On reconstruction program

(Based on a paper to appear in the special issue of CMP in memory of R. Haag)

Feng Xu

Dept of Math

UCR

OUTLINE

1. MOTIVATION

OUTLINE

1. Motivation
2. Conformal nets, rationality, orbifolds, induction

OUTLINE

1. MOTIVATION
2. CONFORMAL NETS, RATIONALITY, ORBIFOLDS, INDUCTION
   - Möbius covariant net

OUTLINE

1. MOTIVATION
2. CONFORMAL NETS, RATIONALITY, ORBIFOLDS, INDUCTION
   - Möbius covariant net
   - Complete rationality

OUTLINE

1. MOTIVATION
2. CONFORMAL NETS, RATIONALITY, ORBIFOLDS, INDUCTION
   - Möbius covariant net
   - Complete rationality
   - Subfactors from conformal nets

OUTLINE

OUTLINE

1. MOTIVATION
2. CONFORMAL NETS, RATIONALITY, ORBIFOLDS, INDUCTION
   - Möbius covariant net
   - Complete rationality
   - Subfactors from conformal nets
3. BANTAY'S ORBIFOLD COVARIANCE PRINCIPLE
4. $3^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ COME FROM CFT

OUTLINE

1. Motivation
2. Conformal nets, rationality, orbifolds, induction
   - Möbius covariant net
   - Complete rationality
   - Subfactors from conformal nets
3. Bantay's Orbifold Covariance Principle
4. $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ come from CFT
5. A puzzle

OUTLINE

OUTLINE

OUTLINE

Motivation

Motivation

Recently there have been many new subfactors constructed by many people (S. Morrison, N. Snyder, D. Penny, Z. Liu , E.Peters, P. Grossmann, ...) using Planar Algebras pioneered by V. Jones. M. Izumi has constructed a large class of subfactors generalizing Haagerup subfactors using Cuntz algebras. T. Gannon and D. Evans have provided evidence suggesting that Haagerup subfactor may come from CFT. V. Jones has devised a renormalization program based on planar algebras as an attempt to show that all finite depth subfactors are related to CFT, i.e., the double of a finite depth subfactor should be related CFT. More generally, the program is the following: given a unitary Modular Tensor Category (MTC) $\mathcal{M}$, can we construct a CFT whose representation category is isomorphic to $\mathcal{M}$?

Motivation

Motivation

A few words on MTC: this is a tensor category with more interesting braiding structures than say reps of a compact Lie group. It is motivated by Rehsethkin-Turaev-Witten 3-manfold invaraints/TQFT in 3 dimensions. Representations of quantum groups at roots of unity provide interesting classes of MTC. Another rich source of such examples come from double of finite depth subfactors. Doplicher-Roberts reconstruction theorem roughly states that a symmetric tensor category comes from reps of compact groups. The question is whether one can have a general result for MTC.

Motivation

Motivation

We shall call such a program “reconstruction program”, analogue to a similar program in higher dimensions by Doplicher-Roberts ,even though in low dimensions the (quantum) symmetries are much richer. M. Bischoff has shown that this can be done for all subfactors with index less than 4. In view of these recent developments, it is natural to examine subfactors from known CFT. In fact, it is already known that so called 2221 subfactor are related to subfactor from conformal inclusions, and it is an interesting question to see if any of these recently constructed subfactors are related to CFT. We do find a few more interesting examples.

Motivation

Motivation

Another motivation for our work is that it is clear from the work of Marcel that holomorphic CFT play an important role in the reconstruction program. Holomorphic CFT have trivial MTC, hence any reconstruction program would have to somehow modulo tensor product with holomorphic CFT.

We construct new subfactors from holomorphic CFT with central charge 24 based on recent work.

Motivation

Motivation

A major progress on reconstruction program would be to identify the origin of Haagerup subfactor in CFT. Despite the evidence , this remains a challenging question. On the other hand, in (one version of) Doplicher-Roberts Theorem a group is constructed first, and then a suitable local net is chosen for the group to act on. In other words the net is not constructed directly. It takes lots of efforts to construct conformal net or chiral algebra from MTC which are not related to groups , even in concrete examples such as the examples we will discuss later. Since conformal net seems to contain more than MTC, it is possible that a general reconstruction program may not work. We will discuss a source of such possible obstructions.

Motivation

The key point: Reconstruction program may or may not work, but there are numerous questions/results that come out of it which may not even have anything to do with the program (e.g.: Vaughan's construction of reps of Thompson group among other things), as such it will keep us busy for some time.
Obvious questions: Does Haagerup appear in CFT? Can one construct $D^{\omega}(G)$ for all ω in CFT (True for $\omega = 1$)?

OUTLINE

OUTLINE

Möbius covariant net

This is an adaption of DHR analysis to chiral CFT which is most suitable for our purposes.

By an *interval* we shall always mean an open connected subset I of S^1 such that I and the interior I' of its complement are non-empty. We shall denote by $\mathcal{I}$ the set of intervals in S^1.

