Categorical duality for quantum group actions

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The 9th MSJ-SI: Operator Algebras and Mathematical Physics
2016.08.08

Overview: categorical duality for actions

1. Tannaka-Krein duality principle (elaboration of 'predual')
   - Compact quantum groups
     $\leftrightsquigarrow$ C*-tensor categories with fiber functor (Woronowicz)
   - Actions of compact quantum group G
     $\leftrightsquigarrow$ module categories over $\mathrm{Rep}\, G$ (DC.-Y.)
2. Classification of actions through category
   1. $\mathrm{SU}_q(2)$-actions and weighted graphs (DC.-Y.)
   2. Braided commutative Yetter-Drinfeld algebras (N.-Y.)
   3. Compact quantum groups with the same fusion structure of $\mathrm{SU}(n)$ (N.-Y.)

Compact quantum groups

1. Compact topological space X
   $\rightsquigarrow$ function algebra $C(X)$ (unital commutative C*-algebra)
2. Compact group $G \times G \to G$
   $\rightsquigarrow$ bialgebra structure $C(G) \to C(G \times G) \simeq C(G) \otimes C(G)$

Woronowicz: forget about the commutativity!

Definition

A *compact quantum group* G is given by

1. a unital C*-algebra $A =$ "$C(G)$",
2. a $*$-homomorphism $\Delta\colon A \to A \otimes A$, which is *(co)cancellative*:
   - span of $(A \otimes 1)\Delta(A)$ is dense in $A \otimes A$, ditto for $(1 \otimes A)\Delta(A)$.

$\exists!$ invariant state h: $(h \otimes \iota)\Delta = h = (\iota \otimes h)\Delta$ (*Haar state*)

Unitary representation of CQGs

If G is a compact group,
its unitary representation $\pi\colon G \to U(H_\pi)$ can be presented by

1. unitary element $U_\pi \in B(H_\pi) \otimes C(G) = C(G; B(H_\pi))$,
   corresponding to $g \mapsto \pi_g$,
2. $\pi_g \pi_h = \pi_{gh}$ translates to $(\iota \otimes \Delta)(U_\pi) = (U_\pi)_{12}(U_\pi)_{13}$.
3. contragredient: $(j \otimes \iota)(U^*) \in B(\bar{H}_\pi) \otimes C(G)$, $j(T)\bar{\xi} = \overline{T^*\xi}$
4. tensor product: $\pi \otimes \sigma : (U_\pi)_{13}(U_\sigma)_{23} \in B(H_\pi \otimes H_\sigma) \otimes C(G)$

Definition

If G is a CQG with $C(G) = (A, \Delta)$, its *unitary representation* π is:

1. a Hilbert space H_π,
2. a unitary $U_\pi \in B(H_\pi) \otimes A$, such that $(\iota \otimes \Delta)(U_\pi) = (U_\pi)_{12}(U_\pi)_{13}$.

Rigid C*-tensor category

C*-tensor category

- C*-structure on morphisms: norm $\|T\|$ and $*\colon \mathscr{C}(X,Y) \to \mathscr{C}(Y,X)$

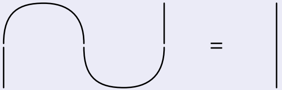
$$\|T^*T\| = \|T\|^2, \quad \|S\|\,\|T\| \geq \|ST\|$$

- Tensor product bifunctor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, (simple) unit $1_{\mathscr{C}}$

Duality (conjugates)

Dual object $\bar{X}$ comes with $R\colon 1_{\mathscr{C}} \to \bar{X} \otimes X$, $\bar{R}\colon 1_{\mathscr{C}} \to X \otimes \bar{X}$ s.t.

$$(R^* \otimes \iota_{\bar{X}})(\iota_{\bar{X}} \otimes \bar{R}) = \iota_{\bar{X}}, \quad (\bar{R}^* \otimes \iota_X)(\iota_X \otimes R) = \iota_X$$



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- Intrinsic (statistical) dimension: $d(X) = \min_{(R,\bar{R})} \|R\| \cdot \|\bar{R}\|$
- Standard solution: $\|R\| = \sqrt{d(X)} = \|\bar{R}\|$ unique up to unitary

Tannaka-Krein duality

If G is a CQG, $U_\pi \mapsto H_\pi$ (forgetful functor) is a C*-tensor functor $\mathrm{Rep}\, G \to \mathrm{Hilb}$ (category of fin. dim. Hilbert spaces).

