in \Delta_+$, we denote by $\mathbf{e}(\alpha,\Omega)$ the corresponding indecomposable representation of (I,Ω).

- $\Delta_+ = \left\{ \alpha_{i,j} := \sum_{k=i}^{j-1} \alpha_k \;\middle|\; 1 \le i < j \le n+1 \right\}.$

○ Realization I ⇒ Realization II

Consider the following orientation:

Ω_0 :



For $\mathbf{a} = (a_{i,j})_{1 \le i < j \le n+1} \in \mathcal{B}$, set

$$\mathbf{e}(\mathbf{a}, \Omega_0) := \bigoplus_{1 \le i < j \le n+1} \mathbf{e}(\alpha_{i,j}, \Omega_0)^{\oplus a_{i,j}}.$$

We denote by $\mathcal{O}_{\mathbf{a}} \subset E_{V,\Omega_0}$ the G_V-orbit through $\mathbf{e}(\mathbf{a}, \Omega_0)$ and let

$$\Lambda_{\mathbf{a}} := \overline{T^*_{\mathcal{O}_{\mathbf{a}}} E_{V,\Omega_0}}.$$

Proposition .

The map $\mathcal{B} \to \bigsqcup_V \mathrm{Irr}\ \Lambda_V$ *defined by*

$$\mathbf{a} \mapsto \Lambda_{\mathbf{a}}$$

is an isomorphism of crystals in usual sense. Moreover this map is compatible with the $*$-crystal structures.

§ Realization III : MV polytopes

◦ Definition of MV polytopes

$K \subset [1, n+1]$: a *Maya diagram of size n*

$\mathcal{M}_n$: the set of all Maya diagrams of size n

$\mathcal{M}_n^\times := \mathcal{M}_n \setminus \{\phi, [1, n+1]\}$

$\mathbf{M} = (M_K)_{K \in \mathcal{M}_n^\times}$: a family of integers indexed by $\mathcal{M}_n^\times$

• $W = \mathfrak{S}_{n+1} \curvearrowright \mathcal{M}_n,\ \mathcal{M}_n^\times$.

We can identify $\mathcal{M}_n^\times$ with $\Gamma_n := \bigsqcup_{w \in W,\ i \in I} W\Lambda_i$ via

$$[1, i] \ \leftrightarrow \Lambda_i.$$

• For $\mathbf{M} = (M_K)_{K \in \mathcal{M}_n^\times}$, consider a polytope in $\mathfrak{h}_\mathbb{R}$

$$P(\mathbf{M}) := \{h \in \mathfrak{h}_\mathbb{R} \mid \langle h, K \rangle \geq M_K \ (\forall K \in \mathcal{M}_n^\times)\}.$$

• A polytope $P(\mathbf{M})$ is called a *pseudo-Weyl polytope* if it satisfies the following condition:

(BZ-1) for every two indices $i \neq j$ in $[1, n+1]$ and every $K \in \mathcal{M}_n$ with $K \cap \{i, j\} = \phi$,

$$M_{Ki} + M_{Kj} \leq M_{Kij} + M_K.$$

Here we denote $Kij = K \cup \{i, j\}$ *etc.*, and we set $M_\phi = M_{[1,n+1]} = 0$.

Remark .
$P(\mathbf{M})$: a pseudo-Weyl polytope

$\Rightarrow P(\mathbf{M})$ is the convex hull of $\mu_\bullet := (\mu_w)_{w\in W} \subset \mathfrak{h}_\mathbb{R}$ (GGMS datum) where

$$\mu_w := \sum_{i=1}^{n} M_{w\Lambda_i} w\alpha_i^\vee \in \mathfrak{h}_\mathbb{R} \qquad (w \in W).$$

That is, for a pseudo-Weyl polytope,

$$P(\mathbf{M}) \quad \leftrightarrow \quad \mathbf{M} = (M_K)_{K\in\mathcal{M}_n^\times}.$$

Definition .
(1) $\mathbf{M}$ *is called a BZ datum if it satisfies* (BZ-1) *and*

