On tensor category arising from representation theory of the restricted quantum universal enveloping algebra associated to $\mathfrak{sl}_2$

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§ Introduction

<u>Background</u>

• Kazhdan-Lusztig (1993∼1994):

$$\begin{array}{c}\text{Category of}\\ \text{representation of}\\ \text{affine Lie algebra } \widehat{\mathfrak{g}}\end{array} \stackrel{\sim}{\longleftrightarrow} \begin{array}{c}\text{Category of}\\ \text{representation of } U_{\mathfrak{q}}(\mathfrak{g})\\ \text{at a root of unity}\end{array}$$

Main tool : Conformal Field Theory (WZW-model)

• Recently, a "log-version" of the above correspondence is considered.

What is a logarithmic CFT?

- Roughly speaking, a log CFT is a CFT such that "KZ-type equations" have logarithmic singularities.
- But, in mathematical sense, there is <u>no definition</u>. That is, there are only some examples.

As an example of log-CFTs, there is a CFT based on representation of *the triplet vertex operator algebra* $W(p)$ ($p \in \mathbb{Z}_{\geq 2}$).

Conjecture 1 (Feigin et al.)**.** *There is a "log-version" of KL-equivalence. That is, as <u>braided tensor categories</u>,*

$$\begin{array}{c}\textit{Category of}\\ W(p)\textit{-modules}\end{array} \stackrel{\sim}{\longleftrightarrow} \begin{array}{c}\textit{Category of}\\ \textit{finite dimensional}\\ \overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)\textit{-modules},\end{array}$$

where $\overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$ *is the restricted quantum group associated* $\mathfrak{sl}_2$ *and* $\mathfrak{q} = \exp(\frac{\pi\sqrt{-1}}{p})$.

They proved the conjecture for $p = 2$ case.

In 2009, Tsuchiya-Nagatomo proved the following theorem.

Theorem 2 (Tsuchiya-Nagatomo)**.** *As abelian categories, these are equivalent.*

• In this talk, we only treat the quantum group side.

Aim :

Study tensor structure of $\overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$-**mod**.

Main result :

Indecomposable decomposition of all tensor products of $\overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$-modules is completely determined in explicit formulas.

As a by-product, we show that $\overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$-**mod** is not a braided tensor category for $p \geq 3$.

$\Rightarrow$ It needs a *rectification* for Conjecture 1.

• Recently, Feigin and Tipunin (arXiv:1002.5047) introduce an *new* tensor structure on $\overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$, and conjecture that it solves the above "contradiction" (but it has not done).

§ Preliminaries

Notations

Let $p \geq 2$ be an integer and $\mathfrak{q}$ be a primitive $2p$-th root of unity. For any integer n, we set

$$[n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}}.$$

Note that $[n] = [p-n]$ for any n.

- $\overline{U} = \overline{U}_{\mathfrak{q}}(\mathfrak{sl}_2)$: The restricted quantum $\mathfrak{sl}_2$

An unital associative $\mathbb{C}$-algebra with generators E, F, K, K^{-1} and relations ;

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$K^{2p} = 1, \quad E^p = 0, \quad F^p = 0.$$

This is a $2p^3$-dimensional $\mathbb{C}$-algebra and has a Hopf algebra structure defined by

$$\begin{aligned}
\Delta\colon\ & E \longmapsto E \otimes K + 1 \otimes E, \quad F \longmapsto F \otimes 1 + K^{-1} \otimes F, \\
& K \longmapsto K \otimes K, \quad K^{-1} \longmapsto K^{-1} \otimes K^{-1}, \\
\varepsilon\colon\ & E \longmapsto 0, \quad F \longmapsto 0, \quad K \longmapsto 1, \quad K^{-1} \longmapsto 1, \\
S\colon\ & E \longmapsto -EK^{-1}, \quad F \longmapsto -KF, \quad K^{\pm 1} \longmapsto K^{\mp 1}.
\end{aligned}$$

The category $\overline{U}$-**mod** of finite-dimensional left $\overline{U}$-modules has a structure of a monoidal category associated with this Hopf algebra structure on $\overline{U}$.

§ Structure of $\overline{U}$-mod

This is a survey of known results on $\overline{U}$-**mod** which were studied by Reshetikhin-Turaev, Suter, Xiao, Gunnlaugsdóttir, Chari-Premet, Feigin-Gainutdinov-Semikhatov-Tipunin, Arike.

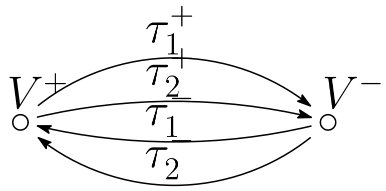
Strategy

• We can compute a complete set of mutually orthogonal primitive idempotents for$\overline{U}$ in expricit way.

$\Rightarrow$ **the expricit form of the basic algbera** $B_{\overline{U}}$.

$B_{\overline{U}} \cong \prod_{s=0}^{p} B_s$ and one can describe each B_s as follows:

- $B_0 \cong B_p \cong \mathbb{C}$. (1-dimensional algebra)
- For each $s = 1, \ldots, p-1$, B_s is isomorphic to the 8-dimensional algebra B defined by the following quiver



with relations $\tau_i^{\pm}\tau_i^{\mp} = 0$ for $i = 1, 2$, and $\tau_1^{\pm}\tau_2^{\mp} = \tau_2^{\pm}\tau_1^{\mp}$.

