QUANTUM DOUBLES AND ORBIFOLD SUBFACTORS

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1. Introduction

Since V. F. R. Jones initiated subfactor theory in [21], the theory of operator algebras has experienced several new relations to other branches of mathematics and mathematical physics. Many of these new interactions take places on combinatorial levels and it is Ocneanu's paragroup theory, initiated in [30], that dictates combinatorial structures of subfactors from the operator algebraic side. (We certainly need a deep analytic theory in order to reduce the classification theory of subfactors to that of paragroups, and it has been studied to full extent by Popa in [40], [41].)

A *paragroup* is a certain quantization of a notion of a group in a different sense from that in the quantum group theory of Drinfel′d [7] and Jimbo [20]. It is a new and natural algebraic system closely related to fusion rule algebras in rational conformal field theory.

Relations of a paragroup to various mathematical objects are as follows. A paragroup arises from a subfactor of a II_1 factor with finite index and finite depth as a quantized version of a Galois group. It gives a complete invariant of subfactors of the hyperfinite II_1 factor with finite index and finite depth, up to isomorphism, as in [40]. A finite group or a finite dimensional Hopf C^*-algebra produces a paragroup as in [30]. A quantum group $\mathcal{U}_q(\mathcal{G})$ with q being an appropriate root of unity produces a paragroup as in [52]. For a connected, simply connected, compact and simple Lie group G, the Wess–Zumino–Witten model of G at level k gives a paragroup as in [2]. A paragroup produces a topological quantum field theory of Turaev–Viro type [46] based on triangulation in 3-dimensions as in [32], [10], [12]. A general paragroup does not give a link invariant directly, but we have a procedure, called the asymptotic inclusion, to make a new paragroup from a given one. After this procedure, a paragroup gives a link invariant and a Reshetikhin–Turaev type topological quantum field theory [43] based on surgery in 3-dimensions, and also a set of combinatorial data satisfying the Moore–Seiberg axioms [29] of rational conformal field theory. This construction of the asymptotic inclusion is an analogue of the quantum double construction of Drinfel′d [7] as explained below.

In this article, we survey two topics in paragroup theory. One is this analogue of the quantum double construction of Drinfel′d and the other is the orbifold construction in the subfactor theory initiated by us [9], [22]. The orbifold construction is a method to get a new paragroup from a given one with a certain symmetry. This has a natural relation to the quantum doubles. Before these two topics, we review the basics of the paragroup theory.

2. Ocneanu's paragroups

We will make a quick review of Ocneanu's paragroup theory [30], [31], [32] here. See [12, Chapters 9–13] for details of the theory.

Ocneanu has found two equivalent ways to define paragroups. One is based on *flat connections* as in [30] and the other is on *quantum* $6j$*-symbols* as in [32].

Suppose we have a subfactor $N \subset M$ of type II_1 with finite index and finite depth. (The index $[M : N]$, studied in [21], measures the relative size of M with respect to N. The definition of the finite depth assumption is given below.) We apply the basic construction successively to get the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$. We then look at the following commuting square.

$$\begin{array}{ccc} M' \cap M_{2k} & \subset & M' \cap M_{2k+1} \\ \cap & & \cap \\ N' \cap M_{2k} & \subset & N' \cap M_{2k+1} \end{array}$$

The finite depth assumption means that the Bratteli diagram for these inclusions does not depend on k if k is sufficiently large. The above commuting square is described by a *bi-unitary connection*, in the sense of Ocneanu [30], which assigns a complex number to each square consisting of four edges from the Bratteli diagram. (Instead of the name "bi-unitary connection", we often use a shorter form "connection".) It is enough to have one commuting square for sufficiently large k in order to recover the original subfactor. So the subfactor is recovered from the corresponding connection. Actually, it is not a connection itself but an equivalence class of connections that is determined uniquely from a commuting square. This equivalence class is similar to a cohomology class and the ambiguity involved in the definition is called a *gauge choice*. (See [30] or [12, Definition 10.11] for details.) We often simply say a connection, when we really mean an equivalence of connections.