A *Möbius covariant* net $\mathcal{A}$ of von Neumann algebras on the intervals of S^1 is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$ to the von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the following:

Möbius covariant

Definition(Möbius covariant net)

Möbius covariant

Definition(Möbius covariant net)

A. Isotony;

Möbius covariant

Definition(Möbius covariant net)

A. Isotony;

B. Möbius covariance;

Möbius covariant

Definition (Möbius covariant net)

A. Isotony;

B. Möbius covariance;

C. Positivity of the energy;

Möbius covariant

Definition (Möbius covariant net)

A. Isotony;

B. Möbius covariance;

C. Positivity of the energy;

D. Locality;

Möbius covariant

Definition(Möbius covariant net)

A. Isotony;

B. Möbius covariance;

C. Positivity of the energy;

D. Locality;

E. Existence of the vacuum;

Möbius covariant

Definition(Möbius covariant net)

A. Isotony;

B. Möbius covariance;

C. Positivity of the energy;

D. Locality;

E. Existence of the vacuum;

F. Uniqueness of the vacuum (or irreducibility);

Möbius covariant

Definition(Möbius covariant net)

A. Isotony;

B. Möbius covariance;

C. Positivity of the energy;

D. Locality;

E. Existence of the vacuum;

F. Uniqueness of the vacuum (or irreducibility);

G. Conformal covariance.

A. ISOTONY

If I_1, I_2 are intervals and $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)\,.$$

A. Isotony

If I_1, I_2 are intervals and $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \,.$$

B. Möbius covariance

There is a nontrivial unitary representation U of $\mathbf{G}$ (the universal covering group of $PSL(2, \mathbf{R})$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI) \,, \qquad g \in \mathbf{G}, \quad I \in \mathcal{I} \,.$$

The group $PSL(2, \mathbf{R})$ is identified with the Möbius group of S^1, i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore $\mathbf{G}$ has a natural action on S^1.

C. Positivity of the energy

The generator of the rotation subgroup $U(R)(\cdot)$ is positive. Here $R(\vartheta)$ denotes the (lifting to $\mathbf{G}$ of the) rotation by an angle ϑ.

C. Positivity of the energy

The generator of the rotation subgroup $U(R)(\cdot)$ is positive. Here $R(\vartheta)$ denotes the (lifting to **G** of the) rotation by an angle ϑ.

D. Locality

If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $A(I)$ commute.
The lattice symbol $\vee$ will denote 'the von Neumann algebra generated by'.

C. Positivity of the energy

The generator of the rotation subgroup $U(R)(\cdot)$ is positive. Here $R(\vartheta)$ denotes the (lifting to **G** of the) rotation by an angle ϑ.

D. Locality

If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $A(I)$ commute.
The lattice symbol $\vee$ will denote 'the von Neumann algebra generated by'.

E. Existence of the vacuum

There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$-invariant and cyclic for $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.

C. Positivity of the energy

The generator of the rotation subgroup $U(R)(\cdot)$ is positive. Here $R(\vartheta)$ denotes the (lifting to $\mathbf{G}$ of the) rotation by an angle ϑ.

D. Locality

If I_0, I are disjoint intervals then $\mathcal{A}(I_0)$ and $A(I)$ commute.
The lattice symbol $\vee$ will denote 'the von Neumann algebra generated by'.

E. Existence of the vacuum

There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$-invariant and cyclic for $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$.

F. Uniqueness of the vacuum (or irreducibility)

The only $U(\mathbf{G})$-invariant vectors are the scalar multiples of Ω.

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:
G. Conformal covariance There exists a projective unitary representation U of $Diff(S^1)$ on $\mathcal{H}$ extending the unitary representation of **G** such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in Diff(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in Diff(I'), \end{aligned}$$

where $Diff(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $Diff(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$.

Möbius covariant *representation*

Assume $\mathcal{A}$ is a Möbius covariant net. A Möbius covariant *representation* π of $\mathcal{A}$ is a family of representations π_I of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation U_π of the covering group **G** of $PSL(2, \mathbf{R})$, with *positive energy*, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$I \supset \bar{I} \Rightarrow \pi_{\bar{I}} \mid_{\mathcal{A}(I)} = \pi_I \quad \text{(isotony)}$$

$$\mathrm{ad} U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \mathrm{ad} U(g) \text{(covariance)} .$$

A unitary equivalence class of Möbius covariant representations of $\mathcal{A}$ is called *superselection sector*.

Connes's fusion

The composition of two superselection sectors are known as Connes's fusion . The composition is manifestly unitary and associative, and this is one of the most important virtues of the above formulation. The main question is to study all superselection sectors of $\mathcal{A}$ and their compositions. Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let G be a group. Let $V : G \to U(\mathcal{H})$ be a faithful unitary representation of G on $\mathcal{H}$. If $V : G \to U(\mathcal{H})$ is not faithful, we can take $G' := G/kerV$ and consider G' instead.