The next problem is :

What is the structure of B-mod ?

In the following, we will give you

- the complete list of indecomposable B-modules and
- Auslander-Reiten quiver of B-**mod**.

Classification of indecomposable B-modules

We can identify a B-module with data

$$\mathcal{Z} = (V_{\mathcal{Z}}^+, V_{\mathcal{Z}}^-; \tau_{1,\mathcal{Z}}^+, \tau_{2,\mathcal{Z}}^+, \tau_{1,\mathcal{Z}}^-, \tau_{2,\mathcal{Z}}^-),$$

where

- $V_{\mathcal{Z}}^{\pm}$ is a vector space over $\mathbb{C}$ (attached to the vertices $\pm$).
- $\tau_{i,\mathcal{Z}}^{\pm}\colon V_{\mathcal{Z}}^{\pm} \to V_{\mathcal{Z}}^{\mp}$ ($i = 1, 2$) are $\mathbb{C}$-linear maps (attached to the arrows) satisfying $\tau_{i,\mathcal{Z}}^{\pm}\tau_{i,\mathcal{Z}}^{\mp} = 0$, $\tau_{1,\mathcal{Z}}^{\pm}\tau_{2,\mathcal{Z}}^{\mp} = \tau_{2,\mathcal{Z}}^{\pm}\tau_{1,\mathcal{Z}}^{\mp}$.

For positive integers m, n and $i = 1, \ldots, m$, $j = 1, \ldots, n$, we denote the composition of j-th projection and i-th embedding

$$e_{i,j} : \mathbb{C}^n \to \mathbb{C} \to \mathbb{C}^m.$$

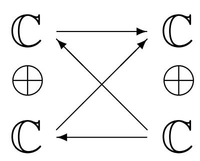
Proposition 3. *Any indecomposable B-module is isomorphic to exactly one of modules in the following list:*

- *Simple modules* :

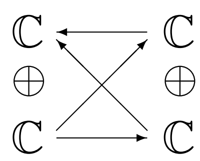
$$\mathcal{X}^+ = (\mathbb{C}, \{0\}; 0, 0, 0, 0), \quad \mathcal{X}^- = (\{0\}, \mathbb{C}; 0, 0, 0, 0).$$

- *Projective-injective modules* :

$$\mathcal{P}^+ = (\mathbb{C}^2, \mathbb{C}^2; e_{1,1}, e_{2,1}, e_{2,2}, e_{2,1}) =$$



$$\mathcal{P}^- = (\mathbb{C}^2, \mathbb{C}^2; e_{2,2}, e_{2,1}, e_{1,1}, e_{2,1}) =$$



- *For each integer $n \geq 2$,*

$$\mathcal{M}^+(n) = \left(\mathbb{C}^{n-1}, \mathbb{C}^n; \textstyle\sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}, 0, 0\right)$$

$$= \mathbb{C}^{n-1} \underset{\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}}{\rightrightarrows}} \mathbb{C}^n, \qquad \left(\begin{array}{c}\textit{Here we omit} \\ 0\textit{-arrows.}\end{array}\right)$$

$$\mathcal{M}^-(n) = \left(\mathbb{C}^n, \mathbb{C}^{n-1}; 0, 0, \textstyle\sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i+1,i}\right),$$

$$\mathcal{W}^+(n) = \left(\mathbb{C}^n, \mathbb{C}^{n-1}; \textstyle\sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i,i+1}, 0, 0\right),$$

$$\mathcal{W}^-(n) = \left(\mathbb{C}^{n-1}, \mathbb{C}^n; 0, 0, \textstyle\sum_{i=1}^{n-1} e_{i,i}, \sum_{i=1}^{n-1} e_{i,i+1}\right).$$

- *For each integer $n \geq 1$ and $\lambda \in \mathbb{P}^1(\mathbb{C})$,*

$$\mathcal{E}^+(n;\lambda) = \left(\mathbb{C}^n, \mathbb{C}^n; \varphi_1(n;\lambda), \varphi_2(n;\lambda), 0, 0\right),$$

$$\mathcal{E}^-(n;\lambda) = \left(\mathbb{C}^n, \mathbb{C}^n; 0, 0, \varphi_1(n;\lambda), \varphi_2(n;\lambda)\right),$$