Ocneanu has determined which connections indeed arise from subfactors by axiomatizing them in a combinatorial way. Roughly speaking, the axioms are unitarity, renormalization, and flatness. (See [30] or [12, Chapter 10] for the exact definitions.) The unitarity and renormalization axioms correspond to the commuting square condition and are rather easy to verify for a given set of data. (See [12, Section 11.2].) The flatness axiom is the most important and subtle. It characterizes commuting squares arising as higher relative commutants of subfactors. (They are called "canonical" commuting squares in Popa's terminology.) Since we have two kinds of higher relative commutants $\{N' \cap M_k\}_k$ and $\{M' \cap M_k\}_k$, the flatness axiom actually consists of two kinds of identities. (See [30] or [12, Theorem 10.10]. We sometimes use the name flat "connection" even when only one kind of flatness holds as in [12, Theorem 11.17], but here we mean a bi-unitary connection satisfying both flatness relations by a flat connection.) A paragroup is an equivalence class of flat connections. (Strictly speaking, we have to identify flat connections up to graph isomorphism. See [30] or [12, Definition 10.11].) S. Popa [42] has characterized commuting squares arising from subfactors without the finite depth assumption, and this flatness is one way to express the commutation relation in Popa's axioms.

Connections are quite similar to Boltzmann weights in the theory of exactly solvable lattice models, especially IRF models (without spectral parameters) in [1]. (A Boltzmann weight of an IRF model also assigns a complex number to each square arising from a certain graph.) The unitarity and renormalization axioms are essentially the same as the first and second inversion relations in the theory of solvable lattice models, respectively. The flatness is closely related to the Yang–Baxter equation of the IRF models in some typical examples, but we have no direct relations between the two in general.

The other way to describe paragroups is in terms of *fusion rule algebras* and *quantum $6j$-symbols*. (See [32] or [12, Chapter 12].) The inclusion $N \subset M$ gives a left N- right M-module $L^2(M)$, which we simply denote by ${}_N M_M$. Bimodules over two factors are quite similar to representations of compact groups, and we can define a dimension of a bimodule and a relative tensor product of two bimodules over a factor as analogues of a dimension of a representation and a tensor product of two representations. (See [39] for a general theory of bimodules. Chapter 9 of [12] describes a necessary part of the bimodule theory for subfactor theory.) It is easy to define irreducibility and irreducible decompositions of bimodules. By making finite tensor powers $\cdots {}_N M \otimes_M M \otimes_N M \otimes_M M \cdots$ and their irreducible bimodules, we get a system of four kinds of bimodules, that is, N-N, N-M, M-N, M-M bimodules. This system is closed under the relative tensor product and irreducible decompositions and determines the fusion rule algebra similar to a representation ring of a compact group. Note that this relative tensor product operation is not commutative at all in general, unlike the tensor product of group representations. So we do *not* assume commutativity when we say a fusion rule algebra. (We can make a relative tensor product only when the two bimodules have the matching right action and left action — we cannot make a relative tensor product of an N-N bimodule and an M-M bimodule, for example. In this sense, this fusion rule algebra has a restricted multiplication.) The finite depth assumption is equivalent to the condition that this system has only finitely many (isomorphism classes of) bimodules.

If the subfactor is of finite depth, then we have a finite closed system of the four kinds of bimodules under the relative tensor product. We choose six bimodules A, B, C, D, X, Y from the system and four intertwiners $\xi_1, \xi_2, \xi_3, \xi_4$ so that the following diagram makes sense.

$$\begin{tikzcd} X \otimes A \otimes Y \arrow[r, "\mathrm{id}_X \otimes \xi_1"] \arrow[d, "\xi_3 \otimes \mathrm{id}_Y"'] & X \otimes B \arrow[d, "\xi_2"] \\ C \otimes Y \arrow[r, "\xi_4"'] & D \end{tikzcd}$$

By making the composition $\xi_4(\xi_3 \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \xi_1)^* \xi_2^* \in \mathrm{End}(D)$, we get a complex number. This assignment of complex numbers to six bimodules and four intertwiners, up to certain normalization, is called a *quantum $6j$-symbol*. (See [12, Section 12.4].) Again we have natural equivalence classes of quantum $6j$-symbols arising from *gauge choices* and we often simply say a quantum $6j$-symbol when we really mean an equivalence classes of quantum $6j$-symbols. Ocneanu has axiomatized quantum $6j$-symbols arising from subfactors with three axioms; unitarity, tetrahedral symmetry, and the pentagon relation. (See [32] or [12, Section 12.2].) By the other definition different from the one based on flat connections, a paragroup is an equivalence class of quantum $6j$-symbols on a fusion rule algebra with restricted multiplication of four kinds of objects, where the equivalence is defined in terms of certain gauge choices. (Note that by quantum $6j$-symbols here we mean some set of data satisfying the three axioms, and we do not mean that we get these data from a Lie group with a deformation parameter q.)