Proper Action

We say that G acts properly on $\mathcal{A}$ if the following conditions are satisfied:
(1) For each fixed interval I and each $s \in G$,
$\alpha_s(a) := V(s)aV(s^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;
(2) For each $s \in G$, $V(s)\Omega = \Omega, \forall s \in G$. We will denote by $\mathrm{Aut}(\mathcal{A})$ all automorphisms of $\mathcal{A}$ which are implemented by proper actions.
Define $\mathcal{A}^G(I) := \mathcal{B}(I)P_0$ on $\mathcal{H}_0$, where $\mathcal{H}_0$ is the space of G invariant vectors and P_0 is the projection onto $\mathcal{H}_0$. The unitary representation U of **G** on $\mathcal{H}$ restricts to a unitary representation (still denoted by U) of **G** on $\mathcal{H}_0$. Then :

Proper Action

We say that G acts properly on $\mathcal{A}$ if the following conditions are satisfied:
(1) For each fixed interval I and each $s \in G$,
$\alpha_s(a) := V(s)aV(s^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;
(2) For each $s \in G$, $V(s)\Omega = \Omega, \forall s \in G$. We will denote by $\mathrm{Aut}(\mathcal{A})$ all automorphisms of $\mathcal{A}$ which are implemented by proper actions.
Define $\mathcal{A}^G(I) := \mathcal{B}(I)P_0$ on $\mathcal{H}_0$, where $\mathcal{H}_0$ is the space of G invariant vectors and P_0 is the projection onto $\mathcal{H}_0$. The unitary representation U of **G** on $\mathcal{H}$ restricts to a unitary representation (still denoted by U) of **G** on $\mathcal{H}_0$. Then :

Proposition

The map $I \in \mathcal{I} \to \mathcal{A}^G(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by U) of **G** on $\mathcal{H}_0$ is an irreducible conformal net. We say that $\mathcal{A}^G$ is obtained by *orbifold* construction from $\mathcal{A}$.

OUTLINE

Complete rationality

We first recall some definitions from KLM . By an interval of the circle we mean an open connected proper subset of the circle. If I is such an interval then I' will denote the interior of the complement of I in the circle. We will denote by $\mathcal{I}$ the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1.. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible conformal net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_2), \hat{\mathcal{A}}(E) := (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I.

Definition

$\mathcal{A}$ is said to be completely rational, or μ-rational, if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. $\mathcal{A}$ is holomorphic if $\mu_{\mathcal{A}} = 1$.
The following theorem is proved :

Definition

$\mathcal{A}$ is said to be completely rational, or μ-rational, if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. $\mathcal{A}$ is holomorphic if $\mu_{\mathcal{A}} = 1$.
The following theorem is proved :

Theorem

Let $\mathcal{A}$ be an irreducible conformal net and let G be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational. Then:
(1): $\mathcal{A}^G$ is completely rational and $\mu_{\mathcal{A}^G} = |G|^2 \mu_{\mathcal{A}}$;
(2): There are only a finite number of irreducible covariant representations of $\mathcal{A}^G$ and they give rise to a unitary modular category.

For a modular tensor category $\mathcal{M}$, we define $\mathrm{Aut}(\mathcal{M})$ to be the collection of automorphisms of $\mathcal{M}$. One can do equivariantization of $\mathcal{M}$ with elements of $\mathrm{Aut}(\mathcal{M})$. We define $\mathrm{Out}(\mathcal{M})$ to be $\mathrm{Aut}(\mathcal{M})/N$ where N is the normal subgroup consisting these automorphisms fixing the isomorphism classes of each simple objects in $\mathcal{M}$. When $\mathcal{M}$ is the representation category of $\mathrm{Rep}(\mathcal{A})$ for a complete rational net $\mathcal{A}$, it may happen that $\mathrm{Out}(\mathcal{M})$ contain elements which do not come from $\mathrm{Aut}(\mathcal{A})$. Will have an example later.

Let $\mathcal{B}$ be a Möbius covariant net and $\mathcal{A}$ a subnet. We assume that $\mathcal{A}$ is strongly additive and $\mathcal{A} \subset \mathcal{B}$ has finite index. Fix an interval $I_0 \in \mathcal{I}$ and canonical endomorphism γ associated with $\mathcal{A}(I_0) \subset \mathcal{B}(I_0)$. Then we can choose for each $I \subset \mathcal{I}$ with $I \supset I_0$ a canonical endomorphism γ_I of $\mathcal{B}(I)$ into $\mathcal{A}(I)$ in such a way that the restriction of γ_I on $\mathcal{B}(I_0)$ is γ_{I_0} and ρ_{I_1} is the identity on $\mathcal{A}(I_1)$ if $I_1 \in \mathcal{I}_0$ is disjoint from I_0, where $\rho_I \equiv \gamma_I$ restricted to $\mathcal{A}(I)$.

Given a DHR endomorphism λ of $\mathcal{A}$ localized in I_0, the inductions $\alpha_\lambda, \alpha_\lambda^-$ of λ are the endomorphisms of $\mathcal{B}(I_0)$ given by

$$\alpha_\lambda \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \varepsilon(\lambda, \rho) \cdot \lambda \cdot \gamma \ , \alpha_\lambda^- \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \tilde{\varepsilon}(\lambda, \rho) \cdot \lambda \cdot \gamma$$

where ε (resp. $\tilde{\varepsilon}$) denotes the right braiding .
Note that $\mathrm{Hom}(\alpha_\lambda, \alpha_\mu) =: \{x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0)\}$ and $\mathrm{Hom}(\lambda, \mu) =: \{x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0)\}$.

