Teoria dei Campi Superconformi e Algebre di Operatori

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Superconformal Field Theory and Operator Algebras

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Operator Algebraic Approach to Quantum Field Theory, particularly to Chiral Superconformal Field Theory.

1. **Local conformal nets and the vertex operator algebras**
2. **Jones theory of subfactors, modular tensor category and α-induction**
3. **Modular invariants and classification theory**
4. **Supersymmetry and the Connes noncommutative geometry**

(with J. Böckenhauer, S. Carpi, D. E. Evans, R. Hillier, R. Longo, M. Müger, U. Pennig, K.-H. Rehren, F. Xu, 1999–)

Quantum Field Theory: (mathematical aspects)

Mathematical ingredients: Spacetime, its symmetry group, quantum fields on the spacetime
$\longrightarrow$ mathematical axiomatization: Wightman fields (operator-valued distributions on the spacetime)

Wightman fields and test functions supported in a space time region O gives observables in O
$\longrightarrow$ von Neumann algebra $A(O)$ of bounded linear operators. (Closed in the $*$-operation and strong-operator topology)

Study a net $\{A(O)\}$ of von Neumann algebras. (Algebraic Quantum Field Theory — Haag, Araki, Kastler)

Chiral Conformal Field Theory:

We study the $(1+1)$-dimensional Minkowski space with higher symmetry, where we see much recent progress and connections to many different topics in mathematics. We restrict a quantum field theory to compactifications of the light rays $\{t = x\}$ and $\{t = -x\}$. One S^1 is now our spacetime.

$\mathrm{Diff}(S^1)$: the orientation preserving diffeomorphism group of S^1. This is our spacetime symmetry group.

This setting is called a chiral conformal field theory. With an operator algebraic axiomatization, we deal with families of von Neumann algebras acting on the same Hilbert space.

We now list our operator algebraic axioms for a chiral conformal field theory.

We have a family $\{A(I)\}$ of von Neumann algebras parameterized by open non-empty non-dense connected sets $I \subset S^1$. (Such an I is called an interval.)

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy condition
5. Vacuum vector Ω

The locality axiom comes from the Einstein causality.

Such a family $\{A(I)\}$ is called a local conformal net.

In all explicitly known examples, each $A(I)$ is always isomorphic to the unique Araki-Woods factor of type III_1.
$\Rightarrow$ Each $A(I)$ carries no information, but it is the family $\{A(I)\}$ that contains information on QFT.

A vertex operator algebra is another mathematical axiomatization of a chiral conformal field theory. The most famous example is the Moonshine vertex operator algebra of Frenkel-Lepowsky-Meurman.

Vertex operator algebras and local conformal nets are expected to have a bijective correspondence (under some nice extra assumptions).

We have a construction of the Moonshine local conformal net (K-Longo 2006), whose automorphism group is the Monster.

Representation theory: Superselection sectors

We now consider a representation theory for a local conformal net $\{A(I)\}$. Each $A(I)$ acts on the initial Hilbert space from the beginning, but consider a representation on another Hilbert space (without a vacuum vector).

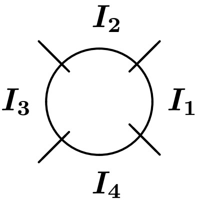
Each representation is given with an endomorphism of one factor $A(I_0)$. The image of the endomorphism is a subfactor of the factor $A(I_0)$, and it has the Jones index. Its square root is defined to be the dimension of the representation π, whose value is in $[1, \infty]$.

We compose the two endomorphisms. This gives a notion of a tensor product. We have a braided tensor category.

(Doplicher-Haag-Roberts + Fredenhagen-Rehren-Schroer)

Only finitely many irreducible representations: rationality

K-Longo-Müger (2001) gave an operator algebraic characterization of such rationality for a local conformal net $\{A(I)\}$ as follows, and it is called complete rationality.



Split the circle into I_1, I_2, I_3, I_4. Then complete rationality is given by the finiteness of the Jones index for a subfactor

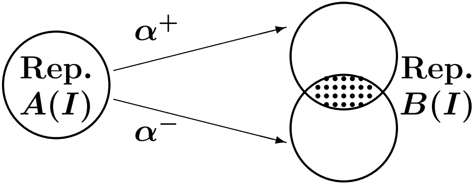
$$A(I_1) \vee A(I_3) \subset (A(I_2) \vee A(I_4))'$$

where $'$ means the commutant.

We have a classical notion of induction. Now introduce a similar construction for local conformal nets.

Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets. We extend an endomorphism of $A(I)$ to a larger factor $B(I)$, using a braiding. ($\alpha^{\pm}$-induction: Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

The intersection of endomorphisms arising from α^+-induction and those from α^--induction are exactly the representations of $\{B(I)\}$.



In the above setting, we automatically get a modular tensor category as the representation category of a local conformal net, and it produces a unitary representation π of $SL(2,\mathbb{Z})$ through its braiding. Its dimension is the number of irreducible representations.

Böckenhauer-Evans-K (1999) have shown that the matrix $(Z_{\lambda,\mu})$ defined by

$$Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$$

is in the commutant of π (using Ocneanu's graphical calculus). Such a matrix Z is called a modular invariant.

In many important examples, modular invariants have been explicitly classified by Cappelli-Itzykson-Zuber and Gannon.

Apply the above machinery to classify local conformal nets.