(BZ-2) *for every three indices* $i < j < k$ *in* $[1, n+1]$ *and every* $K \in \mathcal{M}_n$ *with* $K \cap \{i, j, k\} = \phi$,

$$M_{Kik} + M_{Kj} = \min\{M_{Kij} + M_{Kk},\ M_{Kjk} + M_{Ki}\}.$$

(2) $P(\mathbf{M})$ *is called a MV polytope if* $\mathbf{M}$ *is a BZ datum.*

That is,

$$\text{(BZ-1)} \quad \Rightarrow \quad P(\mathbf{M}) : \text{ a pseudo-Weyl polytope}$$

$$\text{(BZ-1) \& (BZ-2)} \quad \Rightarrow \quad P(\mathbf{M}) : \text{ a MV polytope}$$

◦ Crystal structure on BZ data

A BZ datum $\mathbf{M}$ is called a w_0-*BZ datum* if

$$M_{w_0\Lambda_i} = M_{[n-i+1,n+1]} = 0 \quad \text{for } 1 \leq \forall i \leq n.$$

$\mathcal{BZ}^{w_0}$: the set of all w_0-BZ data

Let us define a crystal structure on $\mathcal{BZ}^{w_0}$. For $\mathbf{M} \in \mathcal{BZ}^{w_0}$,

$$\mathrm{wt}(\mathbf{M}) := \sum_{1\leq i\leq n} M_{[1,i]}\alpha_i,$$

$$\varepsilon_i(\mathbf{M}) := -\left(M_{[1,i]} + M_{[1,i+1]\setminus\{i\}} - M_{[1,i+1]} - M_{[1,i]\setminus\{i\}}\right),$$

$$\varphi_i(\mathbf{M}) := \varepsilon_i(\mathbf{M}) + \langle h_i, \mathrm{wt}(\mathbf{M})\rangle.$$

If $\varepsilon_i(\mathbf{M}) = 0$, we set $\widetilde{e}_i\mathbf{M} = 0$.

Otherwise, there exists a unique w_0-BZ datum $\widetilde{e}_i\mathbf{M}$ s.t.
(i) $(\widetilde{e}_i\mathbf{M})_{[1,i]} = M_{[1,i]} + 1$,
(ii) $(\widetilde{e}_i\mathbf{M})_K = M_K$ for all $K \in \mathcal{M}_n^\times \setminus \mathcal{M}_n^\times(i)$.
Here $\mathcal{M}_n^\times(i) = \{K \in \mathcal{M}_n^\times \mid i \in K \text{ and } i+1 \notin K\} \subset \mathcal{M}_n^\times$.

There exists a unique a unique w_0-BZ datum $\widetilde{f}_i\mathbf{M}$ s.t.
(iii) $(\widetilde{f}_i\mathbf{M})_{[1,i]} = M_{[1,i]} - 1$,
(iv) $(\widetilde{f}_i\mathbf{M})_K = M_K$ for all $K \in \mathcal{M}_n^\times \setminus \mathcal{M}_n^\times(i)$.

Theorem (Kamniter)**.**
$\mathcal{BZ}^{w_0}$ is a crystal which is isomorphic to $B(\infty)$.

- This theorem gives us the 3-rd realization of $B(\infty)$ in terms of BZ data (or MV polytopes).

◦ AM conjecture

Anderson and Mirković conjectured the explicit form of the action of $\widetilde{f}_i$ on $\mathcal{BZ}^{w_0}$ (AM conjecture). This conjecture is proved by Kamnitzer.

Theorem (Kamniter)**.**
For each $i \in I$, we have

$$(\widetilde{f}_i\mathbf{M})_K = \begin{cases} \min\{M_K,\ M_{s_iK} + c_i(\mathbf{M})\} & (K \in \mathcal{M}_n^{\times}(i)), \\ M_K & \text{(otherwise)}. \end{cases}$$

Here $c_i(\mathbf{M}) = M_{[1,i]} - M_{[1,i+1]\setminus\{i\}} - 1.$

In the last of this talk, we will give a sketch of a new proof of this theorem.

§ From Realization I or II to Realization III

∘ e-BZ data and w_0-BZ data

To make a bridge form the 1-st or 2-nd realization of $B(\infty)$ to the 3-rd one, we introduce a notion of e-BZ data.