Given a DHR endomorphism λ of $\mathcal{A}$ localized in I_0, the inductions $\alpha_\lambda, \alpha_\lambda^-$ of λ are the endomorphisms of $\mathcal{B}(I_0)$ given by

$$\alpha_\lambda \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \varepsilon(\lambda, \rho) \cdot \lambda \cdot \gamma \;, \alpha_\lambda^- \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \tilde{\varepsilon}(\lambda, \rho) \cdot \lambda \cdot \gamma$$

where ε (resp. $\tilde{\varepsilon}$) denotes the right braiding .
Note that $\mathrm{Hom}(\alpha_\lambda, \alpha_\mu) =: \{x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0)\}$ and $\mathrm{Hom}(\lambda, \mu) =: \{x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0)\}$.

Theorem

Given a DHR endomorphism λ of $\mathcal{A}$ localized in I_0, the inductions $\alpha_\lambda, \alpha_\lambda^-$ of λ are the endomorphisms of $\mathcal{B}(I_0)$ given by

$$\alpha_\lambda \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \varepsilon(\lambda, \rho) \cdot \lambda \cdot \gamma \;, \alpha_\lambda^- \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \tilde{\varepsilon}(\lambda, \rho) \cdot \lambda \cdot \gamma$$

where ε (resp. $\tilde{\varepsilon}$) denotes the right braiding .
Note that $\mathrm{Hom}(\alpha_\lambda, \alpha_\mu) =: \{x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0)\}$ and $\mathrm{Hom}(\lambda, \mu) =: \{x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0)\}$.

Theorem

- (1) $[\lambda] \to [\alpha_\lambda]$, are ring homomorphisms;

Given a DHR endomorphism λ of $\mathcal{A}$ localized in I_0, the inductions $\alpha_\lambda, \alpha_\lambda^-$ of λ are the endomorphisms of $\mathcal{B}(I_0)$ given by

$$\alpha_\lambda \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \varepsilon(\lambda, \rho) \cdot \lambda \cdot \gamma \ , \alpha_\lambda^- \equiv \gamma^{-1} \cdot \mathrm{Ad}\, \tilde{\varepsilon}(\lambda, \rho) \cdot \lambda \cdot \gamma$$

where ε (resp. $\tilde{\varepsilon}$) denotes the right braiding .
Note that $\mathrm{Hom}(\alpha_\lambda, \alpha_\mu) =: \{x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0)\}$ and $\mathrm{Hom}(\lambda, \mu) =: \{x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0)\}$.

Theorem

- (1) $[\lambda] \to [\alpha_\lambda]$, are ring homomorphisms;
- (2) $\langle \alpha_\lambda, \alpha_\mu \rangle = \langle \lambda\rho, \mu \rangle$.

OUTLINE

There are three general classes of subfactors:
(i) If π is a covariant representation of $\mathcal{A}$, then by locality we have the following subfactor $\pi_I(\mathcal{A}(I)) \subset \pi_{I'}(\mathcal{A}(I'))'$. These are known as Jones-Wassermann subfactors;
(ii) Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1. Suppose that I_1, I_2 are disjoint and let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

$$\mathcal{A}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_2), \hat{\mathcal{A}}(E) := (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))'.$$

Note that by locality we have subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. These are Jones-Wassermann subfactors for disjoint intervals.
(iii) If $\mathcal{B} \subset \mathcal{A}$ is a subnet, then we have subfactors $\mathcal{B}(I) \subset \mathcal{A}(I), \forall I$, and under certain conditions we get irreducible finite index subfactors, and in such cases we can induce a representation of $\mathcal{B}$ to a soliton of $\mathcal{A}$: i.e., it is only a representation of net $\mathcal{A}$ restricted to a punctured circle which is isomorphic to the real line. By locality such solitons will also give subfactors .

There are close relations between the index of subfactors in (i) and (ii). This is related to the notion of complete rationality in KLM. As for subfactors coming from (iii), a notable class of such examples come from conformal inclusions and simple current extensions. For an example, one can construct all subfactors of index less than 4 with principal graphs of type D, E from such constructions. Subfactors induced from simple current extensions also provide examples with interesting lattice of intermediate subfactors.

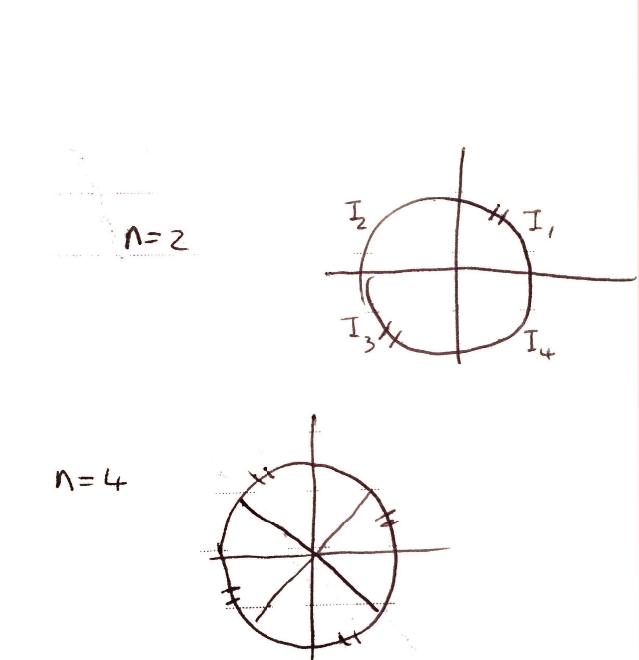
OUTLINE

Bantay's Orbifold Covariance Principle

Given a rational conformal net $\mathcal{A}$ and any finite group G. Assume that $G \leq S_n$ where S_n is a symmetric group on n letters. Note that S_n acts on $\mathcal{A}^{\otimes^n}$ by permuting tensors, and by Th. 3 the fixed point algebra $(\mathcal{A}^{\otimes^n})^G$ is also rational. A particular interesting case is when G is generated by an cycle, in this case one can relate the chiral quantities of $(\mathcal{A}^{\otimes^n})^G$ with that of $\mathcal{A}$, and this leads to many interesting equations due to the rationality of $(\mathcal{A}^{\otimes^n})^G$. In fact P. Bantay proposes that any MTC $\mathcal{M}$ will have an associated class of MTCs coming from the orbifolds as in the case of conformal nets.

