The hypergroupoid of boundary conditions in QFT

Karl-Henning Rehren

Institut für Theoretische Physik, Universität Göttingen

“OA & MP”, MSJ-SI Sendai, August 8, 2016

Abstract

Hypergroups (acting by completely positive maps) are generalized symmetries of quantum field theory in 1 and 2 dimensions. We show that boundary conditions between a pair of QFTs (with common stress-energy tensor) can be viewed as the morphisms of a hypergroupoid. Their completely positive action is linear on the charged generators, and naturally generalizes gauge transformations.

Joint project with **Marcel Bischoff** (Vanderbilt)

Plan:

Subfactors and hypergroups

Quantum field theory

Boundaries and boundary conditions

The hypergroupoid of boundary conditions

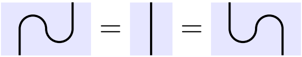
SUBFACTORS and HYPERGROUPS

Let $N \subset M$ be a subfactor (finite index, type III), $\iota : N \to M$ the embedding homomorphism, $\bar{\iota} : M \to N$ a conjugate homomorphism.

$\gamma = \iota\bar{\iota} \in \mathrm{End}_0(M)$ is called the **canonical endomorphism** (with range inside N), $\theta = \bar{\iota}\iota \in \mathrm{End}_0(N)$ the **dual canonical endomorphism**.

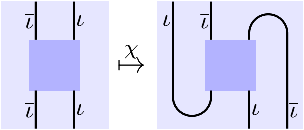
(Type II: θ corresponds to ${}_N M_N$, γ to ${}_M {M_1}_M$ where $N \subset M \subset M_1$ is the Jones basic construction.

We fix intertwiners $w \in \mathrm{Hom}(\mathrm{id}_N, \theta)$ and $v \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ satisfying the **conjugacy ("zig-zag") relations**

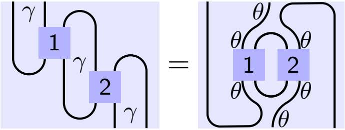
$v^*\iota(w) = 1_M$  $\bar{\iota}(v^*)w = 1_N$

normalized by $w^*w = d \cdot 1_N$, $v^*v = d \cdot 1_M$, where $d = [M : N]^{1/2}$ is called the dimension (of ι).

Frobenius duality ("Fourier transform") $\mathrm{Hom}(\theta,\theta) \leftrightarrow \mathrm{Hom}(\gamma,\gamma)$:



turns the **concatenation product** $\circ$ into the **convolution product** $*$:



i.e., $\chi(x_1) \circ \chi(x_2) = \chi(x_1 * x_2)$.

This was exploited earlier (KHR 1997, cf. also Böhm-Szlachanyi 1995) in order to view $N \subset M$ as fixed points w.r.t. a "Weak C* Hopf symmetry" of M, which suffered from a "depth 2 obstruction" for the coproduct (namely $\alpha \prec \gamma^2 \not\Rightarrow \alpha \prec \gamma$ entailing $\Delta(1) \neq 1 \otimes 1$).

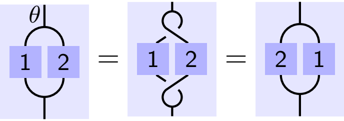
Lemma:

If θ is an object in a **braided** subcategory of $\mathrm{End}_0(N)$, and the Frobenius algebra $[\theta, w, x = \bar{\iota}(v)]$ is **commutative**, then $[\mathrm{Hom}(\theta,\theta), *]$ is commutative $\Leftrightarrow$ $[\mathrm{Hom}(\gamma,\gamma), \circ]$ is commutative $\Leftrightarrow$ γ is **multiplicity-free**:

$$\gamma \simeq \bigoplus_a \alpha_a$$

($\alpha_a \in \mathrm{End}_0(M)$ irreducible, pairwise inequivalent).

Proof of the first statement:



We shall now **assume that γ is multiplicity-free** (for the reason given in the Lemma, or for some other reason).

For subfactor theory, this condition may seem artificial.

But for certain subfactors in Quantum Field Theory, it is a consequence of **Locality** (see below).

We want to turn this feature into a **hypergroup action** on M, such that $N \subset M$ are the fixed points w.r.t. this action.