Theorem (Woronowicz's Tannka-Krein duality)

The following things are equivalent:

1. *A CQG G, taken in the GNS representation of the invariant state (reduced CQG)*
2. *A rigid C*-tensor category $\mathscr{C}$ with $(U_1, U_1) \simeq \mathbb{C}$, and a C*-tensor functor $F\colon \mathscr{C} \to \mathrm{Hilb}$*

Key construction for $2 \to 1$: taking the irreducible classes $(U_\pi)_\pi$ of $\mathscr{C}$, $\rightsquigarrow$ Peter-Weil presentation "$\mathscr{O}(G)$" $= \oplus_\pi \overline{F(\pi)} \otimes F(\pi)$

- algebra structure from decomposition of tensor products
- coalgebra structure by transposition of operator composition

Actions of CQGs

If G is a cpt. group, we are also interested in symmetry given by G:
Group homomorphism $\alpha\colon G \to \mathrm{Aut}(X)$ for some structure X
If X is a cpt. top. space, the product map $G \times X \to X$ can be encoded by the $*$-hom. $C(X) \to C(G \times X) \simeq C(G) \otimes C(X)$.

Definition

An action of a CQG G on a C*-algebra B is given by a $*$-homomorphism $\beta\colon B \to C(G) \otimes B$, satisfying $(\Delta \otimes \iota)\beta = (\iota \otimes \beta)\beta$.

Example (Quantum homogeneous space by quantum subgroups)

Closed quantum subgroup "$G' \subset G$" given by a surjective Hopf algebra hom. $r\colon \mathscr{O}(G) \to \mathscr{O}(G')$
$\rightsquigarrow$ "Quotient" G/G', represented by

$$C(G)^{G'} = \{f \in C(G) \mid (\iota \otimes r)\Delta(f) = f \otimes 1\}$$

First ideas of duality

von Neumann algebraic model:

- $L^\infty(G)$: closure of $C(G)$ in the GNS representation of Haar state h
- coproduct $\Delta\colon L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G)$ as normal $*$-hom
- predual $L^1(G) = L^\infty(G)_*$ has 'algebra structure', which is

$$L^1(G) \supset \bigoplus_{\pi\colon \mathrm{Irr}\, G} B(H_\pi)$$

Ergodic action: $C(X) \to C(G) \otimes (X)$ such that $C(X)^G = \mathbb{C}$

- von Neumann algebraic model $L^\infty(X)$ from the invariant state
- coaction map $L^\infty(X) \to L^\infty(G) \bar{\otimes} L^\infty(X)$
- predual $L^1(X) = L^\infty(X)_*$ is a module over $L^1(G)$
  $\rightsquigarrow$ captures multiplicity of each $\pi \in \mathrm{Irr}\, G$ in $L^\infty(G)$

...towards categorical duality

We want to capture:

- the 'coalgebra' structure on $L^1(X)$ (corresponds to the algebra structure of $C(X)$)
- compatibility condition with the coalgebra structure of $L^1(G)$

What we do:

- take the category of G-equivariant modules on $C(X)$ ('equivariant vector bundles')
- $(\mathrm{Rep}\, G)$-*module category* structure of such modules

Why?: multiplicity can be captured by

$$\mathrm{Hom}_G(H_\pi, C(X)) \simeq \mathrm{Hom}_{G\text{-}C(X)}(H_\pi \otimes C(X), C(X))$$

Module category

From G-algebra $(B, \beta\colon B \to C(G) \otimes B)$:

- C∗-category $\mathscr{D}_B = \{$ fin. gen. proj. & G-equivariant B-modules $\}$
  $(E_B, \delta_E\colon E \to C(G) \otimes E) \in \mathscr{D}_B$
- tensor product of $E \in \mathscr{D}_B$, $\pi \in \operatorname{Rep} G$: modeled on $H_\pi \otimes E$, with left $C(G)$-comodule structure by $U^*_{21}\beta_{13}$
  $\rightsquigarrow$ right associative product $E \times (U_\pi \otimes U_\sigma) \simeq (E \times U_\pi) \times U_\sigma$

Definition (Yetter, Häring-Oldenburg, Ostrik, DC.-Y.)