Following Bantay, we shall say such MTC verify Orbifold Covariance Principle. This suggests the following possible obstructions to reconstruction program: if one can find a MTC which does not verify Orbifold Covariance Principle, then such a MTC will not come from CFT. For an example, if one can find a MTC for which some of those identities are not verified, then such a MTC will not come from CFT. However it is not clear which identity to check, and in fact some properties of MTC such as certain equations among chiral identities which may follow from Orbifold Covariance Principle are in fact proved By Richard Ng without constructing the associated class of oribifold MTCs.

Let us consider one case when n=2. This was considered in Kac-Longo-Xu (2005):
Given a rational net $\mathcal{A}$, let $P = T^{\frac{1}{2}} S T^2 S T^{\frac{1}{2}}$.
Then it follows that $\frac{1}{2}(N_{\lambda_1\lambda_1\lambda_2\lambda_3} \pm \sum_\mu \frac{S_{\lambda_1\mu} P_{\lambda_2\mu} P_{\lambda_3\mu}}{S_{1\mu}^2})$ is the fusion coefficients of some twisted representation of $\mathcal{A} \otimes \mathcal{A}$, therefore is a non-negative integer.
Richard Ng et. al proved that it is a non-negative integer for general modular tensor category, but it is not clear what these integers are. They should be fusion coefficients of some associated MTC if reconstruction works.
Hence if you take your favorite modular tensor category $\mathcal{M}$, there should be an infinite sequence of MTC $\mathcal{M}_n$ corresponding to cyclic permutations if reconstruction works. Can we do this for the double of Izumi-Haagerup and near group subfactors? We can if they come from CFT.


n=2
I2
I1
I3
I4
n=4

Another striking fact is that
$[\pi^n_{1,\{0,1,\ldots,n-1\}}] = \bigoplus_{\lambda_1,\ldots,\lambda_n} M_{\lambda_1,\ldots,\lambda_n}[(\lambda_1,\ldots,\lambda_n)]$ where
$M_{\lambda_1,\ldots,\lambda_n} := \sum_\lambda S_{1,\lambda}^{2-2g} \prod_{1\leq i\leq n} \frac{S_{\lambda_i,\lambda}}{S_{1,\lambda}}$ with $g = \frac{(n-1)(n-2)}{2}$, and $\pi_{1,\{0,1,\ldots,n-1\}}$ is the soliton. Here $M_{\lambda_1,\ldots,\lambda_n}$ is the dimension of genus g conformal block with n marked points. $\pi_{1,\{0,1,\ldots,n-1\}}$ is the Jones-Wassermann subfactors for n disjoint intervals. when $n = 2$ this is Longo-Rehren.
From this description it is easy to see that the Jones-Wassermann subfactors for n disjoint intervals are isomorphic to its dual as subfactors.
Here is a proof using orbifold theory: the dual of $\pi_{1,\{0,1,\ldots,n-1\}}$ is $\pi_{1,\{n-1,n-2,\ldots,0\}}$, but $\{n-1, n-2, \ldots, 0\}$ is conjugate to $\{0, 1, \ldots, n-1\}$ in S_n via $g(i) = n - i - 1, i = 0, 1, \ldots, n - 1$, hence
$\pi_{1,\{n-1,n-2,\ldots,0\}} \simeq g\pi_{1,\{0,1,\ldots,n-1\}}g^{-1}$.

Based on reconstruction program, one can conjecture that same results hold for MTC. It is true! (joint work in progress with Zhengwei Liu.) When n=2 this is just Longo-Rehren construction, but when $n > 2$ braiding is crucial for our construction. Hence there are Jones-Wassermann subfactors for double of Haagerup!
Physicists like Greg Moore has done what they called 2nd string quantization to add up all the n's. Is it possible to combine these n-interval subfactors and consider large n limit to construct CFT?

OUTLINE

Note that Haagerup subfactor is $3^{\mathbb{Z}_3}$. We will consider $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$, and show that (1): it comes from CFT in an interesting way ; (2): even in this case there are puzzling questions about how to find CFT related to orbifold of $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$.

