Structure and classification of free Araki-Woods factors

Operator Algebras and Mathematical Physics

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- **Open problem:**
  classify these von Neumann algebras M in terms of $(U_t)_{t\in\mathbb{R}}$.

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- If $\xi \perp \eta$, then $s(\xi)$ and $s(\eta)$ are $*$-free w.r.t. φ.
- For $H = \mathbb{C}^n$, we have $L(\mathbb{F}_n) \cong \{\ell(e_i) + \ell(e_i)^* \mid i = 1, \ldots, n\}''$.

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Basic question: classify $\Gamma(K_{\mathbb{R}} \subset H)''$ in terms of $K_{\mathbb{R}} \subset H$;

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Note: conversely $S(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta$ for all $\xi, \eta \in K_\mathbb{R}$ and then $S = J\Delta^{1/2}$.

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$\mathrm{Sd}(M) = \mathrm{Sd}(U) :=$ subgroup of $\mathbb{R}_+^*$ generated by the eigenvalues of U.

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Attention: only the "non trivial" case, because $\Gamma(\mathrm{id}, H_\mathbb{R})'' \cong L(\mathbb{F}_{\dim(H_\mathbb{R})})$.

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A full factor M is called **almost periodic** if it admits a faithful normal state φ such that $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ is almost periodic.

Then, $\mathrm{Sd}(M) \subset \mathbb{R}_+^*$ is defined such that the compactification given by $t \mapsto \sigma_t^\varphi \in \mathrm{Out}(M)$ corresponds to $\mathbb{R} \subset \widehat{\mathrm{Sd}(M)}$.

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Note: unique free Araki-Woods factor of type III_λ, $\lambda \in (0, 1)$.

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Note: almost periodic = atomic measure μ.

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Further applications: many free products of amenable von Neumann algebras are **not** isomorphic to free Araki-Woods factors.

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Note: only consider subalgebras that are the range of a faithful normal conditional expectation.

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