where

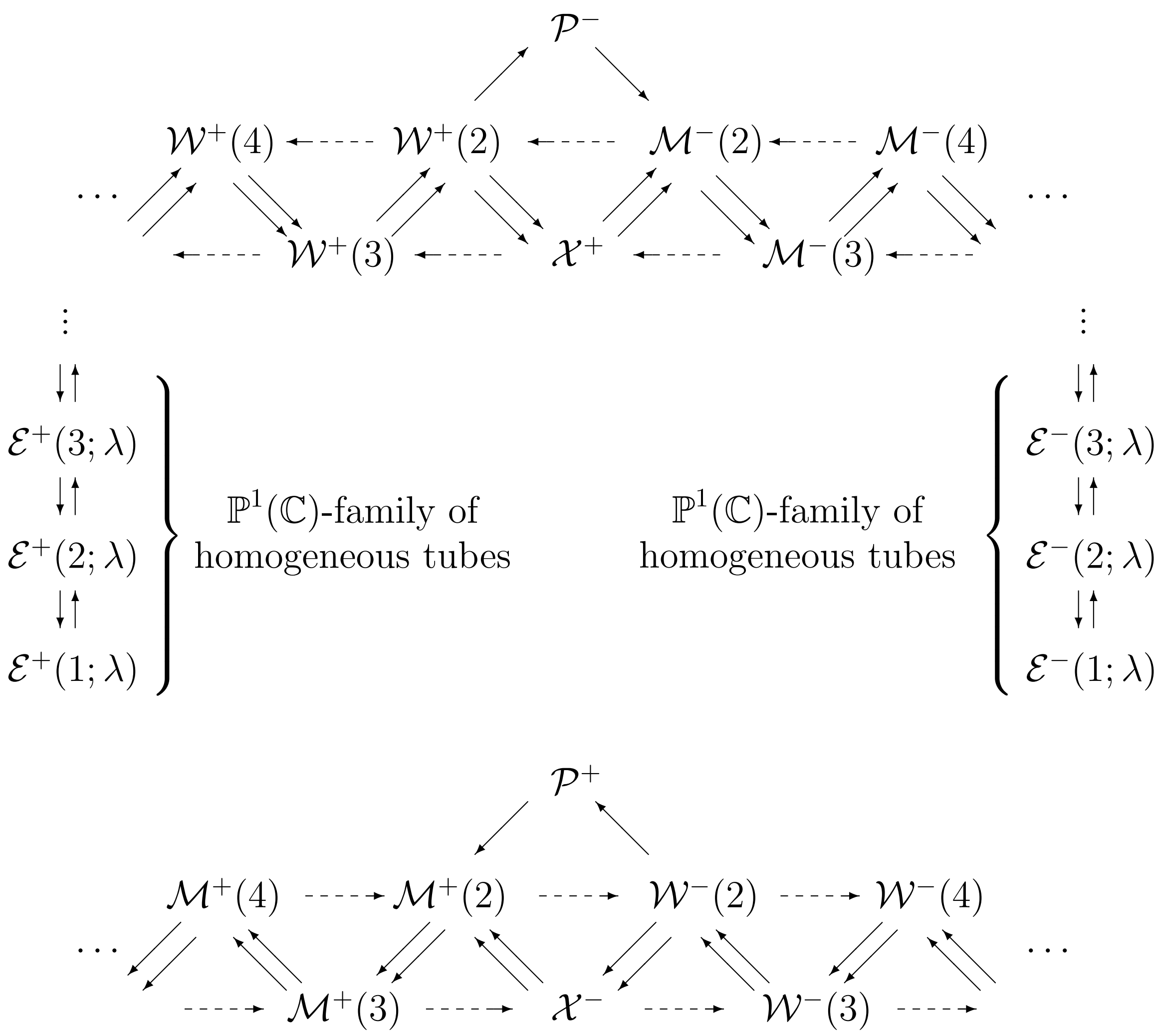
$$\left(\varphi_1(n;\lambda), \varphi_2(n;\lambda)\right) = \begin{cases} \left(\beta \cdot \mathrm{id} + \sum_{i=1}^{n-1} e_{i,i+1}, \mathrm{id}\right) & (\lambda = [\beta : 1]), \\ \left(\mathrm{id}, \sum_{i=1}^{n-1} e_{i,i+1}\right) & (\lambda = [1 : 0]). \end{cases}$$

i.e,

$$\mathcal{E}^+(n;\lambda) = \begin{cases} \mathbb{C}^n \underset{\mathrm{id}}{\overset{J(\beta;n)}{\rightrightarrows}} \mathbb{C}^n & (\lambda = [\beta : 1]), \\ \mathbb{C}^n \underset{J(0;n)}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{C}^n & (\lambda = [1 : 0]). \end{cases}$$

Here $J(\beta;n)$ is the $(n \times n)$-Jordan cell with eigenvalue β.

<u>Auslander-Reiten quiver of B-mod</u>



Remark . We "*divide*" the quiver of B into the following two pieces which are isomorphic to the Kronecker quiver:

$$Q^+ := \overset{+}{\circ} \underset{\tau_2^+}{\overset{\tau_1^+}{\rightrightarrows}} \overset{-}{\circ} \quad \text{“}+\text{”} \quad Q^- := \overset{+}{\circ} \underset{\tau_2^-}{\overset{\tau_1^-}{\leftleftarrows}} \overset{-}{\circ}$$

Consider AR-quivers of Q^+ and Q^- (*i.e.* two copies of AR-quiver of the Kronecker quiver), and "*paste*" the above two copies.

$\Rightarrow$ AR-quiver of B-**mod**

Structure of $\overline{U}$-mod

Recall a decomposition of the basic algebra $B_{\overline{U}}$ of $\overline{U}$:

$$B_{\overline{U}} = \bigoplus_{s=0}^{p} B_s$$

where

$$B_0 \cong B_p \cong \mathbb{C}, \qquad B_s \cong B \quad (1 \le s \le p-1).$$

Denote by $\mathcal{C}(s)$ the full subcategory of $\overline{U}$-**mod** corresponding to B_s-modules (considered as $B_{\overline{U}}$-modules) for $s = 0, \ldots, p$.