The branching rules for the conformal inclusion $SU(5)_5 \subset Spin(24)_1$ are given by:

$$[1] = ([(0,0,0,0)]) + ([(0,1,0,2)]), [\sigma_1] = [(1,1,1,1)] + ([(1,0,0,1)]),$$
$$[\sigma_2] = [\sigma_3] = 2[(1,1,1,1)]$$

where $\sigma_1, \sigma_2, \sigma_3$ denote the vector and spinor representations which form $\mathbb{Z}_2 \times \mathbb{Z}_2$ under fusion, and $([(\lambda_1, \lambda_2, \lambda_3, \lambda_4)])$ denotes the orbit of sector $[(\lambda_1, \lambda_2, \lambda_3, \lambda_4)]$ under the center $\mathbb{Z}_4$ of $SU(4)$. Here by slightly abusing notations the left hand side of the equations above are understood as the restrictions from representations of $Spin(24)_1$ to $SU(5)_5$. The fusion of the adjoint representation is given by

By using the above we have

$$\langle \alpha_{v_0}, \alpha_{v_0} \rangle = 3.$$

It follows that $[\alpha_{v_0}] = [\sigma_1] + [A] + [\sigma_1 A]$ where A is irreducible. Since $[\sigma_1 \alpha_v] = [\alpha_v]$, α_v has an intermediate subfactor denoted by ρ_1 and $[\rho_1 \bar{\rho}_1] = [1] + [A]$.
We have the following fusion rules:

$$[A^2] = [1] + [A\sigma_1] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3]$$

Let $\mathcal{A}_1$ be the simple current extensions of $\mathcal{A}_{SU(5)_5}$. Note that $\mathcal{A}_{SU(5)_5} \subset \mathcal{A}_1 \subset \mathcal{A}_{Spin(24)_1}$. Note that color 0 irreducible representations of $\mathcal{A}_{SU(5)_5}$ induce to DHR representations of $\mathcal{A}_1$. We enumerate these 10 irreducible representations of $\mathcal{A}_1$ as follows: $1, z_1, z_2, z_3, \mathrm{ad}, b_i, 1 \leq i \leq 5$ where 1 is the vacuum representation, z_1, z_2, z_3 are induced from $(0,0,2,1), (0,1,0,2)$ and $(0,1,1,0)$ respectively , ad is induced from $(1,0,0,1)$, and $b_i, 1 \leq i \leq 5$ are irreducible components of the representation induced from $(1,1,1,1)$. Let $g \in \mathbb{Z}_5$ be the generator of $\mathrm{Aut}(\mathcal{A}_1)$ due to the simple current extension. We have $[\mathrm{Ad}_g\, b_i] = [b_{i+1}], 1 \leq i \leq 5$ and Ad_g fix the rest 5 irreducible representations of $\mathcal{A}_1$. We consider the induction from $\mathcal{A}_1$ to $\mathcal{A}_{Spin(24)_1}$. The branching rules of the inclusion $\mathcal{A}_1 \subset \mathcal{A}_{Spin(24)_1}$ are given by :

$$[\Lambda_0] = [1] + [z_2], [\sigma_1] = [b_1] + [\mathrm{ad}], [\sigma_2] = [b_2] + [b_3], [\sigma_3] = [b_4] + [b_5]$$

The following can be determined from the fusion rules and Th above:

$$
\begin{aligned}
&[\alpha_{\mathrm{ad}}] = [\sigma_1] + [A] + [\sigma_1 A], [\alpha_{b_1}] = [\sigma_1] + [\sigma_2 A] + [\sigma_3 A], \\
&[\alpha_{b_2}] = [\sigma_2] + [A] + [\sigma_2 A], [\alpha_{b_3}] = [\sigma_2] + [\sigma_1 A] + [\sigma_3 A], \\
&[\alpha_{b_4}] = [\sigma_3] + [A] + [\sigma_3 A], [\alpha_{b_5}] = [\sigma_3] + [\sigma_1 A] + [\sigma_2 A] \\
&[\alpha_{z_1}] = [\alpha_{z_3}] = [A] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3], \\
&[\alpha_{z_2}] = [1] + [A] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3]
\end{aligned}
$$

We see the intermediate subfactor, denoted by ρ_1 above, is exactly $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactor constructed by Izumi . The double of this subfactor is computed by Grossmann and Izumi. We can now see the double is $\mathrm{Rep}\mathcal{A}_1 \otimes \mathrm{Rep}\mathcal{B}^{\mathrm{rev}}$. Consider inclusions $\mathcal{B}\otimes\mathcal{B} \subset \mathcal{A}_{Spin(48)_1} \subset \mathcal{B}_1$ where $\mathcal{B}_1$ is $\mathbb{Z}_2$ extension of $\mathcal{A}_{Spin(48)_1}$ which is holomorphic. Inspecting the spectrum of $\mathcal{B}\otimes\mathcal{B} \subset \mathcal{B}_1$ we see that the inclusion $\mathcal{B} \subset \mathcal{B}_1$ is normal, and we conclude that $\mathrm{Rep}\mathcal{B}^{\mathrm{rev}} \simeq \mathrm{Rep}\mathcal{B}$ as braided tensor categories. So we have shown the following theorem:

We see the intermediate subfactor, denoted by ρ_1 above, is exactly $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactor constructed by Izumi . The double of this subfactor is computed by Grossmann and Izumi. We can now see the double is $\mathrm{Rep}\mathcal{A}_1 \otimes \mathrm{Rep}\mathcal{B}^{\mathrm{rev}}$. Consider inclusions $\mathcal{B}\otimes\mathcal{B} \subset \mathcal{A}_{Spin(48)_1} \subset \mathcal{B}_1$ where $\mathcal{B}_1$ is $\mathbb{Z}_2$ extension of $\mathcal{A}_{Spin(48)_1}$ which is holomorphic. Inspecting the spectrum of $\mathcal{B}\otimes\mathcal{B} \subset \mathcal{B}_1$ we see that the inclusion $\mathcal{B} \subset \mathcal{B}_1$ is normal, and we conclude that $\mathrm{Rep}\mathcal{B}^{\mathrm{rev}} \simeq \mathrm{Rep}\mathcal{B}$ as braided tensor categories. So we have shown the following theorem:

Theorem

The double of $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ subfactor is the category of representations of $\mathcal{A}_1 \otimes \mathcal{A}_{Spin(24)_1}$ and verifies the Orbifold Covariance Principle.

