A new look at C^*-simplicity and the unique trace property

(based on work by Uffe Haagerup)

Mikael Rørdam
rordam@math.ku.dk

Department of Mathematics
University of Copenhagen

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- G = (countable) discrete group; X = compact Hausdorff space.

Definition (Furstenberg, 1963)

A compact G-space X is *strongly proximal* if

$$\forall \mu \in \mathrm{Prob}(X) : \overline{G.\mu} \cap \{\delta_x : x \in X\} \neq \emptyset.$$

$G \curvearrowright X$ is a *boundary action* if it is strongly proximal and minimal.

- Given $G \curvearrowright X$. TFAE:

(a) $G \curvearrowright X$ is a boundary action,

(b) $\forall \mu \in \mathrm{Prob}(X) : \{\delta_x : x \in X\} \subseteq \overline{G.\mu}$,

(c) $\forall x \in X\ \exists (g_i) \subseteq G\ \forall \mu \in \mathrm{Prob}(X) : g_i.\mu \to \delta_x$.

- If a bdry action $G \curvearrowright X$ admits an invariant prob. measure μ, then $X = \{\mathrm{pt}\}$. In particular, the only bdry action $G \curvearrowright X$ of an amenable group G is the trivial one: $X = \{\mathrm{pt}\}$.

Furstenberg made the following further observations:

- If $G \curvearrowright Y$ is a quotient of boundary action $G \curvearrowright X$, i.e., $Y = q(X)$ for some cts G-map q, then $G \curvearrowright Y$ is a boundary action.
- If (X_i) are strongly proximal G-spaces, then so is $\prod_i X_i$ (wrt the diagonal action).
- There is a universal boundary action $G \curvearrowright \partial_F G$, i.e., every other boundary action is a quotient of $G \curvearrowright \partial_F G$ (now called the Furstenberg boundary).
- $\partial_F G \neq \{\mathrm{pt}\}$ iff G is non-amenable.

Proposition (Furman, 2003).
$g \in G$ acts non-trivially on $\partial_F G \iff g \notin \mathrm{Rad}(G)$.

$\mathrm{Rad}(G) =$ the largest normal amenable subgroup of G
$=$ the amenable radical of G

Definition (Laca–Spielberg, Glasner). An action $G \curvearrowright X$ is a strong boundary action if for every open set $\emptyset \neq U \subseteq X$ and compact set $K \subset X$ there exists $g \in G$ st $g.K \subseteq U$.

- Strong boundary $\Rightarrow$ boundary. ($\Leftarrow$ does not hold.)

Example: The action of a non-elementary word hyperbolic group G on its Gromov boundary ∂G is a strong boundary action (and hence a boundary action).

Theorem (Laca–Spielberg): If $G \curvearrowright X$ is a strong boundary action, then $C(X) \rtimes_{\mathrm{red}} G$ is simple and purely infinite. If, furthermore, the action is amenable, G is countable and X is metrizable, then $C(X) \rtimes_{\mathrm{red}} G$ is a Kirchberg algebra.

- In the theorem above, one can relax "strong boundary action" to the statement that each clopen set is G-paradoxical relatively to the clopen subsets of X, provided that X is totally disconnected.

- τ_0 = the canonical (faithful) tracial state on $C^*_\lambda(G)$.

Theorem (Powers, 1975): $C^*_\lambda(\mathbb{F}_2)$ is simple (and has a unique tracial state). Moreover, $\forall a \in C^*_\lambda(\mathbb{F}_2)\ \forall \varepsilon > 0\ \exists g_1, \dots, g_n \in \mathbb{F}_2$:

$$\left\| \tau_0(a)\mathbf{1} - \frac{1}{n}\sum_{j=1}^{n} \lambda(g_j)a\lambda(g_j)^* \right\| < \varepsilon.$$

Question: For which groups G is $C^*_\lambda(G)$ simple? has unique tracial state? Partial answer (de la Harpe):

$$C^*_\lambda(G) \text{ simple} \Rightarrow \mathrm{Rad}(G) = \{e\} \Leftarrow C^*_\lambda(G) \text{ unique trace}.$$

Theorem (Kalantar–Kennedy, 2014).
$C^*_\lambda(G)$ simple $\iff$ $G \curvearrowright \partial_F G$ is (topologically) free.

- Breuillard–Kalantar–Kennedy–Ozawa (BKKO):
"topological freeness" $\Longrightarrow$ "freeness" for actions $G \curvearrowright \partial_F G$.

Theorem (Furman, 2003).
$\mathrm{Rad}(G) = \{e\}$ $\iff$ $G \curvearrowright \partial_F G$ is faithful.

Theorem (Breuillard–Kalantar–Kennedy–Ozawa, 2014). $C^*_\lambda(G)$ has unique trace $\iff \mathrm{Rad}(G) = \{e\}$.

- As a consequence, BKKO can conclude: BKKO + Le Boudec can conclude:

$$C^*_\lambda(G) \text{ simple } \Rightarrow \overset{\not\Leftarrow}{\Rightarrow} \mathrm{Rad}(G) = \{e\} \Leftrightarrow C^*_\lambda(G) \text{ unique trace.}$$

- Using the Kalantar–Kennedy theorem, BKKO established C^*-simplicity for large classes of groups (with simpler proofs, when already known).

Theorem (Le Boudec, 2015). There exists a class of groups G st $\mathrm{Rad}(G) = \{e\}$, while $C^*_\lambda(G)$ is non-simple.