The relation Ocneanu has found for flat connections and quantum $6j$-symbols is as follows. Setting two bimodules X, Y of the six bimodules as above to be the generating bimodules ${}_N M_M$ or ${}_M M_N$ of the system, we get an assignment of a complex number to four bimodules and four intertwiners. This is exactly a flat connection of the corresponding paragroup because irreducible bimodules and intertwiners correspond to vertices and edges of the Bratteli diagram. (See [12, Section 12.5].) This correspondence is easy because this is just a restriction of the definition domain of a map. Conversely, if we have a paragroup, then the flatness axiom implies that we can extend the connection to a quantum $6j$-symbol via the corresponding subfactor. In this way, we have a bijective correspondence between equivalence classes of flat connections and those of quantum $6j$-symbols on fusion rule algebras with four kinds of objects and restricted multiplications. The quantum $6j$-symbol approach to paragroups require more data than the flat connection approach, while it is much easier to verify the axioms in the quantum $6j$-symbol approach than in the flat connection approach. It is often very difficult to write down the formulae for quantum $6j$-symbols explicitly for a concrete subfactor, while it is usually easier to write down the formulae for flat connections explicitly.

The above three axioms of the quantum $6j$-symbols are the same as those for "initial data" of the 3-dimensional Turaev–Viro type *topological quantum field theory* [46] based on triangulation, but we have the following difference between our quantum $6j$-symbols for subfactors and those in the Turaev–Viro theory.

One is the positivity of dimensions. In our setting, each bimodule has a dimension and it is positive, but in the purely algebraic setting of Turaev–Viro, the "dimensions" do not have to be positive. In this sense, our setting is more restrictive.

The other is that we have four kinds of bimodules. Actually it is enough to have only one kind of bimodules, N-N or M-M, to get a topological quantum field theory, and the quantum $6j$-symbols of N-N bimodules and those of M-M bimodules give the same topological quantum field theory. (See [12, Section 12.4]. In Ocneanu's recent terminology, two finite systems of irreducible bimodules are said to be *equivalent* if they can be realized as the systems of N-N and M-M bimodules of a single subfactor $N \subset M$ with finite index and finite depth. Then one can say that equivalent systems of bimodules give the same topological quantum field theory.) In this sense, our quantum $6j$-symbols have redundancy from the viewpoint of topological quantum field theory. For example, the Jones subfactor of type A_5 and the subfactor $N \subset N \times S_3 = M$ give the same topological quantum field theory, where the symmetric group S_3 of order 3 acts on N freely.

We explain the notion of the global index of Ocneanu here. It is sometimes said that a subfactor is regarded as a "fixed point algebra" of an action of a paragroup in some vague sense, and the Jones index measures the size of the paragroup. This is an important viewpoint of the quantized Galois theory, but it is not very appropriate to think of the Jones index as the size of a paragroup. We have the notion of the global index as a better way to measure the size of a paragroup. For a subfactor $N \subset M$, its *global index* is defined to be the square sum of the dimensions of the irreducible N-N bimodules appearing in the fusion rule algebra. Or, we can also define the global index by the square sum of the Perron–Frobenius eigenvector entries of the principal graph. Note that the global index is always greater than or equal to the Jones index and that the global index is finite if and only if the subfactor is of finite depth. The global index is also important for normalization in the topological quantum field theory. Note that if the subfactor arises from a free action of a finite group or a finite dimensional Hopf C^*-algebra, then the global index is the same as the order of the group or the dimension of the Hopf algebra, respectively, which is also equal to the Jones index of the subfactor. See Sato's work [44], [45] for more on significance of the global index.

Today we have many mathematical objects with names containing the word "quantum", so it is not very surprising that Wess–Zumino–Witten models or quantum groups $\mathcal{U}_q(\mathcal{G})$ produce paragroups, because they are among the "best" of the quantum objects. General axioms of paragroups require "less" quantum symmetry than those well-known examples in a sense, so we expect more "exotic" paragroups than those studied in the quantum group theory. Haagerup has tried to get a list of paragroups with small index in [18] and listed candidates of possible paragroups. He proved that the first example in his list with index $(5+\sqrt{13})/2$ is indeed realized in [18], but it is open whether the other candidates are realized or not, except for that D. Bisch has recently shown with fusion rule computations that case (4) in [18] is impossible. Recently, K. Ikeda has made numerical computations to check the paragroup axioms in [19] and his computations strongly suggest that Haagerup's candidate for index $(5+\sqrt{17})/2$ in case (3) in [18] should really be a paragroup, though the numerical computations do not give a rigorous proof. It is one main advantage of the paragroup approach to the study of quantum mathematics that we can handle this type of exotic objects.