Let $\mathscr{C}$ be a C*-tensor category. A *$\mathscr{C}$-module category* is a C*-category (involution on morphisms, C*-identity) $\mathscr{D}$ endowed with a bifunctor $\mathscr{D} \times \mathscr{C} \to \mathscr{D}, (X, U) \mapsto X \cdot U$, satisfying the associativity conditions.

Tannaka–Krein duality for actions

Theorem (DC.-Y., N., N.-Y.)

The following categories are equivalent:

1. *{G-actions on C^*-algebras; G-equivariant $*$-homs }*
2. $\{(\mathscr{D}, X) \mid \mathscr{D} : (\mathrm{Rep}\, G)$*-module category*, $X \in \mathscr{D}$; $(\mathrm{Rep}\, G)$*-module homomorphism functors* $\}$

Key construction for 2 → 1: Given $(\mathscr{D}, X)$,

- form $\mathscr{B}_X = \oplus_{\pi \in \mathrm{Irr}\, G} \bar{H}_\pi \otimes \mathscr{D}(X, X \cdot U_\pi)$
- algebra structure from decomposition of tensor product
- G-action on $\bar{H}_\pi$
- $*$-structure from action of $(R_\pi, \bar{R}_\pi)$ (Frobenius duality argument)
- completely positive map $\mathscr{B}_X \to \mathscr{D}(X, X)$ = summand for $\pi = 1$

Dictionary

- G-C*-algebras $\leftrightarrow$ $(\mathrm{Rep}\, G)$-module categories
- G-equivariant $*$-homomorphisms $\leftrightarrow$ $(\mathrm{Rep}\, G)$-homomorphism functors
- Coideals of $C(G)$ $\leftrightarrow$ factorization of $\mathrm{Rep}\, G \to \mathrm{Hilb}_\mathrm{f}$ as $(\mathrm{Rep}\, G)$-module categories
- Quantum subgroups of G $\leftrightarrow$ factorization of $\mathrm{Rep}\, G \to \mathrm{Hilb}_\mathrm{f}$ as tensor functors
- induction along $C(G) \to C(H)$ $\leftrightarrow$ restriction along $\mathrm{Rep}\, G \to \mathrm{Rep}\, H$
- restriction along $C(G) \to C(H)$ $\leftrightarrow$ induction along $\mathrm{Rep}\, G \to \mathrm{Rep}\, H$
- braided commutative Yetter-Drinfeld G-algebras $\leftrightarrow$ tensor functors from $\mathrm{Rep}\, G$

Comparison with other theories

Subfactors, Conformal Field Theory

Original structure → tensor category
module category → (easier to) classify
algebras in category

Quantum group actions

(easier to) classify
associated structures ← module category ← algebras in Rep G

- G-algebras: (morally) algebra objects in $\operatorname{Rep} G$
- (sometimes) easier to control module categories and tensor functors on $\operatorname{Rep} G$

Ergodic actions

From $(\operatorname{Rep} G)$-module category $\mathscr{D}$ and $X \in \mathscr{D}$:

- associated G-algebra $A_{\mathscr{D},X}$ has $A_{\mathscr{D},X}^G = \mathscr{D}(X,X)$
- $\dim A_{\mathscr{D},X}^G < \infty \Leftrightarrow X$ spans a *semisimple* category: $\dim \mathscr{D}(Y,Z) < \infty$

From semisimple C*-category $\mathscr{D}$ with $I = \operatorname{Irr} \mathscr{D}$:

- Rigid subcategory of $\operatorname{End}(\mathscr{D}) \equiv \operatorname{Hilb}_{\mathrm{f}}^{I \times I}$: row and column-finite $I \times I$-graded Hilbert spaces & uniformly bounded maps

$$(T\colon \mathscr{D} \to \mathscr{D}) \leftrightarrow (H_{st} = \mathscr{D}(T(X_s), X_t))_{s,t \in I}$$

- $(\operatorname{Rep} G)$-module structure on $\mathscr{D} \equiv$ C*-tensor functor $\operatorname{Rep} G \to \operatorname{End}(\mathscr{D})$

Ergodic actions of $\mathrm{SU}_q(2)$

Theorem (folklore)

When $q \in \mathbb{R}^\times$, the C^*-tensor category $\operatorname{Rep}\mathrm{SU}_q(2)$ is universally generated by the object $U_{1/2}$ and the morphism $R\colon 1 = U_0 \to U_{1/2} \otimes U_{1/2}$ satisfying

$$(\iota_{1/2} \otimes R^*)(R \otimes \iota_{1/2}) = -\operatorname{sgn}(q), \quad R^*R = \left|q + q^{-1}\right| \iota_0$$

$\operatorname{Rep}\mathrm{SU}_q(2)$: the Temperley-Lieb category $\mathscr{T}\mathscr{L}_{|q+q^{-1}|,\operatorname{sgn}(q)}$

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To classify C*-tensor functors $\operatorname{Rep}\mathrm{SU}_q(2) \to \mathrm{Hilb}_{\mathrm{f}}^{I\times I}$

- specify how $U_{1/2}$ is represented (as an $I \times I$-graded Hilbert space H_{**})
- specify how R is represented: $\mathbb{C} \to \bigoplus_t H_{st} \otimes H_{ts}$
- sort out relation in terms of associated antilinear maps $J_{st}\colon H_{st} \to H_{ts}$

$\rightsquigarrow$ weighted graphs from eigendecomposition of $J_{st}^* J_{st}$

Classification by graphs

Definition (Fair & balanced graph)

Let $t \in \mathbb{R}$. A *fair and balanced* t-graph is:

- an oriented graph $\Gamma = (V, E; s, t\colon E \to V)$
- 'weight' function $w\colon E \to \mathbb{R}_{>0}$ such that
  1. $|t| = \sum_{e\,:\,s(e)=v} w(e)$ for any $v \in V$
  2. $\exists$ ori. reversing involution $e \to \bar{e}$ on E such that $w(\bar{e}) = w(e)^{-1}$
  3. if $t > 0$, this involution is fixed-point free

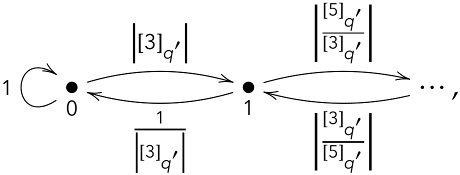
Theorem (DC.-Y.)

The following categories are equivalent:

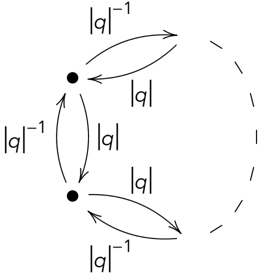
1. *Ergodic* $\mathrm{SU}_q(2)$*-algebras*
2. *'Fair and balanced'* $(q + q^{-1})$*-graphs, solution of certain quadratic equations associated with the graphs*

Fair and balanced $[2]_q$-graphs

Finding fair and balanced $[2]_q$-graphs:



A'_∞ (q negative, $q' = i(-q)^{1/2}$)



$A_n^{(1)}$ ($n+1$ vertices)

Poisson boundary

Izumi: noncommutative Poisson boundary of $\hat{G}$

- prob. measure μ on $\mathrm{Irr}\, G$ defines $P_\mu = \sum_{\pi\colon \mathrm{Irr}\, G} \mu(\pi)(\phi_\pi \otimes \iota)\hat{\Delta}$ on $\ell^\infty \hat{G} = \ell^\infty\text{-}\prod_\pi B(H_\pi)$
- Poisson boundary: $H^\infty(\hat{G};\mu) = \{T \in \ell^\infty \hat{G} \mid P_\mu(T) = T\}$ as an injective operator system
- (with Choi-Effros product) $H^\infty(\hat{G};\mu)$ is a Yetter-Drinfeld G-algebra