The hypergroup (cf. below for the definition) will be given by the minimal projections of $\mathrm{Hom}(\gamma, \gamma)$, equipped with the $*$-product.

We know that for an irreducible subfactor there is a unique **conditional expectation** $\mu : M \to N$ with fixed points N.

The unique conditional expectation $\mu : M \to N$ can be written as

$$\mu(m) = d^{-1} \cdot w^* \bar{\iota}(m) w.$$

Assume that γ is multiplicity-free, and decompose $\gamma = \sum_a t_a \alpha_a(\cdot) t_a^*$ with the help of isometries $t_a \in \mathrm{Hom}(\alpha_a, \gamma)$.

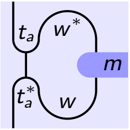
Then the map $\iota \circ \mu : M \to M$ can be decomposed as

$$\iota \circ \mu = d^{-1} \sum_a \iota(w^*) t_a \alpha_a(\cdot) t_a^* \iota(w),$$

where $t_a^* \iota(w) \in \mathrm{Hom}(\iota, \alpha_a \iota)$ are multiples of isometries.

Because $\iota(w^*)t_a t_a^* \iota(w) = \frac{d_a}{d} \cdot 1_M$, the completely positive maps

$$\phi_a(\cdot) = \frac{d}{d_a} \cdot \iota(w^*)t_a\alpha_a(\cdot)t_a^*\iota(w)$$



are normalized. Moreover, ϕ_a are N-N-bimodule maps, and $\mu \circ \phi_a = \mu$, hence ϕ_a preserve every μ-invariant state.

Therefore

$$\boxed{\iota \circ \mu = \frac{1}{[M:N]} \sum_a d_a \cdot \phi_a}$$

splits the conditional expectation $\mu : M \to N$ into a convex sum of **N-linear stochastic maps** $\phi_a : M \to M$.

ϕ_0 corresponding to $\alpha_0 = \mathrm{id}_M \prec \gamma$ is the identity map.

Definition. (see V.S. Sunder and N.J. Wildberger, 2003)

(i) A (finite-dimensional) **hypergroup** K is a unital associative algebra with a basis k_a $(a = 0, \cdot|K| - 1)$ such that $k_0 = 1$, a product that is a convex sum:

$$k_a k_b = \sum\nolimits_c C_{ab}^c \, k_c \quad \text{with} \quad C_{ab}^c \geq 0, \ \sum\nolimits_c C_{ab}^c = 1,$$

and a conjugation $k_a \to k_{\bar{a}}$ such that

$$C_{ab}^0 > 0 \quad \Leftrightarrow \quad b = \bar{a}.$$

(ii) $w_a = 1/C_{a\bar{a}}^0$ is called the weight of k_a. The Haar measure of the hypergroup is the element $(\sum_a w_a)^{-1} \sum_a w_a \cdot k_a$.

Well-known examples:

The cosets of a group w.r.t. a normal subgroup

The double cosets of a group w.r.t. a subgroup

The conjugacy classes of a group

Fusion algebras

Proposition. (Bischoff 2016)

(i) The normalized minimal projections $k_a = \frac{d}{d_a} \cdot t_a t_a^* \in \mathrm{Hom}(\gamma, \gamma)$, equipped with the $*$-product, form a finite hypergroup K, where the conjugation is induced by the sector conjugation $\alpha_a \prec \gamma \Rightarrow \overline{\alpha}_a \prec \gamma$. The weights are given by the dimensions $w_a = d_a$.

(ii) ϕ_a define a *-action of K on M.

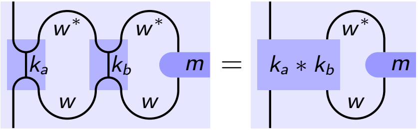
(iii) The action of the Haar measure of the hypergroup coincides with $\iota \circ \mu$, and $N = M^K$.

Example: If $N = M^G$ under the action of a finite group G, then $\gamma = \bigoplus_{g \in G} \alpha_g$, $K = G$, and μ is the group average. In this case, α_g are actually automorphisms, and $\phi_g = \alpha_g$.

