Automorphisms with the Rohlin property on nuclear C^*-algebras

Yasuhiko Sato

Kyoto University/Purdue University

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Theorem

Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state τ. Assume that A absorbs the Jiang-Su algebra $\mathcal{Z}$. Then the following conditions are equivalent.

(i) α **is strongly outer,**

(ii) $\overline{\alpha}$ **(the extension on $\pi_\tau(A)''$) has the Rohlin property in von Neumann algebraic sense,**

(iii) α **has the weak Rohlin property,**

(iv) $\alpha \otimes \mathrm{id}_B$ **has the (cyclic) tracial Rohlin property for any UHF algebra B,**

(v) **the Rohlin dimension of α is at most 1,**

(vi) $\alpha \otimes \mathrm{id}_B$ **has the Rohlin property in C^*-algebraic sense for any UHF algebra B.**

(i)$\Longleftrightarrow$ (ii)

Theorem (A. Connes '75)

Let H be a separable Hilbert space, $\mathrm{M} \subset B(H)$ a factor von Neumann algebra such that $\mathrm{M}\bar{\otimes}\mathcal{R} \cong \mathrm{M}$, and $\alpha \in \mathrm{Aut}(M)$. Then we see that α is aperiodic in $\mathrm{Aut}(\mathrm{M})/\mathrm{Inn}(\mathrm{M})$ (strongly outer) if and only if α has the Rohlin property in the following sense :
for any $k \in \mathbb{N}$ there exist projections p_j, $j = 1, 2, ..., k$ in the central sequence algebra M_ω satisfying

$$\alpha(p_j) = p_{j+1}, \text{ for } j = 1, 2, ..., k-1, \quad \sum_{j=1}^{k} p_j = 1.$$

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Theorem (A. Connes '75)

If α, $\beta \in \mathrm{Aut}(M)$ are approximately inner with the above Rohlin property, then α and β are conjugate in $\mathrm{Aut}(\mathrm{M})/\mathrm{Inn}(\mathrm{M})$.

For a unital C^*-algebra A with a unique tracial state τ, we put $M := \pi_\tau(A)''$,

$$A_\omega := A' \cap \ell^\infty(A)/\{(a_n)_n \ : \ \lim_{n\to\omega} \|a_n\| = 0\},$$
$$M_\omega := M' \cap \ell^\infty(M)/\{(a_n)_n \ : \ \lim_{n\to\omega} \tau(a_n^* a_n) = 0\}.$$

Theorem (Kirchberg-Rørdam, 2012)
The canonical morphism $\pi : A_\omega \to M_\omega$ is surjective.

Definition (weak Rohlin property)
We say that an automorphism α of A has the weak Rohlin property, if for any $k \in \mathbb{N}$ there exists a central sequence $(a_n)_n \in A_\omega$ of positive contractions such that

$$\lim_{n\to\omega} \max_{j=1,2,\ldots,k-1} \|\alpha^j(a_n)a_n\| = 0, \quad \lim_{n\to\omega} \tau(1_A - \sum_{j=0}^{k-1} \alpha^j(a_n)) = 0,$$

- **Applying the above theorem, we see that (ii) $\Longrightarrow$ (iii).**

We denote by $\mathcal{Z}$, the Jiang-Su algebra.

Theorem (known as Toms-Winter conjecture)
For a unital separable simple nuclear C^*-algebra A with a unique tracial state, the following conditions are equivalent

(i) $A \otimes \mathcal{Z} \cong A$,
(ii) A **has strict comparison,**
(iii) **nuclear dimension of A is at most one.**

Theorem **Let A be as in above.**
Assume that $A \cong A \otimes \mathcal{Z}$ and α has the weak Rohlin property. Then we have $(A \times_\alpha \mathbb{Z}) \otimes \mathcal{Z} \cong A \times_\alpha \mathbb{Z}$.

- α **has the weak Rohlin property** $\Longleftrightarrow$ $A \times_\alpha \mathbb{Z}$ **has a unique tracial state.**

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- α has the weak Rohlin property $\iff A \times_\alpha \mathbb{Z}$ has a unique tracial state.