At the end of this Section, we mention a most recent development in the paragroup theory, Ocneanu's new rigidity theorem he presented in [38] with proof. His theorem is as follows.

2.1 Theorem. *On a given fusion rule algebra, possibly with restricted multiplication, we have only finitely many equivalence classes of quantum $6j$-symbols.*

Ocneanu works on the compact space of all the quantum $6j$-symbols on the fusion rule algebra and then shows that a "small" perturbation of a quantum $6j$-symbol gives an equivalent quantum $6j$-symbol by "differentiating" the pentagonal relation in a certain sense. Then the following corollaries, all due to Ocneanu, follow from this theorem.

2.2 Corollary. *For any given constant C, we have only finitely many paragroups with global index less than C. (Here we assume that the corresponding subfactors are irreducible.)*

Proof. If the global index is bounded, we have only finitely many choices of the structure constants for fusion rule algebras, so we have only finitely many of them. For each of them, we have finitely many $6j$-symbols by Theorem 2.1.

2.3 Corollary. *The total number of the isomorphism classes of subfactors with finite index and finite depth of the hyperfinite* II_1 *factor is countable.*

Proof. This immediately follows from Corollary 2.1.

2.4 Corollary. *For any finite graph, we have only finitely many flat connections on it.*

Proof. The given graph determines the global index. So the claim follows from Corollary 2.1.

The following corollary, which had been proved by Stefan earlier recently, also follows from Theorem 2.1.

2.5 Corollary. *For a given dimension, we have only finitely many Kac (or Hopf* C^**-) algebras.*

Proof. The dimension of a Hopf C^*-algebra is the global index of the corresponding paragroup of depth 2. So the claim follows from Corollary 2.1.

3. Asymptotic inclusions, quantum doubles, and TQFT

We next explain Ocneanu's machinery constructing a new and "better" paragroup from a given paragroup. His basic observation in [31], [33], [34], [35], [36] was that the following four constructions are mutually analogous.

(1) The asymptotic inclusion in [30].
(2) The central sequence subfactor in [30].
(3) The quantum double construction in [7].
(4) The topological quantum field theory based on triangulation [46].

We now explain these constructions. For (1) and (2), we start with a hyperfinite type II_1 subfactor $N \subset M$ with finite index and finite depth. Then the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ is called the *asymptotic inclusion* of the original inclusion. This new subfactor has a finite index and finite depth, and the index is given by the global index of the original subfactor. (See [30], [12, Theorem 12.23]. If the original subfactor is of infinite depth, then the asymptotic inclusion does not have a finite index.) Then we look at only the system of the M_∞-M_∞ bimodules arising from the asymptotic inclusion. This system gives the "quantum double" of the original systems of both M-M and N-N bimodules simultaneously in the sense explained below. (Longo and Rehren found a similar construction in [27] from a different motivation. Masuda has shown in [28] that the Longo–Rehren construction is the same as the asymptotic inclusion from the viewpoint of tensor categories.)

Let ω be a free ultrafilter over a countable set. Then the inclusion $N^\omega \cap M' \subset M_\omega$ gives a type II_1 subfactor with finite index and finite depth. (See [30], [31], [12, Theorem 15.32]. These II_1 factors are not hyperfinite — they are not even separable.) This subfactor is called the *central sequence subfactor*. The paragroup arising from this subfactor is dual to that from the asymptotic inclusion. (See [31], [12, Theorem 15.32].) That is, the basic construction of this subfactor gives the same paragroup as the one from the asymptotic inclusion. Thus we consider the system of $N^\omega \cap M'$-$N^\omega \cap M'$ bimodules for this central sequence subfactor and this system is isomorphic to the system of the M_∞-M_∞ bimodules arising from the asymptotic inclusion.

For (3), we start with a Hopf algebra without R-matrix and after the *quantum double* construction, we get an R-matrix, or a braiding.