A BZ datum $\mathbf{M}'$ is called a *e-BZ datum* if

$$M'_{\Lambda_i} = M'_{[1,i]} = 0 \quad \text{for } 1 \le \forall i \le n.$$

$\mathcal{BZ}^e$: the set of all e-BZ data

• For $\mathbf{M} = (M_K)_{K \in \mathcal{M}_n^\times} \in \mathcal{BZ}^{w_0}$, set $\mathbf{M}^* = (M_K^*)_{K \in \mathcal{M}_n^\times}$ by

$$M_K^* := M_{K^c}$$

where $K^c := [1, n+1] \setminus K$ is the compliment of $K \in \mathcal{M}_n^\times$.

Then, it is easy to check $\mathbf{M}^* \in \mathcal{BZ}^e$ and the map

$$* : \mathcal{BZ}^{w_0} \to \mathcal{BZ}^e$$

gives a bijection. The inverse is also denoted by $*$.

• We can define a crystal structure on $\mathcal{BZ}^e$:

$$\widetilde{e}_i^* : \mathcal{BZ}^e \xrightarrow{*} \mathcal{BZ}^{w_0} \xrightarrow{\widetilde{e}_i} \mathcal{BZ}^{w_0} \xrightarrow{*} \mathcal{BZ}^e,$$

$$\widetilde{f}_i^* : \mathcal{BZ}^e \xrightarrow{*} \mathcal{BZ}^{w_0} \xrightarrow{\widetilde{f}_i} \mathcal{BZ}^{w_0} \xrightarrow{*} \mathcal{BZ}^e,$$

etc.

○ From I to III

Definition . *Let* $K = \{k_1 < k_2 < \cdots < k_l\} \in \mathcal{M}_n^\times$ *be a Maya diagram. For such* K*, we define a* K*-tableau as an upper-triangular matrix* $C = (c_{p,q})_{1\leq p\leq q\leq l}$ *with integer entries satisfying*

$$c_{p,p} = k_p \qquad (1 \leq p \leq l),$$

and the usual monotonicity conditions for semi-standard tableaux:

$$c_{p,q} \leq c_{p,q+1}, \qquad c_{p,q} < c_{p+1,q}.$$

Example .
$K = \{1, 3, 4\} \Rightarrow K$-tableaux are :

$$\begin{pmatrix} 1 & 1 & 1 \\ & 3 & 3 \\ & & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ & 3 & 3 \\ & & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 2 \\ & 3 & 3 \\ & & 4 \end{pmatrix}.$$

Recall $\mathbf{a} = (a_{i,j}) \in \mathcal{B}$: "limit" of semistandard Young tableau.

For a giving $\mathbf{a} = (a_{i,j}) \in \mathcal{B}$, let $\mathbf{M}(\mathbf{a}) = (M_K(\mathbf{a}))_{K \in \mathcal{M}_n^\times}$ be a collection of integers defined by

$$M_K(\mathbf{a}) := -\sum_{j=1}^{l} \sum_{i=1}^{k_j - 1} a_{i,k_j} + \min \left\{ \sum_{1 \le p < q \le l} a_{c_{p,q}, c_{p,q}+(q-p)} \;\middle|\; \begin{array}{c} C = (c_{p,q}) \text{ is} \\ \text{a } K\text{-tableau.} \end{array} \right\}$$

and denote the map $\mathbf{a} \mapsto \mathbf{M}(\mathbf{a})$ by Ψ.

Proposition (Bernstein-Fomin-Zelevinsky)**.**
For any $\mathbf{a} \in \mathcal{B}$, $\Psi(\mathbf{a}) = \mathbf{M}(\mathbf{a})$ *is an e-BZ datum. Moreover* $\Psi : \mathcal{B} \to \mathcal{BZ}^e$ *is a bijection.*

Moreover we can prove

Proposition .
The map $\Psi : \mathcal{B} \xrightarrow{\sim} \mathcal{BZ}^e$ *is an isomorphism of* $*$*-crystals.*

○ From II to III

Any Maya diagram $K \in \mathcal{M}_n^{\times}$ can be written as a disjoint union of intervals

$$K = [s_1+1, t_1] \sqcup [s_2+1, t_2] \sqcup \cdots \sqcup [s_l+1, t_l]$$
$$(0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_l < t_l \leq n+1).$$

$K_m = [s_m + 1, t_m]$ $(1 \leq m \leq l)$: the m-th component of K.