$\Rightarrow$ We have a block decomposition of $\overline{U}$-**mod**:

$$\overline{U}\textbf{-mod} = \bigoplus_{s=0}^{p} \mathcal{C}(s).$$

- For $s = 1, \ldots, p-1$, let Φ_s be the composition of functors

$$\Phi_s : B\textbf{-mod} \to B_{\overline{U}}\textbf{-mod} \to \overline{U}\textbf{-mod}.$$

We denote by

$$\mathcal{X}_s^+, \mathcal{X}_{p-s}^-, \mathcal{P}_s^+, \mathcal{P}_{p-s}^-, \mathcal{M}_s^+(n), \mathcal{M}_{p-s}^-(n), \mathcal{W}_s^+(n), \mathcal{W}_{p-s}^-(n),$$
$$\mathcal{E}_s^+(n;\lambda), \mathcal{E}_{p-s}^-(n;\lambda)$$

the images of

$$\mathcal{X}^+, \mathcal{X}^-, \mathcal{P}^+, \mathcal{P}^-, \mathcal{M}^+(n), \mathcal{M}^-(n), \mathcal{W}^+(n), \mathcal{W}^-(n),$$
$$\mathcal{E}^+(n;\lambda), \mathcal{E}^-(n;\lambda)$$

by Φ_s.

• On the other hand, for $s = 0$ or p, let Φ_s be the composition of functors

$$\Phi_s : \mathbb{C}\textbf{-mod} \to B_{\overline{U}}\textbf{-mod} \to \overline{U}\textbf{-mod}.$$

Let us denote $\mathcal{X} \cong \mathbb{C}$ the unique simple object in $\mathbb{C}$- $\textbf{mod}$. We denote the corresponding object in $\mathcal{C}(0)$ and $\mathcal{C}(p)$ by

$$\mathcal{X}_p^- := \Phi_0(\mathcal{X}) \in \mathcal{C}(0),$$
$$\mathcal{X}_p^+ := \Phi_p(\mathcal{X}) \in \mathcal{C}(p).$$

We remark that both $\mathcal{X}_p^-$ and $\mathcal{X}_p^+$ are also projective. In that sense, we sometimes denote

$$\mathcal{P}_p^\pm := \mathcal{X}_p^\pm.$$

Simple objects in $\mathcal{C}(s)$

The explicit form of $\Phi_s(\mathcal{X}^{\pm})$ are given as follows:

$\circ\ 1 \leq s \leq p-1$

• $\mathcal{X}_s^+ = \Phi_s(\mathcal{X}^+)$ is isomorphic to the s-dimensional module defined by basis $\{a_n\}_{n=0,\ldots,s-1}$ and $\overline{U}$-action given by

$$Ka_n = q^{s-1-2n}a_n,$$

$$Ea_n = \begin{cases} [n][s-n]a_{n-1} & (n \neq 0) \\ 0 & (n=0) \end{cases},$$

$$Fa_n = \begin{cases} a_{n+1} & (n \neq s-1) \\ 0 & (n = s-1) \end{cases}.$$

• $\mathcal{X}_{p-s}^- = \Phi_s(\mathcal{X}^-)$ is isomorphic to the $(p-s)$-dimensional module defined by basis $\{a_n\}_{n=0,\ldots,p-s-1}$ and $\overline{U}$-action given by

$$Ka_n = -q^{p-s-1-2n}a_n,$$

$$Ea_n = \begin{cases} -[n][p-s-n]a_{n-1} & (n \neq 0) \\ 0 & (n=0) \end{cases},$$

$$Fa_n = \begin{cases} a_{n+1} & (n \neq p-s-1) \\ 0 & (n = p-s-1) \end{cases}.$$

Remark . Since we consider all finite dimensional $\overline{U}$-modules, modules which are not of type I are appeared. For example, $\mathcal{X}_s^+$ is a $\overline{U}$-module of type I. On the other hand $\mathcal{X}_{p-s}^-$ is not.

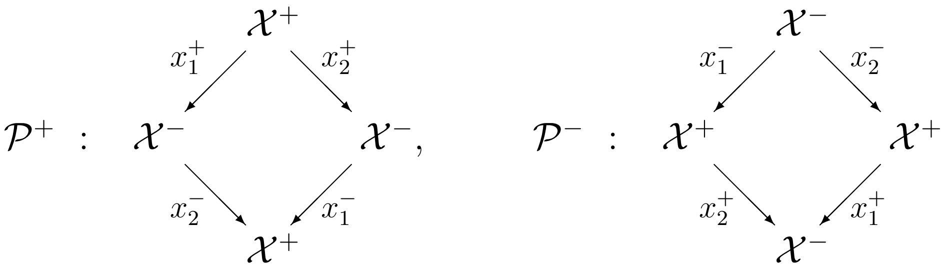
$\circ\ s = 0$ or p

$\mathcal{X}_p^+ = \Phi_p(\mathcal{X})$ (*resp.* $\mathcal{X}_p^- = \Phi_0(\mathcal{X})$) is the p-dimensional irreducible module of $\overline{U}$ defined as similar way.

Other indecomposable objects in $\mathcal{C}(s)$ $(1 \le s \le p-1)$

• Since $\mathcal{C}(s)$ is equivalent to B-**mod** as an abelian category, all information of indecomposable objects in $\mathcal{C}(s)$ can be obtained form one of the corresponding objects in B-**mod**.