Theorem

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Tracial AF algebras

A C^*-algebra A is called tracial AF
if for any finite subset $F \subset A$ and $\epsilon > 0$ there exists a finite dimensional C^*-subalgebra B of A such that

$$\max_{x \in F} \inf_{b \in B} \|1_B x - b\| < \varepsilon, \quad \max_{x \in F} \|x 1_B - 1_B x\| < \varepsilon,$$

$$\max_{\tau \in T(A)} \tau(1_A - 1_B) < \varepsilon.$$

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- **Popa algebra A : $\exists$ non-zero B such that**

$$\max_{x \in F} \inf_{b \in B} \|1_B x - b\| < \varepsilon, \quad \max_{x \in F} \|x 1_B - 1_B x\| < \varepsilon.$$

By using the same technique as his alternative proof of Connes' approximation result, Popa showed that any nuclear quasi-diagonal C^*-algebra satisfies the condition of Popa algebras, (1986, '97).

Theorem (Matui-S '13) Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state. Assume that A is quasi-diagonal. Then for any UHF algebra B, we see that $A \otimes B$ is TAF.

Theorem Let A be as in above (u.s.s.n., unique trace). Assume that $\alpha \in \mathrm{Aut}(A)$ has the weak Rohlin property. Then for any UHF algebra B, we see that $\alpha \otimes \mathrm{id}_B$ has (cyclic) tracial Rohlin property introduced by Osaka-Phillips, i.e.,
for any $k \in \mathbb{N}$ there is a central sequence of projections $(p_n)_n \in A_\omega$ such that ($\lim_n \|\alpha^k \otimes \mathrm{id}_B(p_n) - p_n\| = 0$ and)

$$\lim_{n\to\omega} \max_{j=1,2,\ldots,k-1} \|\alpha^j \otimes \mathrm{id}_B(p_n)p_n\| = 0,$$

$$\lim_{n\to\omega} \tau(1_A - \sum_{j=0}^{k-1} \alpha^j \otimes \mathrm{id}_B(p_n)) = 0.$$

Property (SI) (small isometry, sin, cos, small ideal)

Definition

Let A be a separable C^*-algebra.
We say that A has *property (SI)* if for any positive contractions $(e_n)_n$, $(f_n)_n \in A_\omega$ satisfying

$$\max_{\tau \in T(A)} \tau(e_n) \to 0, \qquad \lim_{m\to\infty} \lim_{n\to\omega} \min_{\tau \in T(A)} \tau(f_n^m) > 0,$$

there exists $(s_n)_n \in A_\omega$ such that

$$(s_n^* s_n)_n = (e_n)_n, \quad (f_n s_n)_n = (s_n)_n \text{ in } A_\omega.$$

- **If A is a nuclear C^*-algebra with a unique tracial state, A has property (SI) $\iff A \otimes \mathcal{Z} \cong A$.**

By using property (SI), we see (iii) $\Longrightarrow$ (v), (iv) $\Longrightarrow$ (vi).

Theorem (Liao '15) If α has the weak Rohlin property, then Rohlin dimension of α, introduced by Hirshberg-Winter-Zacharias, is at most one : for $k \in \mathbb{N}$ there are central sequences a, b, $c \in A_\omega$ of positive contractions such that

$$\alpha^j(a)a = 0,\ \alpha^j(b)b = 0,\ j = 1, ..., k-1, \quad \alpha^j(c)c = 0,\ j = 1, 2, ..., k.$$

$$\alpha^k(a) = a, \quad \alpha^k(b) = b, \quad \alpha^{k+1}(c) = c, \quad \alpha^j(b)c = 0 \text{ for any } j,$$

$$\sum_{j=0}^{k-1} \alpha^j(a) + \sum_{j=0}^{k-1} \alpha^j(b) + \sum_{j=0}^{k} \alpha^j(c) = 1.$$

Theorem If α has tracial Rohlin property, then for any UHF-algebra B, $\alpha \otimes \mathrm{id}_B$ has the Rohlin property, by Bratteli-Evans-Kishimoto, Kishimoto, i.e., for any $k \in \mathbb{N}$ there are central sequences of projections $p, q \in (A \otimes B)_\omega$ such that

$$\sum_{j=0}^{k-1} \alpha^j \otimes \mathrm{id}_B(p) + \sum_{j=0}^{k} \alpha^j \otimes \mathrm{id}_B(q) = 1.$$

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Classification of automorphisms

By using the Rohlin property, in particular (vi), we obtain the following result.

Theorem (Matui-S.)

Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state, and α, β are automorphisms on A with the Rohlin property. Assume that A absorbs the Jiang-Su algebra $\mathcal{Z}$ and satisfies the condition of UCT, and that α and β are asymptotically unitarily equivalent.
Then α and β are conjugate in $\mathrm{Aut}(A)/\mathrm{Inn}(A)$.