$\mathrm{out}(K) := \{t_m |\ 1 \leq m \leq l\} \cap [1, n]$,

$\mathrm{in}(K) := \{s_m |\ 1 \leq m \leq l\} \cap [1, n]$.

$\Omega(K)$: the orientation so that

- an element of $\mathrm{out}(K)$ is a source,
- an element if $\mathrm{in}(K)$ is a sink.

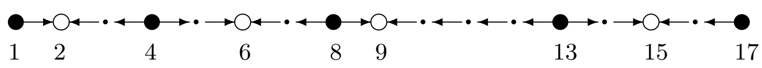
Example . Let $n = 17$ and

$$K = [3, 4] \sqcup [7, 8] \sqcup [10, 13] \sqcup [16, 17].$$

Then we have

$$\mathrm{out}(K) = \{4, 8, 13\}, \qquad \mathrm{in}(K) = \{2, 6, 9, 15\}.$$

In this case, the orientation $\Omega(K)$ is given as follows:

$\Omega(K) =$  .

Here ○ is a sink and ● is a source. That is,

$$\mathrm{sink}(\Omega(K)) = \mathrm{in}(K) = \{2, 6, 9, 15\},$$
$$\mathrm{source}(\Omega(K)) = \mathrm{out}(K) \cup \{1, 17\} = \{1, 4, 8, 13, 17\}.$$

For $B = (B_\tau)_{\tau \in H} \in X_V$, we set

$$M_K(B) := -\dim_{\mathbb{C}} \operatorname{Coker} \left(\bigoplus_{k \in \mathrm{out}(K)} V_k \stackrel{\oplus B_\sigma}{\longrightarrow} \bigoplus_{l \in \mathrm{in}(K)} V_l \right),$$

where σ is a path in $\Omega(K)$, and for $\Lambda \in \operatorname{Irr} \Lambda_V$, set

$$M_K(\Lambda) := M_K(B) \qquad (B \text{ is a generic point of } \Lambda).$$

Proposition . *The family of integers* $\{M_K(\Lambda)\}_{K \in \mathcal{M}_n^\times}$ *is a e-BZ datum and the map* $\bigsqcup_V \operatorname{Irr} \Lambda_V \to \mathcal{BZ}^e$ *defined by*

$$\Lambda \mapsto \{M_K(\Lambda)\}_{K \in \mathcal{M}_n^\times}$$

is an isomorphism of $*$*-crystals.*

In particular, for $\Lambda = \Lambda_{\mathbf{a}}$ $(\mathbf{a} = (a_{i,j}) \in \mathcal{B})$, *we have*

$$M_K(\mathbf{a}) = M_K(\Lambda_{\mathbf{a}}).$$

○ Conclusions

There are three realizations of $B(\infty)$:

$$
\begin{array}{ccc}
 & \mathcal{B} : \text{“limit” of SST} & \\
(b) \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow (a) \\
\mathcal{BZ}^e : e\text{-BZ data} & \overset{(c)}{\longleftrightarrow} & \bigsqcup_V \mathrm{Irr}\,\Lambda_V : \text{Irred. Lagrangians.}
\end{array}
$$

(a) orbits $\leftrightarrow$ conormal bundles

(b) $\mathcal{B} \xrightarrow{\sim} \mathcal{BZ}^e : \mathbf{a} \mapsto \mathbf{M}(\mathbf{a}) = (M_K(\mathbf{a}))_{\mathbf{K}\in\mathcal{M}_{\mathbf{n}}^{\times}}$,

$$
M_K(\mathbf{a}) = -\sum_{j=1}^{l}\sum_{i=1}^{k_j-1} a_{i,k_j} + \min\left\{ \sum_{1\le p<q\le l} a_{c_{p,q},c_{p,q}+(q-p)} \;\middle|\; \begin{array}{l} C=(c_{p,q}) \text{ is} \\ \text{a } K\text{-tableau.} \end{array} \right\}.
$$

(c) $\bigsqcup_V \mathrm{Irr}\,\Lambda_V \xrightarrow{\sim} \mathcal{BZ}^e$: $\Lambda \mapsto (M_K(\Lambda))_{K\in\mathcal{M}_n^{\times}}$,

$$
M_K(\Lambda) = -\dim_{\mathbb{C}} \mathrm{Coker}\left(\bigoplus_{k\in \mathrm{out}(K)} V_k \xrightarrow{\oplus B_\sigma} \bigoplus_{l\in \mathrm{in}(K)} V_l \right)
$$

- $M_K(\mathbf{a}) = M_K(\Lambda_{\mathbf{a}})$

$\Rightarrow$ The above is a commutative diagram.