Example. In B-**mod**, the structure of the projective modules $\mathcal{P}^{\pm}$ are given as:



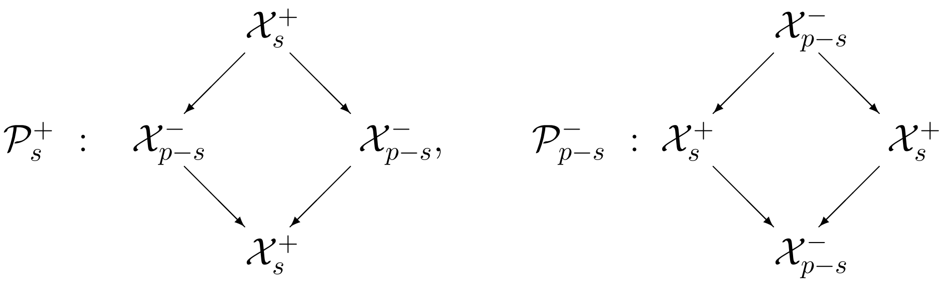
By easy computation, we have

$$\mathrm{Ext}^1_B(\mathcal{X}^{\pm}, \mathcal{X}^{\mp}) = \mathbb{C}^2.$$

We fix basis of $\mathrm{Ext}^1_B(\mathcal{X}^{+}, \mathcal{X}^{-})$ and $\mathrm{Ext}^1_B(\mathcal{X}^{-}, \mathcal{X}^{+})$ by $\{x_1^+,\ x_2^+\}$ and $\{x_1^-,\ x_2^-\}$ respectively.
(We omit to give the explicit form of them.)

In the above diagram, we denote $\mathcal{X}_1 \xrightarrow{x} \mathcal{X}_2$ by the extension by $x \in \mathrm{Ext}^1_B(\mathcal{X}_1, \mathcal{X}_2)$.

Applying the functor Φ_s, we have



As a corollary, we have

$$\dim \mathcal{P}_s^+ = 2p \ (= 2s + 2(p-s)), \quad \dim \mathcal{P}_{p-s}^- = 2p.$$

§ Calculation of tensor products

Main tools

(a) Some (basic) short exact sequences.
(It is enough to show the existence of them in B-**mod**.)

(b) Exactness of the functors $- \otimes \mathcal{Z}$ and $\mathcal{Z} \otimes -$.
($\because \otimes$ in a tensor product over a field $\mathbb{C}$.)

(c) For a projective module $\mathcal{P}$, both $\mathcal{P} \otimes \mathcal{Z}$ and $\mathcal{Z} \otimes \mathcal{P}$ are also projective.

(d) $\overline{U}$ is a Frobenius algebra. As a by-product,

$$\mathcal{Z} \text{ is projective } \Leftrightarrow \mathcal{Z} \text{ is injective.}$$

(e) Rigidity : For $n \geq 0$,

$$\mathrm{Ext}^n_{\overline{U}}(\mathcal{Z}_1, \mathcal{Z}_2 \otimes \mathcal{Z}_3) \cong \mathrm{Ext}^n_{\overline{U}}(D(\mathcal{Z}_2) \otimes \mathcal{Z}_1, \mathcal{Z}_3),$$

$$\mathrm{Ext}^n_{\overline{U}}(\mathcal{Z}_1 \otimes \mathcal{Z}_2, \mathcal{Z}_3) \cong \mathrm{Ext}^n_{\overline{U}}(\mathcal{Z}_1, \mathcal{Z}_3 \otimes D(\mathcal{Z}_2)).$$

Here $D(\mathcal{Z})$ the standard dual of $\mathcal{Z} \in \overline{U}$-**mod**. More precisely, define $\overline{U}$-module structure on the dual space $D(\mathcal{Z}) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{Z}, \mathbb{C})$ as:

$$(a \cdot f)(z) := f(S(a)z) \quad (a \in \overline{U}, f \in D(\mathcal{Z}), z \in \mathcal{Z}),$$

where S is the antipode of $\overline{U}$.

Remark . It is known that the properties (c), (d) and (e) hold in more general setting. Namely, for any finite dimensional Hopf-algebra A over a field, these properties hold in A–**mod**.

(Of course, (b) is also satisfied.)

Tensor products of simple modules

The following proposition is proved by Reshetikhin-Turaev.

Proposition 4 (Reshetikhin-Traev). *For $s, s' = 1, \ldots, p$,*

$$\mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^+ \cong \begin{cases} \displaystyle\bigoplus_{\substack{t=|s-s'|+1,\\ 2\text{-}steps}}^{s+s'-1} \mathcal{X}_t^+ & (s+s'-1 \le p), \\ \left(\displaystyle\bigoplus_{\substack{t=|s-s'|+1,\\ 2\text{-}steps}}^{2p-s-s'-1} \mathcal{X}_t^+\right) \oplus \left(\displaystyle\bigoplus_{\substack{t'=2p-s-s'+1,\\ 2\text{-}steps}}^{p-\delta} \mathcal{P}_{t'}^+\right) \\ \qquad\qquad\qquad\qquad (s+s'-1 > p), \end{cases}$$

where

$$\delta = \begin{cases} 1 & (s+s'-p-1 \ \textit{is odd}), \\ 0 & (s+s'-p-1 \ \textit{is even}). \end{cases}$$