Any local conformal net $\{A(I)\}$ comes with a projective unitary representation of $\mathrm{Diff}(S^1)$. This gives a unitary representation of the Virasoro algebra defined as follows.

It is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a single central element c, the central charge, with the following relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

An irreducible unitary representation maps c to a real number, also called the central charge, in

$$\{1 - 6/m(m+1) \mid m = 3, 4, 5, \dots\} \cup [1, \infty).$$

(Friedan-Qiu-Shenker + Goddard-Kent-Olive)

Consider diffeomorphisms of S^1 trivial on the complement of an interval I. Their unitary images generate a von Neumann subalgebra of $A(I)$, which gives a Virasoro subnet $\{\mathrm{Vir}_c(I)\}$ with the same central charge value.
For $c < 1$, Xu's coset construction shows that Virasoro subnets are completely rational.

The corresponding unitary representations of $SL(2, \mathbb{Z})$ have been well-known, and their modular invariants have been classified by Cappelli-Itzykson-Zuber. They are labeled with pairs of the A-D-E Dynkin diagrams whose Coxeter numbers differ by 1.
Here we have different appearance of modular invariants.

Classification of local conformal nets with $c < 1$

(K-Longo 2004):

We now apply the above theory to classify local conformal nets with $c < 1$, since they are extensions of the Virasoro nets with $c < 1$. Here is the classification list.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.
(2) Their simple current extensions with index 2.
(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.
Three exceptionals in the above (3) are identified with coset constructions, but the other one does not seem to be related to any other known constructions. (Xu's mirror extensions)

We also have a formulation and a classification for a full conformal field theory based on a 2-dimensional local conformal net $\{B(I \times J)\}$ where I, J are intervals on S^1. That is, we completely classify all extensions of $\{\mathrm{Vir}_c(I) \otimes \mathrm{Vir}_c(J)\}$ for $c < 1$ (K-Longo 2004).

We also have a formulation of a boundary conformal field theory on the $1+1$-dimensional half Minkowski space $\{(x,t) \mid x > 0\}$ based on nets on the half space (Longo-Rehren). Based on this framework, we also have a complete classification of such nets on the half-space with $c < 1$ (K-Longo-Pennig-Rehren 2007).

The Longo-Rehren subfactors play an important role.

Geometric aspects of local conformal nets

Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. Recall the Weyl formula:

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

where the coefficients have a geometric meaning.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1. For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo 2005)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

Noncommutative geometry:

Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.

In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

The Dirac operator D is a “square root” of the Laplacian Δ. Expect some square root of the conformal Hamiltonian L_0 plays a similar role to the Dirac operator in noncommutative geometry. Supersymmetry produces such a square root.

The $N = 1$ super Virasoro algebras (Neveu-Schwarz, Ramond) are generated by the central charge c, the even elements L_n, $n \in \mathbb{Z}$, and the odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}$$

Our construction in superconformal field theory: We construct a family $(\mathcal{A}(I), H, D)$ of θ-summable spectral triples parameterized by intervals $I \subset S^1$ from a representation of the Ramond algebra. (Carpi-Hillier-K-Longo 2010)

One of the Ramond relations gives $G_0^2 = L_0 - c/24$. So G_0 should play the role of the Dirac operator, which is a "square root" of the Laplacian.

The representation space of the Ramond algebra is our Hilbert space H for the spectral triples (without a vacuum vector). The image of G_0 is now the Dirac operator D, common for all the spectral triples.

Then $\mathcal{A}(I) = \{x \in A(I) \mid [D, x] \in B(H)\}$ gives a net of spectral triples $\{\mathcal{A}(I), H, D\}$ parameterized by I.

$N = 2$ super Virasoro algebras (Ramond/N-S for $a = 0, 1/2$)

Generated by central element c, even elements L_n and J_n, and odd elements $G^{\pm}_{n\pm a}$, $n \in \mathbb{Z}$, with the following.

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n}, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^{+}_{n+a}, G^{-}_{m-a}] &= 2L_{m+n} + (n - m + 2a)J_{n+m} + \\
&\qquad \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}.
\end{aligned}
$$

For the discrete values of the central charge c, that is $c < 3$ now, we label the irreducible representations of the even part of the net with triples (j, k, l).
The chiral ring and the spectral flow are given by $\{(j, j, 0)\}$ and is by $(0, 1, 1)$ respectively.

We classify all $N = 1$ superconformal nets with discrete values of c, that is $c < 3/2$ now (Carpi-K-Longo 2008) and also all $N = 2$ superconformal nets with $c < 3$ (Carpi-Hillier-K-Longo-Xu). In the $N = 2$ superconformal case, we have a mixture of the coset construction and the mirror extension, which give a new type of simple current extensions with cyclic groups of large orders. This is a new feature in this $N = 2$ superconformal case.

Further studies:

Gepner model: Make a fifth tensor power of the $N = 2$ super Virasoro net with $c = 9/5$. This should give a setting of the Gepner model with mirror symmetry. The mirror symmetry appears as an isomorphism of two $N = 2$ super Virasoro algebras sending J_n to $-J_n$ and $G^{\pm}_m$ to $G^{\mp}_m$.

We have given a formula for the Witten index in terms of the Jones index (Carpi-K-Longo 2008). Further developments?

Computations of noncommutative geometric invariants such as entire cyclic cohomology with Jaffe-Lesniewski-Osterwalder cocycle: Possible connections to invariants in superconformal field theory. (Recent progress by Carpi-Hillier)