OUTLINE

In a paper by Pinhas Grossmann and M. Izumi, a $\mathbb{Z}_3$ equivarization of $3^{\mathbb{Z}_2\times\mathbb{Z}_2}$ is found. However there is no such $\mathbb{Z}_3$ in $\mathrm{Aut}(\mathcal{A}_1)$ which lifts to $\mathbb{Z}_3$ on $\mathcal{A}_{Spin(24)_1}$ permuting σ_i. If reconstruction program works, there must be a conformal net $\mathcal{B}_1$ with $\mathbb{Z}_3 \in \mathrm{Aut}(\mathcal{B}_1)$ such that the category of representations of $\mathcal{B}_1^{\mathbb{Z}_3}$ is braided equivalent to the category of representations of $\mathcal{A}_1 \otimes \mathcal{A}_{Spin(24)_1}$. It is also an interesting question to see if one can relate $\mathcal{A}_1$ to the double of $D2D$ subfactor.
Here is a simple example: $\mathrm{Rep}(\mathcal{A}_{Spin(24)_1}) \sim \mathrm{Rep}(\mathcal{A}_{Spin(8)_1})$, $\mathcal{A}_{Spin(8)_1}$ has triality while $\mathcal{A}_{Spin(24)_1}$ does not. The puzzle above ask for a highly nontrivial generalization of this simple case.

OUTLINE

Among subfactors coming from $SU(4)_6 \subset SU(10)_1$, there is a closed fusion tensor category generated by A and η with

$$[A\eta] = [\eta A], [\eta^5] = [1], [A^2] = [1] + [\eta A] + [\eta^2 A] + [\eta^3 A] + [\eta^4 A]$$

This is a quadratic category which is not in M. Izumi's list but should be accessible by his techniques. Is this part of an infinite series? i.e., are there such quadratic category for abelian groups such as $\mathbb{Z}_n$ for infinitely many n's? Apart from $n = 5$ example above one can also check that $n = 2, n = 3$ come from CFT.
Also note that complex conjugation acts on $SU(4)_6 \subset SU(10)_1$, and we get inclusions $\mathcal{A}^{\mathbb{Z}_2}_{SU(4)_6} \subset \mathcal{A}^{\mathbb{Z}_2}_{SU(10)_1}$. This is related to $\mathbb{Z}_2$ equivariantization of the fusion category above.

OUTLINE

In 1996 A. Schellekens gave a conjectured list of 71 holomorphic CFT of central charge 24. Many examples on this list have been realized . One of the special property of such holomorphic CFT $\mathcal{B}$ is that if the weight 1 subspace is non-zero, then it generates a Kac-Moody subnet $\mathcal{A} \subset \mathcal{B}$ such that the spectrum of $\mathcal{A} \subset \mathcal{B}$ is finite. Hence we can consider subfactors associated with such $\mathcal{A} \subset \mathcal{B}$. To apply the results of induction we will also need that the index $[\mathcal{B} : \mathcal{A}] < \infty$. We select a few examples from VOA which have been constructed recently, in a similar order as in the previous section.

Example 1. This case is constructed by C. Lam and his collaobrators. We have $[\mathcal{B} : \mathcal{A}] < \infty$, and so we have a new finite index subfactor. If we take the commutant of the subnet generated by $SU(2)_1^2$, then we see that we get a local extension of $\mathcal{A}_{Spin(12)_5} \subset \mathcal{B}_1$ whose spectrum is given by

$$[1] + [010002] + [010020] + [100111] + [002000] + [200100].$$

Example 1. This case is constructed by C. Lam and his collaobrators. We have $[\mathcal{B} : \mathcal{A}] < \infty$, and so we have a new finite index subfactor. If we take the commutant of the subnet generated by $SU(2)_1^2$, then we see that we get a local extension of $\mathcal{A}_{Spin(12)_5} \subset \mathcal{B}_1$ whose spectrum is given by

$$[1] + [010002] + [010020] + [100111] + [002000] + [200100].$$

Theorem

There is a local extension $\mathcal{A}_{Spin(12)_5} \subset \mathcal{B}_1$ whose spectrum is given by

$$[1] + [010002] + [010020] + [100111] + [002000] + [200100]$$

We can now consider the induced subfactor α_v where v denotes the vector representation of $\mathcal{A}_{Spin(12)_5}$. The centralizer algebras $\mathrm{Hom}(\alpha_v^n, \alpha_v^n), n \geq 0$ will contain BMW algebra . We know that $d_v = 7.7396813$. It is an interesting question to analyze the nature of such algebras.