In (4), we start with a fusion rule algebra with a quantum $6j$-symbol and apply the Turaev–Viro construction [46]. Then consider the Hilbert space $H_{S^1\times S^1}$ for the 2-dimensional torus as in [46]. We have a natural basis for this finite dimensional Hilbert space and then a new fusion rule, called the *convolution*, on the set of these vectors as in [34], [35], [12, Section 12.6]. This gives a tensor category regarded as the "quantum double" of the original data, as explained below.

These constructions are mutually analogous in the following sense. Constructions (1) and (2) give essentially same paragroups as mentioned above. For constructions (1) and (4), we can identify the irreducible M_∞-M_∞ bimodules and the natural basis vectors for the $H_{S^1\times S^1}$ of the topological quantum field theory arising from $N\subset M$ as in [35], [12, Theorems 12.21, 12.26, 12.28]. In this identification, the relative tensor product of M_∞-M_∞ bimodules is also identified with the convolution product. (Strictly speaking, for this identification, we need an assumption of connectedness of the fusion graph, which is defined below. If this connectedness does not hold, then the set of the M_∞-M_∞ bimodules is mapped to a proper subset of the basis vectors of $H_{S^1\times S^1}$.) In this sense, the constructions (1) and (4) are essentially the same.

The relation of construction (3) to the other construction is less direct. Start with a subfactor $N\subset N\times G=M$ where a finite group G acts on the hyperfinite II_1 factor N freely. Then Izumi's computation in [12, Section 12.8] shows that the asymptotic inclusion $M\vee(M'\cap M_\infty)\subset M_\infty$ is given as $R^{G\times G}\subset R^G$, where R is the hyperfinite II_1 factor and G is embedded into $G\times G$ by $g\mapsto(g,g)$ with free action of $G\times G$ on R. It then follows that the M_∞-M_∞ bimodules are described with the quantum double $D(G)$ of the function algebra $C(G)$ on the group G. (See [25].) These also correspond to the natural basis vectors of the Hilbert space $H_{S^1\times S^1}$ of the corresponding topological quantum field theory as in [5], [6]. In this sense, we can say that constructions (1), (2), (4) are generalizations of the quantum double construction.

The quantum double construction of a Hopf algebra produces a braiding, and an analogue of this fact holds for the asymptotic inclusions. Using construction (4), we can prove that there exists a natural braiding on the system of M_∞-M_∞ bimodules as in [36], [12, Section 12.7]. (Here a *braiding* means a systematic choice of certain intertwiners between relative tensor products of M_∞-M_∞ bimodules. See [36].) Actually, we can construct a set of data satisfying the combinatorial axioms of Moore–Seiberg [29] for rational conformal field theory as in [12, Section 13.5]. Then we can construct a 3-dimensional Reshetikhin–Turaev type topological quantum field theory [43] based on surgery from these data as in [36]. We conjecture that this Reshetikhin–Turaev type topological quantum field theory is the same as the Turaev–Viro type topological quantum field theory arising from the original subfactor $N\subset M$.

The principal graph of the asymptotic inclusion is described easily. We first define the fusion graph of the original subfactor $N\subset N$ as follows. The even vertices are labeled with pairs (X,Y) of irreducible M-M bimodules arising from the original subfactor $N\subset M$. The odd vertices are labeled with irreducible M-M bimodules from $N\subset M$. Then the number of the edges connecting (X,Y) and Z is the multiplicity of the bimodule Z in the relative tensor product $X \otimes_M Y$. This is the fusion graph and the principal graph of the asymptotic inclusion is equal to the connected component of the fusion graph containing the vertex $(*, *)$, where $*$ means the identity bimodule ${}_M M_M$. The dual principal graph of the asymptotic inclusion, whose even vertices correspond to the more important M_∞-M_∞ bimodules, is much harder to compute in general.

There are important examples of subfactors for which the principal graphs are easy to compute, but the dual principal graphs are much more difficult, as these asymptotic inclusions. Ocneanu has noticed that we often have a common strategy to compute the dual principal graphs of $N \subset M$ and the fusion rule on their even vertices in this kind of situations as follows.

(1) Find an appropriate finite dimensional C^*-algebra.
(2) Show that the minimal central projections in this algebra correspond to the irreducible M-M bimodules of the subfactor.
(3) Define another product on the set of these minimal central projections and show that this product corresponds to the relative tensor product of the M-M bimodules.

Examples of such subfactors contain the following.