- Ivanov and Omland produced in 2016 new examples of non-C^*-simple groups with trivial amenable radical arising as amalgamated free products.

- Given $G \curvearrowright X$, we have

$$C^*_\lambda(G) \subseteq C(X) \rtimes_r G, \qquad C(X) \subseteq C(X) \rtimes_r G,$$

satisfying: $\lambda(g)f = \alpha_g(f)\lambda(g)$, where $\alpha_g(f)(x) = f(g^{-1}.x)$, $f \in C(X)$, $g \in G$, $x \in X$.

Lemma. If φ is a state on $C(X) \rtimes_r G$ and $x \in X$ st $\varphi|_{C(X)} = \delta_x$, then $\varphi(\lambda(g)) = 0$, for all $g \in G$ st $g.x \neq x$.
In particular, if $G_x = \{e\}$, then $\varphi|_{C^*_\lambda(G)} = \tau_0$.

Proof: φ is multiplicative on $C(X)$.

Lemma. Let $G \curvearrowright X$ be a bdry action, let τ be a tracial state on $C^*_\lambda(G)$, and let $x \in X$.
Then τ extends to a state φ on $C(X) \rtimes_r G$ st $\varphi|_{C(X)} = \delta_x$.

Theorem (BKKO). If $g \notin \mathrm{Rad}(G)$, then $\tau(\lambda(g)) = 0$, for all tracial states τ on $C^*_\lambda(G)$. In particular,

$$\mathrm{Rad}(G) = \{e\} \iff C^*_\lambda(G) \text{ has unique tracial state.}$$

Theorem (BKKO). If $g \notin \mathrm{Rad}(G)$, then $\tau(\lambda(g)) = 0$, for all tracial states τ on $C^*_\lambda(G)$. In particular,

$$\mathrm{Rad}(G) = \{e\} \iff C^*_\lambda(G) \text{ has unique tracial state.}$$

Theorem (Haagerup). Let $g \in G$:

$$g \notin \mathrm{Rad}(G) \iff 0 \in \overline{\mathrm{conv}}\{\lambda(hgh^{-1}) : h \in G\}.$$

- The proof uses Hahn–Banach and Furman's characterization of $\mathrm{Rad}(G)$ in terms of boundary actions.

Corollary (Haagerup). $C^*_\lambda(G)$ has unique tracial state iff $\forall g \in G \setminus \{e\}$ $\forall \varepsilon > 0$ $\exists h_1, \ldots, h_m \in G$ st

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(h_j g h_j^{-1}) \right\| < \varepsilon.$$

Theorem (Furman, Haagerup, BKKO)

Let G be a group. TFAE:

1. $C^*_\lambda(G)$ *has unique tracial state,*
2. G *admits a faithful boundary action,*
3. $\mathrm{Rad}(G) = \{e\}$,
4. $\forall g \in G \setminus \{e\}\ \ \forall \varepsilon > 0\ \ \exists h_1, \ldots, h_m \in G$ *st*
$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(h_j g h_j^{-1}) \right\| < \varepsilon.$$

Lemma (from before). If φ is a state on $C(X) \rtimes_r G$ and $x \in X$ st $\varphi|_{C(X)} = \delta_x$ and $G_x = \{e\}$, then $\varphi|_{C^*_\lambda(G)} = \tau_0$.

Lemma. Let G be a C^*-simple group and let φ be a state on $C^*_\lambda(G)$. Then $\exists (g_i) \subseteq G$ st $g_i.\varphi \to \tau_0$.

Proposition. Let G be a C^*-simple group. Then $\exists (g_i) \subseteq G$ st $g_i.\varphi \to \tau_0$, for *all* states φ on $C^*_\lambda(G)$.
Moreover, $g_i.\omega \to \omega(\mathbf{1})\tau_0$ for all $\omega \in C^*_\lambda(G)^*$.

Theorem (Haagerup, Kennedy). $C^*_\lambda(G)$ is simple iff $\forall g_1, \ldots, g_n \in G \setminus \{e\}$ $\forall \varepsilon > 0$ $\exists h_1, \ldots, h_m \in G$ st

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(h_j g_i h_j^{-1}) \right\| < \varepsilon, \qquad i = 1, \ldots, n.$$

Theorem (Kennedy-Kalantar, Haagerup, BKKO)

Let G be a group. TFAE:

1. $C^*_\lambda(G)$ *is simple,*
2. G *admits a (topologically) free boundary action,*
3. $\tau_0 \in \overline{\{g.\varphi : g \in G\}}$*, for all states* φ *on* $C^*_\lambda(G)$*,*
4. $\exists (g_i) \subseteq G$ *st* $g_i.\varphi \to \tau_0$*, for all states* φ *on* $C^*_\lambda(G)$*,*
5. $\forall g_1, \ldots, g_n \in G \setminus \{e\}$ $\forall \varepsilon > 0$ $\exists h_1, \ldots, h_m \in G$ *st*
$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(h_j g_i h_j^{-1}) \right\| < \varepsilon, \qquad i = 1, \ldots, n,$$
6. $C^*_\lambda(G)$ *has the Dixmier property:*
$$\overline{\mathrm{conv}}\{uxu^* : u \text{ unitary in } C^*_\lambda(G)\} \cap \mathbb{C} \cdot \mathbf{1} \neq \emptyset,$$
for all $x \in C^*_\lambda(G)$*.*