We can now consider the induced subfactor α_v where v denotes the vector representation of $\mathcal{A}_{Spin(12)_5}$. The centralizer algebras $\mathrm{Hom}(\alpha_v^n, \alpha_v^n), n \geq 0$ will contain BMW algebra . We know that $d_v = 7.7396813$. It is an interesting question to analyze the nature of such algebras.

Remark

The above local extension of $\mathcal{A}_{Spin(12)_5} \subset \mathcal{B}_1$ should be the mirror extension associated with the conformal inclusion $Spin(5)_{12} \subset E_8$. This example is similar to No. 27 on Schelleken's list, which is also constructed by using the mirror extensions of conformal inclusion $SU(3)_9 \subset E_6$.

The second case is also constructed recently by C.Lam and his colloborators. By examining the spectrum, we can see that we have $\mathcal{A}_1 \otimes \mathcal{A}_1 \subset \mathcal{B}$ and the spectrum is given by

$$[1]\otimes[1]+[z_2]\otimes[z_2]+[z_1]\otimes[z_3]+[z_3]\otimes[z_1]+[b_1]\otimes[\mathrm{ad}]+[\mathrm{ad}]\otimes[b_1]+[b_2]\otimes[b_2]+[b_3]\otimes$$

(up to subtle ambiguities in the numbering of the last 5 terms) where $\mathcal{A}_1$ and its irreducible representations are as in previous section. So the inclusion $\mathcal{A}_1 \subset \mathcal{B}$ is normal, and we have a braided tensor category equivalence $F_1 : \mathrm{Rep}(\mathcal{A}_1) \to \mathrm{Rep}(A_1)^{\mathrm{rev}}$ such that $F_1(z_1) = z_3, F_1(z_3) = z_1, F_1(\mathrm{ad}) = b_1, F_1(b_1) = \mathrm{ad}$.
Now consider the conformal inclusions $SU(5)_5 \times SU(5)_5 \subset SU(25)_1 \subset \mathcal{B}_2$ where $\mathcal{B}_2$ denotes the holomorphic net corresponding to No. 67 on Schelleken's list. By examining the spectrum we find that we have $\mathcal{A}_1 \otimes \mathcal{A}_1 \subset \mathcal{B}_2$ where the spectrum is given by

$$[1]\otimes[1]+[z_2]\otimes[z_2]+[z_1]\otimes[z_3]+[z_3]\otimes[z_1]+[\mathrm{ad}]\otimes[\mathrm{ad}]+[b_1]\otimes[b_1]+[b_2]\otimes[b_2]+[b_3]\otimes$$

So the inclusion $\mathcal{A}_1 \subset \mathcal{B}_2$ is normal, and we have a braided tensor category equivalence $F_2 : \mathrm{Rep}(\mathcal{A}_1) \to \mathrm{Rep}(\mathcal{A}_1)^{\mathrm{rev}}$ such that $F_2(z_1) = z_3, F_2(z_3) = z_1$ and F_2 fix the rest of irreducibles. We note that the generator $g \in \mathrm{Aut}(\mathcal{A}_1)$ induces braided tensor category equivalence of $\mathrm{Rep}(\mathcal{A}_1)$ of order 5. Composing g with $F_1 F_2^{-1}$, and examining actions on the 6 element set of irreducible representations ad, $b_i, 1 \leq i \leq 5$, we get a subgroup of S_6 which acts transitively on these 6 elements, hence we get at least 60 elements in the outer group of braided tensor category equivalences of $\mathrm{Rep}(\mathcal{A}_1)$.
We have therefore proved the following:

So the inclusion $\mathcal{A}_1 \subset \mathcal{B}_2$ is normal, and we have a braided tensor category equivalence $F_2 : \mathrm{Rep}(\mathcal{A}_1) \to \mathrm{Rep}(\mathcal{A}_1)^{\mathrm{rev}}$ such that $F_2(z_1) = z_3, F_2(z_3) = z_1$ and F_2 fix the rest of irreducibles. We note that the generator $g \in \mathrm{Aut}(\mathcal{A}_1)$ induces braided tensor category equivalence of $\mathrm{Rep}(\mathcal{A}_1)$ of order 5. Composing g with $F_1 F_2^{-1}$, and examining actions on the 6 element set of irreducible representations $\mathrm{ad}, b_i, 1 \leq i \leq 5$, we get a subgroup of S_6 which acts transitively on these 6 elements, hence we get at least 60 elements in the outer group of braided tensor category equivalences of $\mathrm{Rep}(\mathcal{A}_1)$.
We have therefore proved the following:

Theorem

$\mathrm{Out}(\mathrm{Rep}(\mathcal{A}_1))$ has at least order 60.

It remains an interesting question to determine the equivarizations of $\mathrm{Rep}(\mathcal{A}_1)$ with respect to the group elements in the above theorem. Except for the case of $g \in \mathrm{Aut}(\mathcal{A}_1)$, where the orbifold net is simply $\mathcal{A}_{SU(5)_5}$, the rest of the cases are not known to be related to CFT, and this includes the $\mathbb{Z}_3$ case considered before.
We note that as soon as central charge is greater than 24, there are a lot more holomorphic CFT since there are many more unimodular even positive definite lattices (there are billions of such in dimension 32 while only 24 in dimension 24). There seem to be lots of room left for interesting subfactors in CFT!