- If $s + s' - 1 \le p$, the formula is nothing but Clebush-Gordan formula.
- It is easy to see that

$$\mathcal{X}_s^\pm \otimes \mathcal{X}_1^- \cong \mathcal{X}_1^- \otimes \mathcal{X}_s^\pm \cong \mathcal{X}_s^\mp,$$
$$\mathcal{P}_s^\pm \otimes \mathcal{X}_1^- \cong \mathcal{X}_1^- \otimes \mathcal{P}_s^\pm \cong \mathcal{P}_s^\mp.$$

By the associativity of tensor products, we can calculate other decompositions. For example,

$$\begin{aligned} \mathcal{X}_s^- \otimes \mathcal{X}_{s'}^+ &\cong (\mathcal{X}_1^- \otimes \mathcal{X}_s^+) \otimes \mathcal{X}_{s'}^+ \\ &\cong \mathcal{X}_1^- \otimes \big((\oplus_t \mathcal{X}_t^+) \oplus (\oplus_{t'} \mathcal{P}_{t'}^+)\big) \\ &\cong \big((\oplus_t \mathcal{X}_t^-) \oplus (\oplus_{t'} \mathcal{P}_{t'}^-)\big). \end{aligned}$$

Main result

Theorem 5. *Indecomposable decomposition of all tensor products in $\overline{U}$-* **mod** *is completely determined in explicit formulas.*

Since there are too many indecomposables in $\overline{U}$-**mod**, we can not list up all formulas in this talk. In the following, we will give some typical examples.

Tensor products of $\mathcal{E}_s^{\pm}(1;\lambda)$ with simple modules

By direct calculation, we have the following.

Proposition 6. *For $s, s' = 1, \ldots, p-1$, $n \geq 1$ and $\lambda = [\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C})$ we have*

$$\mathcal{E}_s^{\pm}(1;\lambda) \otimes \mathcal{X}_1^{-} \cong \mathcal{E}_s^{\mp}(1;-\lambda),$$
$$\mathcal{X}_1^{-} \otimes \mathcal{E}_s^{\pm}(1;\lambda) \cong \mathcal{E}_s^{\mp}\big(1;(-1)^{p-1}\lambda\big),$$

$$\mathcal{E}_s^{+}(1;\lambda) \otimes \mathcal{X}_2^{+} \cong \mathcal{E}_{s-1}^{+}\left(1;\frac{[s]}{[s-1]}\lambda\right) \oplus \mathcal{E}_{s+1}^{+}\left(1;\frac{[s]}{[s+1]}\lambda\right),$$
$$\mathcal{X}_2^{+} \otimes \mathcal{E}_s^{+}(1;\lambda) \cong \mathcal{E}_{s-1}^{+}\left(1;-\frac{[s]}{[s-1]}\lambda\right) \oplus \mathcal{E}_{s+1}^{+}\left(1;-\frac{[s]}{[s+1]}\lambda\right).$$

Here, for $c \in \mathbb{C}$, we set $c\lambda = [c\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C})$.

Remark . This proposition tells us that, in general,

$$\mathcal{E}_s^{\pm}(1;\lambda) \otimes \mathcal{X}_1^{-} \ncong \mathcal{X}_1^{-} \otimes \mathcal{E}_s^{\pm}(1;\lambda),$$
$$\mathcal{E}_s^{+}(1;\lambda) \otimes \mathcal{X}_2^{+} \ncong \mathcal{X}_2^{+} \otimes \mathcal{E}_s^{+}(1;\lambda).$$

That is, $\overline{U}$-**mod** is <u>not</u> a braided tensor category.

Proposition 7. *For $s, s' = 1, \dots, p-1$ and $\lambda \in \mathbb{P}^1(k)$,*

$$\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1 \in I_{s,s'}} \mathcal{E}_{t_1}^+\left(1; \frac{[s]}{[t_1]}\lambda\right) \oplus \bigoplus_{t_2 \in J_{s+s'}} \mathcal{P}_{t_2}^+ \oplus \bigoplus_{t_3 \in J_{p-s+s'}} \mathcal{P}_{t_3}^-.$$

Here, $I_{s,s'}$, $J_{s+s'}$, $J_{p-s+s'}$ are some sets of integers.

(For $\mathcal{X}_{s'}^+ \otimes \mathcal{E}_s^+(1;\lambda)$, we have a similar formula.)

Proof. There is a (basic) exact sequence in $\overline{U}$-**mod**:

$$0 \to \mathcal{X}_{p-s}^- \to \mathcal{E}_s^+(1;\lambda) \to \mathcal{X}_s^+ \to 0.$$

Applying $- \otimes \mathcal{X}_{s'}^+$, we have

$$0 \to \mathcal{X}_{p-s}^- \otimes \mathcal{X}_{s'}^+ \to \mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+ \to \mathcal{X}_s^+ \otimes \mathcal{X}_{s'}^+ \to 0.$$

By Proposition 4, we have

$$\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1} \mathcal{Z}_{t_1}^+ \oplus \bigoplus_{t_2} \mathcal{P}_{t_2}^+ \oplus \bigoplus_{t_2} \mathcal{P}_{t_3}^-$$

with an exact sequence $0 \to \mathcal{X}_{p-t_1}^- \to \mathcal{Z}_{t_1} \to \mathcal{X}_{t_1}^+ \to 0$ for each t_1. We remark that $\mathcal{Z}_{t_1}$ is not projective.