(a) The asymptotic inclusion in [30].
(b) The Goodman–de la Harpe–Jones subfactor in [14].
(c) The group-subgroup pair $R^G \subset R^H$ in [26].

In (c), a finite group G and its subgroup H acts freely on the hyperfinite II_1 factor R.

In (a), the corresponding finite dimensional C^*-algebra is Ocneanu's *tube algebra* introduced in [33], [34], [36]. (See [12, Section 12.6].) In (b), the algebra is Ocneanu's *double triangle algebra* introduced in [37]. In order to describe what kind of procedures we need, we present "trivial" examples $R \subset R \times G$ instead of (c). Note that the even vertices of the principal graph are labeled with the group elements in G and those of the dual principal graph are by the elements of the group dual $\hat{G}$. In this case, both graphs are easy, but for the sake of explanation, suppose that we know only the principal graph. It is certainly impossible to get the dual principal graph only from the principal graph, because the principal graph determines only the order of the group, but if we have the fusion rule of the even vertices of the principal graph, then we can find the dual principal graph as follows. The fusion rule is exactly the multiplication law of the group G, so we can define the group algebra $\mathbf{C}[G]$, and then the minimal central projections in this algebra are in a bijective correspondence to the irreducible unitary representations of G. Furthermore, we can define a product on these minimal central projections corresponding to the tensor product of representations. These correspond to steps (1)–(3) in the above strategy.

This easy example suggests that just the principal graph is not enough to compute the dual principal graph and we need some data about the fusion rule. Indeed, in the above examples of the asymptotic inclusions and the Goodman–de la Harpe–Jones subfactors, we need quantum $6j$-symbols or braiding on the fusion rule algebra and these give definitions of the tube algebra and the double triangle algebra. Then the above strategy works.

4. Orbifold subfactors

In this Section, we explain the orbifold subfactors introduced in [9], [22] and its natural relation to the asymptotic inclusion.

Roughly speaking, the orbifold construction makes a quotient of a paragroup with a certain symmetry. In the flat connection approach, if we have a graph automorphism leaving the connection invariant, then this graph automorphism induces an automorphism of a commuting square and we can take simultaneous fixed point algebras which still make a commuting square. In the quantum $6j$-symbol approach, we take a quotient of the fusion rule algebra with respect to a fusion rule subalgebra.

The most fundamental examples come from the Jones subfactors with principal graph A_{2n+1} in [21]. This graph clearly has a symmetry of order 2 and it is easy to see that the connection is also $\mathbf{Z}/2\mathbf{Z}$-symmetric. (Note that the principal graph A_{2n} also has a symmetry of order 2, but this symmetry switches the even and the odd vertices of the principal graph, so does not act on the commuting square as an automorphism.) In the quantum $6j$-symbol approach, the two end vertices of the principal graph give a group $\mathbf{Z}/2\mathbf{Z}$ and a fusion rule subalgebra. So we take a quotient of the fusion rule subalgebra with respect to this subalgebra. The "quotient" is taken with an operator algebraic method as follows. The two end vertices give bimodules of dimension 1 and they are given by automorphisms of N or M. In this case, this indeed gives an action of the group $\mathbf{Z}/2\mathbf{Z}$ on the inclusion $N \subset M$ and this action is non-strongly outer in the sense of Choda–Kosaki [4]. Then the simultaneous fixed point algebras $N^{\mathbf{Z}/2\mathbf{Z}} \subset M^{\mathbf{Z}/2\mathbf{Z}}$ (or the simultaneous crossed products $N \times \mathbf{Z}/2\mathbf{Z} \subset M \times \mathbf{Z}/2\mathbf{Z}$) give a "quotient" of the fusion rule algebra as follows. (See [53] for a more categorical treatment.)

Two vertices of the principal graph symmetric under the $\mathbf{Z}/2\mathbf{Z}$-action are merged into one vertex and the vertex fixed under the symmetry splits into two vertices. In this way, we get the Dynkin diagram D_{n+2} from the Dynkin diagram A_{2n+1}. (This splitting of a fixed point is an analogue of the orbifold bifurcation in geometry and gives the reason of the name "orbifold subfactors".)