Theorem (N.-Y.; cf. Tomatsu, De Rijdt-Vander Vennet)

The $(\mathrm{Rep}\, G)$-module for $H^\infty(\hat{G};\mu)$ is given by a tensor functor $\Pi\colon \mathrm{Rep}\, G \to \mathscr{P}$ which only depends $\mathscr{C}$ and μ.
If G is coamenable, Π is universal among the tensor functors $F\colon \mathscr{C} \to \mathscr{D}$ such that $d(F(\pi)) = \dim H_\pi$.

Classifying $\mathrm{SU}(n)$-type quantum groups

Woronowicz's problem

classify the compact quantum groups G such that:

1. fusion ring isomorphism $\phi\colon \mathbb{N}[\operatorname{Irr} G] \to \mathbb{N}[\operatorname{Irr} \mathrm{SU}(n)]$
2. (classical) dimension equality $\dim H_\pi = \dim H_{\phi(\pi)}$

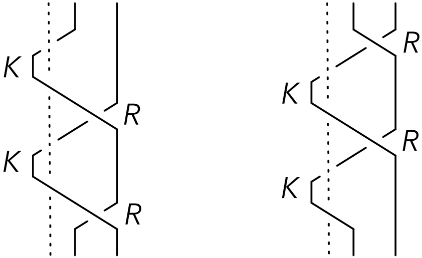
For $\operatorname{Rep} G = \operatorname{Rep} \mathrm{SU}_q(n)$, understand the strict quantizations of Poisson-Lie group structures on $\mathrm{SU}(n)$.

N.-Y.: complete answer for non-Kac ($S^2 \neq \iota$) case

- Kazhdan-Wenzl classification for the candidates of $\operatorname{Rep} G$ ($\mathbb{Z}/n$-parameter other than $q \in \mathbb{R}_{>0}$)
- classify fiber functors on $\operatorname{Rep} G$ through the Poisson boundary $\mathscr{P} \simeq \operatorname{Rep} T$ (maximal torus)

Reflection equation and module category

Cherednik: reflection equation $K_1RK_1R = RK_1RK_1$



- representation of R (braiding) and K (reflection operator): Brieskorn braid group
- 'pure Brieskorn braid group':

$$P(B_n) = \pi_1(\mathbb{C}^n \setminus (\text{type B hyperplane configuration}))$$

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$\operatorname{Rep} G$ braided & solution K in $B(H_U)$ $\rightsquigarrow$ new category by:

1. adding K to $\operatorname{End}_G(H_U)$,
2. generating a right $(\operatorname{Rep} G)$-module category

Example ($G = \mathrm{SU}_q(2)$)

Nonstandard Podleś spheres from $U = U_{1/2}$,

$$K = \begin{pmatrix} 0 & 1 \\ 1 & q^s - q^{-s} \end{pmatrix}$$

Cyclotomic Knizhnik-Zamolodchikov equation

Leibman, Golubeva-Leksin: the reflection operators for $\mathrm{SU}(2)$ can be obtained from the monodromy of

$$\frac{\partial v}{\partial z_i} = \left(\hbar \left(\sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} + \frac{t^+_{ij}}{z_i + z_j} \right) + s \frac{(E+F)_i}{z_i} \right) v$$

for

- invariant 2-tensor $t = \frac{1}{2} H \otimes H + E \otimes F + F \otimes E$
- $t^+ = (\sigma \otimes \iota)(t)$ with Chevalley involution σ

with De Commer, in progress

Generalization to $\mathrm{SU}(n)$: for a *cyclotomic* Knizhnik-Zamolodchikov equation (after Enriquez) on $\mathbb{C}^n \setminus \{z_i = \zeta_n^k z_j, z_i = 0\} \rightsquigarrow$ quantizations of nonstandard Poisson structures on $\mathrm{SU}(n)/T$.