Assume the formula hols for $s'' \le s'-1$. Then,

$$\begin{aligned}
&\big(\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'-1}^+\big) \otimes \mathcal{X}_2^+ \\
&\qquad \cong \mathcal{E}_s^+(1;\lambda) \otimes (\mathcal{X}_{s'-1}^+ \otimes \mathcal{X}_2^+) \\
&\qquad \cong \mathcal{E}_s^+(1;\lambda) \otimes (\mathcal{X}_{s'-2}^+ \oplus \mathcal{X}_{s'}^+) \\
&\qquad \cong \big(\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'-2}^+\big) \oplus \big(\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+\big)
\end{aligned}$$

tells us that a non-projective indecomposable summand of $\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+$ must be of the form $\mathcal{E}_t^+\big(1; \frac{[s]}{[t]}\lambda\big)$ with $t = 1, \dots, p-1$. Then we have $\mathcal{Z}_{t_1} \cong \mathcal{E}_{t_1}^+\big(1; \frac{[s]}{[t_1]}\lambda\big)$ since $\mathcal{Z}_{t_1}$ cannot be projective. Thus we have the formula. □

Tensor products of $\mathcal{E}_s^{\pm}(n;\lambda)$ with simple modules

For computing these combination, we need the rigidity.

Proposition 8. *For $s = 1, \ldots, p-1$ and $\lambda \in \mathbb{P}^1(k)$,*

$$D(\mathcal{X}_s^{\pm}) \cong \mathcal{X}_s^{\pm}, \quad D\big(\mathcal{E}_s^{+}(1;\lambda)\big) \cong \mathcal{E}_{p-s}^{-}\big(1;(-1)^s\lambda\big),$$
$$D\big(\mathcal{E}_s^{-}(1;\lambda)\big) \cong \mathcal{E}_{p-s}^{+}\big(1;(-1)^{p-s}\lambda\big).$$

Proposition 9.

$$D\big(\mathcal{E}_s^{+}(n;\lambda)\big) \cong \mathcal{E}_{p-s}^{-}\big(n;(-1)^s\lambda\big),\ D\big(\mathcal{E}_s^{-}(n;\lambda)\big) \cong \mathcal{E}_{p-s}^{+}\big(n;(-1)^{p-s}\lambda\big).$$

Proof. Since $\dim \mathcal{E}_s^{+}(n;\lambda) = pn$ and D preserves direct sum and dimension, $D\big(\mathcal{E}_s^{+}(n;\lambda)\big)$ is an indecomposable module of dimension pn.

$\Rightarrow$ This is of the form $\mathcal{E}_t^{\pm}(n;\mu)$ or is projective (the latter case could occur only if $n \leq 2$).

$$\begin{aligned}
&\mathrm{ext}^1_{\overline{U}}\big(D\big(\mathcal{E}_s^{+}(n;\lambda)\big), \mathcal{X}_s^{+}\big) \qquad (\mathrm{ext} := \dim_{\mathbb{C}} \mathrm{Ext}.)\\
&\quad = \mathrm{ext}^1_{\overline{U}}\big(D\big(\mathcal{E}_s^{+}(n;\lambda)\big) \otimes \mathcal{X}_1^{+}, \mathcal{X}_s^{+}\big) = \mathrm{ext}^1_{\overline{U}}\big(\mathcal{X}_1^{+}, \mathcal{E}_s^{+}(n;\lambda) \otimes \mathcal{X}_s^{+}\big)\\
&\quad = \mathrm{ext}^1_{\overline{U}}\big(\mathcal{X}_1^{+}, \mathcal{E}_s^{+}(n;\lambda) \otimes D(\mathcal{X}_s^{+})\big) = \mathrm{ext}^1_{\overline{U}}\big(\mathcal{X}_1^{+} \otimes \mathcal{X}_s^{+}, \mathcal{E}_s^{+}(n;\lambda)\big)\\
&\quad = \mathrm{ext}^1_{\overline{U}}\big(\mathcal{X}_s^{+}, \mathcal{E}_s^{+}(n;\lambda)\big) = \mathrm{ext}^1_{B}\big(\mathcal{X}^{+}, \mathcal{E}^{+}(n;\lambda)\big) = n.
\end{aligned}$$

$\Rightarrow$ $D\big(\mathcal{E}_s^{+}(n;\lambda)\big)$ must be of the form $\mathcal{E}_t^{\pm}(n;\mu)$.