Here one subtlety comes in. Not all the Dynkin diagrams D_{n+2} are realized in this way. As Ocneanu announced in [30], the Dynkin diagrams D_{2n} are realized as principal graphs of subfactors and thus have corresponding paragroup structures, but D_{2n+1} are not. (See [12] or [22] for a proof.) This is because the flatness condition is not automatically preserved in this "quotient" procedure. In the flat connection approach, it is easy to see that the unitarity and the renormalization axioms are preserved, but a certain obstruction to the flatness arises in the orbifold construction. This is a new phenomenon which does not occur in the theory of solvable lattice models. This obstruction vanishes for the Dynkin diagrams D_{2n}, but this obstruction kills D_{2n+1}. From the viewpoint of the rational conformal field theory, these subfactors of type A_n correspond to the Wess–Zumino–Witten models $SU(2)_{n-1}$. The orbifold construction was generalized to $SU(n)_k$ with prime n and established as a general machinery in [9] and later Xu identified the obstruction to flatness with the conformal dimension in rational conformal field theory in [51]. Goto has generalized the orbifold construction to non-hyperfinite subfactors with a more algebraic method in [16].

From an analytic viewpoint, this obstruction to flatness appears naturally in the setting of the central sequence subfactors in connection to the relative versions of the Connes invariant χ and the Jones invariant κ as in [23], [24].

We next explain a natural relation of the orbifold construction to the asymptotic inclusion. In Section 3, we have explained the asymptotic inclusion and importance of the M_∞-M_∞ bimodules. Since the Jones subfactors of type A_n in [21] are the most fundamental "quantum" subfactors, we are naturally interested in the asymptotic inclusions of these subfactors. (M. Choda computed the Jones indices of these subfactors in [3].)

These subfactors correspond to the quantum group $\mathcal{U}_q(sl_2)$ at roots of unity in some sense, and nothing interesting happens when we apply the quantum double construction to quantum groups. So one might expect the same situation, that is, nothing interesting happens for the asymptotic inclusions of the Jones subfactors of type A_n, but this is not true. Ocneanu has observed in [36] that the tensor category of the M_∞-M_∞ bimodules for the subfactor $N \subset M$ of type A_{2n} is just a square, or the "double", of the original system of the M-M bimodules, but a more interesting and mysterious orbifold phenomenon occurs for $N \subset M$ of type A_{2n+1}. He attributed this phenomenon to the non-degeneracy of the braiding in his sense in [36], but the relation of his orbifold phenomenon to our orbifold subfactors explained above was not clear, and the reason for this orbifold phenomenon was also unclear. (The A_n subfactors are fairly simple objects and ad hoc arguments are enough for the computations of the asymptotic inclusions, so the real reason of the mysterious phenomenon is rather hidden.)

In order to clarify these natural questions, we worked in [13] on the asymptotic inclusions of the Hecke algebra subfactors of Wenzl [48]. Erlijman [8] had computed the indices of these subfactors constructed in a different way. (It was Goto [17] that proved Erlijman's construction indeed gives the asymptotic inclusions of the Wenzl subfactors.) We have shown in [13] the similar orbifold phenomenon occurs for the Wenzl subfactors with indices converging to 9 and identified Ocneanu's orbifold phenomenon with the orbifold construction in our sense. For this study, we work on the tube algebra and study *Ocneanu projections* labeled by pairs of primary fields of $SU(n)_k$. Then we can naturally understand the orbifold phenomenon and appearence of (the pairs of) the *ghosts* in Ocneanu's terminology in [36]. For these, we need detailed information about the quantum Littlewood–Richardson coefficients studied by Goodman–Wenzl [15].

Roughly speaking, our conclusion in [13] is as follows. The subtle orbifold phenomenon happens because we have a degenerate braiding at the beginning. The asymptotic inclusion "doubles" everything, but automatically invoke the orbifold construction in order to remove degeneracy in the resulting system. (Note that the asymptotic inclusion gives a non-degenerate braiding as long as the fusion graph of the original subfactor is connected. See [36].) We can also show as in [13] that the even vertices of the D_{2n} subfactors have a non-degenerate braiding, while the A_{4n-3} subfactors (before the orbifold construction) has a degenerate braiding. This is another evidence to the conceptual understanding that the orbifold construction removes degeneracy.

As the final remark, we mention the difference between the A_{4n-3} subfactors and the A_{4n-1} subfactors. These two classes have essential difference in the viewpoint of the obstruction to flatness in the orbifold construction, but this difference disappears in the asymptotic inclusions. A conceptual explanation to this fact is that the obstruction -1 to the flatness is squared in the quantum "double" procedure and hence vanishes. See [13] more on this.

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