§ Applications

∘ A new proof of AM conjecture

The AM conjecture (proved by Kamnizter) can be re-written as follows:

Corollary (e-BZ datum version)**.**
Let $\mathbf{M} = (M_K) \in \mathcal{BZ}^e$*. For each* $i \in I$*, we have*

$$(\widetilde{f}_i^* \mathbf{M})_K = \begin{cases} \min\{M_K,\ M_{s_i K} + c_i^*(\mathbf{M})\} & (K \in \mathcal{M}_n^\times(i)^*), \\ M'_K & (\text{otherwise}). \end{cases}$$

Here

$$\mathcal{M}_n^\times(i)^* = \{K \in \mathcal{M}_n^\times \mid i \notin K \ \textit{and}\ i+1 \in K\},$$
$$c_i^*(\mathbf{M}) = M_{[1,i]^c} - M_{([1,i+1]\setminus\{i\})^c} - 1.$$

• By using a Lagrangian realization of (e-)BZ data, we can easily check that

$$(\widetilde{f}_i^* \mathbf{M})_K = M_K \qquad (K \notin \mathcal{M}_n^\times(i)^*).$$

⇒ The remaining problem is:

$$(\widetilde{f}_i^* \mathbf{M})_K = \min\{M_K,\ M_{s_i K} + c_i^*(\mathbf{M})\} \quad (K \in \mathcal{M}_n^\times(i)^*). \qquad (\sharp)$$

Lemma .

(♯) $\Leftrightarrow$ *For any* $K \in \mathcal{M}_n^{\times}(i)^*$,

$$M_K(\Lambda) = \min\left\{M_K(\overline{\Lambda}), M_{s_iK}(\overline{\Lambda}) + \langle h_i, \mathrm{wt}(\overline{\Lambda})\rangle - \varepsilon_i^*(\Lambda)\right\}, \qquad (\sharp\sharp)$$

where $\overline{\Lambda} = \widetilde{e}_i^{*max}\Lambda$.

Proposition .

The formula (♯♯) *holds for any* $K \in \mathcal{M}_n^{\times}(i)^*$.

Key properties

- $\overline{\widetilde{f}_i^*\Lambda} = \overline{\Lambda}$.

- By the definition, i is a source in $\Omega(s_iK)$.

  $\Rightarrow \quad M_{s_iK}(\overline{\Lambda}) = M_{s_iK}(\Lambda)$.

- Let $\pi : V \to \overline{V}$ be a surjective linear map, and $\psi : N \to \overline{V}$ a linear map. Consider a generic map $\varphi : N \to V$ such that $\psi = \pi \circ \varphi$.

$$\begin{array}{ccc} N & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{\psi} & \\ V & \underset{\pi}{\twoheadrightarrow} & \overline{V} \end{array}$$

$\Rightarrow \dim_{\mathbb{C}} \mathrm{Ker}\varphi = \max\{\dim_{\mathbb{C}} \mathrm{Ker}\psi - (\dim_{\mathbb{C}} V - \dim_{\mathbb{C}} \overline{V}), 0\}$.

§ **Future problems**

○ $A_{n-1}^{(1)}$ case

• Realization I and II : known

• Realization III :
There is no corresponding affine Grassmannian.
$\Rightarrow$ There is no MV cycle.

But, there exists an affine analogue of BZ datum.
(Naito-Sagaki's unpublished result : A_∞-case $\to$ n-reduction)

In affine case,

· I $\leftrightarrow$ III : OK.

· II $\leftrightarrow$ III : not yet (partially done).

• Beck-Nakajima's affine PBW basis ?
(In A_n-case, Realization I is closely related to the theory of PBW basis.)

• Other (finite or affine) types?