By the similar argument,

$$\begin{aligned}
&\mathrm{ext}^1_{\overline{U}}\big(D\big(\mathcal{E}_s^{+}(n;\lambda)\big), \mathcal{E}_s^{+}(1;\mu)\big) = \mathrm{ext}^1_{B}\big(\mathcal{E}^{-}\big(1;(-1)^s\mu\big), \mathcal{E}^{+}(n;\lambda)\big)\\
&\qquad = \begin{cases} 1 & ((-1)^s\mu = -\lambda) \\ 0 & ((-1)^s\mu \neq -\lambda) \end{cases}.
\end{aligned}$$

$\Rightarrow$ $D\big(\mathcal{E}_s^{+}(n;\lambda)\big) \cong \mathcal{E}_{p-s}^{-}\big(n;(-1)^s\lambda\big)$. $\square$

Proposition 10.

$$\mathcal{E}_s^+(n;\lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1 \in I_{s,s'}} \mathcal{E}_{t_1}^+\left(n; \frac{[s]}{[t_1]}\lambda\right) \oplus \bigoplus_{t_2 \in J_{s+s'}} (\mathcal{P}_{t_2}^+)^n \oplus \bigoplus_{t_3 \in J_{p-s+s'}} (\mathcal{P}_{t_3}^-)^n.$$

(We have a similar formula for $\mathcal{X}_{s'}^+ \otimes \mathcal{E}_s^+(n;\lambda)$.)

Proof. The same argument as the case of $\mathcal{E}_s^+(1;\lambda) \otimes \mathcal{X}_{s'}^+$ shows that

$$\mathcal{E}_s^+(n;\lambda) \otimes \mathcal{X}_{s'}^+ \cong \bigoplus_{t_1} \mathcal{Z}_{t_1} \oplus \bigoplus_{t_2} (\mathcal{P}_{t_2}^+)^n \oplus \bigoplus_{t_3} (\mathcal{P}_{t_3}^-)^n$$

with an exact sequence $0 \to (\mathcal{X}_{p-t_1}^-)^n \to \mathcal{Z}_{t_1} \to (\mathcal{X}_{t_1}^+)^n \to 0$ for each t_1. Moreover, by the exact sequence

$$0 \to \mathcal{E}_s^\pm(n-1;\lambda) \to \mathcal{E}_s^\pm(n;\lambda) \to \mathcal{E}_s^\pm(1;\lambda) \to 0$$

and induction on n, we have the following exact sequence

$$0 \to \mathcal{E}_{t_1}^+\left(n-1; \frac{[s]}{[t_1]}\lambda\right) \to \mathcal{Z}_{t_1} \to \mathcal{E}_{t_1}^+\left(1; \frac{[s]}{[t_1]}\lambda\right) \to 0.$$

$\Rightarrow \quad \mathcal{Z}_{t_1} \in \mathcal{C}(t_1)$ and $\dim_{\mathbb{C}} \mathcal{Z}_{t_1} = pn$.

By using the rigidity, we have

$$\begin{aligned}
\mathrm{ext}^1_{\overline{U}}\big(\mathcal{E}_s^+(n;\lambda) \otimes \mathcal{X}_{s'}^+, \mathcal{X}_t^+\big) &= 0,\\
\mathrm{ext}^1_{\overline{U}}\big(\mathcal{E}_s^+(n;\lambda) \otimes \mathcal{X}_{s'}^+, \mathcal{X}_{p-t}^-\big) &= n,\\
\mathrm{ext}^1_{\overline{U}}\big(\mathcal{E}_s^+(n;\lambda) \otimes \mathcal{X}_{s'}^+, \mathcal{E}_t^+(1;\mu)\big) &= \begin{cases} 1 & \left(\lambda = \frac{[t]}{[s]}\mu\right) \\ 0 & \left(\lambda \neq \frac{[t]}{[s]}\mu\right) \end{cases}.
\end{aligned}$$

$\Rightarrow \quad$ By the above properties, $\mathcal{Z}_{t_1}$ is uniquely determined. Namely, we have $\mathcal{Z}_{t_1} \cong \mathcal{E}_{t_1}^+\big(n; \frac{[s]}{[t_1]}\lambda\big)$. □

Conclusions

For other combinations, we can compute the explicit formulas by the similar methods.

As a by-product, we have

Corollary 11. (1) *Let* $\mathcal{Z}_1$, $\mathcal{Z}_2$ *be* $\overline{U}_q(\mathfrak{sl}_2)$*-modules. If* $\mathcal{Z}_1$ *nor* $\mathcal{Z}_2$ *do not have any indecomposable summand of type* $\mathcal{E}$*, we have* $\mathcal{Z}_1 \otimes \mathcal{Z}_2 \cong \mathcal{Z}_2 \otimes \mathcal{Z}_1$.

(2) *If* $p = 2$*, for arbitrary* $\overline{U}_q(\mathfrak{sl}_2)$*-modules* $\mathcal{Z}_1$, $\mathcal{Z}_2$ *we have* $\mathcal{Z}_1 \otimes \mathcal{Z}_2 \cong \mathcal{Z}_2 \otimes \mathcal{Z}_1$.

(3) *If* $p \geq 3$*, there exist* $\overline{U}_q(\mathfrak{sl}_2)$*-modules* $\mathcal{Z}_1$, $\mathcal{Z}_2$ *such that* $\mathcal{Z}_1 \otimes \mathcal{Z}_2 \ncong \mathcal{Z}_2 \otimes \mathcal{Z}_1$*. In particular,* $\overline{U}_q(\mathfrak{sl}_2)$-**mod** *is not a braided tensor category.*

Remark . These method can be applied only for $\mathfrak{s}l_2$-case. If $\mathfrak{g} \neq \mathfrak{s}l_2$, it is known that $\overline{U}_q(\mathfrak{g})$-**mod** has a wild representation type.