In general, ϕ_a are **not even homomorphisms**.

For the proof, one first notices that $k_a * k_b$ is a linear combination of k_c, because these span $\mathrm{Hom}(\gamma, \gamma)$, and the coefficients are non-negative numbers, because the $*$-product of two projections is a positive operator.

Second, the composition $\phi_a \circ \phi_b$ of stochastic maps coincides with the $*$-product of k_a:



It only remains to check the proper normalization.

By this result, every finite-index subfactor $N \subset M$ with multiplicity-free canonical endomorphism $\gamma \in \mathrm{End}(M)$ defines a hypergroup K and an action on M, such that $N \subset M$ is the **fixed point subfactor** $N = M^K \subset M$.

Proposition. (Bischoff, KHR 2016)

The maps ϕ_a preserve the subspaces of isometries $H_\rho = \mathrm{Hom}(\iota, \iota\rho) \subset M$ $(\rho \prec \theta)$. Their linear actions on H_ρ are the **matrix representations** of the hypergroup K.

QUANTUM FIELD THEORY

A **quantum field theory** is a collection (= net) of von Neumann algebras (type III factors) $A(O)$ assigned to spacetime regions O, subject to Haag-Kastler axioms. In physics language: $A(O)$ are the observables accessible by experiments inside O.

Locality = Einstein causality: **observables at spacelike separation commute**.

Superselection sectors are (a class of) Hilbert space representations of the net A, that can be described by DHR endomorphisms of $A(O)$. The distinguished vacuum representation has the trivial DHR endomorphism id_A.

DHR endomorphisms (localized in O) are the objects of a unitary **braided tensor category** in $\mathrm{End}_0(A(O))$.

One QFT A is an (irreducible) **subtheory** of another QFT B if

$$A(O) \subset B(O)$$

is an (irreducible) subfactor for all O. The index of the subfactor does not depend on O. With some regularity assumption on A, finite index $[B(O) : A(O)]$ is automatic.

Proposition: (Longo, KHR 1995)

The dual canonical endomorphism $\theta_O \in \mathrm{End}(A(O))$ is a DHR endomorphism of A, restricted to $A(O)$.

The subsectors of θ are regarded as "generalized charges" carried by isometric **charged fields** $\Psi_\rho \in H_\rho = \mathrm{Hom}(\iota, \iota\rho)$ in B, one for every subsector $\rho \prec \theta$ (with multiplicities), while fields in A are "neutral".

In the case of finite index, the net B can be recovered from the net A and a single subfactor $A(O_0) \subset B(O_0)$, more precisely by its Frobenius algebra (Q-system) $[\theta_O, w_O, x_O = \bar{\iota}_O(v_O)]$.

An extension $B(O) \supset A(O)$ of a local QFT is local iff the Frobenius algebra $[\theta, w, x]$ is commutative:

$$\varepsilon_{\theta,\theta}\, x = x.$$

Hence the previous subfactor results apply: the canonical endomorphisms γ_O (associated with the subfactors $A(O) \subset B(O)$) are multiplicity-free, and $A(O) = B(O)^K$ are the **fixed points** of the compatible actions of a finite hypergroup K.

K acts linearly on the charged fields.

(The compatibility of the hypergroup actions for different regions O is a nontrivial issue, that requires some work.)

The fixed-point property $A(O) = B(O)^H$ means that hypergroups naturally arise as **generalized symmetries**.

In 4D (three dimensions of space plus time), where the braiding is always a permutation symmetry for geometric reasons, it follows from the work of Doplicher, Haag and Roberts ("DHR theory"), that the hypergroup is a double quotient $K = H\backslash G/H$ of the gobal gauge group G of the subtheory A by some subgroup H:

$$A = F^G \subset B = F^H \subset F \quad \Rightarrow \quad A = B^{H\backslash G/H}.$$

In contrast, **low-D quantum field theories with proper braidings admit more general hypergroups**.

BOUNDARY CONDITIONS
between TWO QFTs

Consider **two local extensions** $B^L \supset A$ and $B^R \supset A$ of a QFT A in two spacetime dimensions.

Proposition. (Bischoff-Kawahigashi-Longo-KHR 2015)

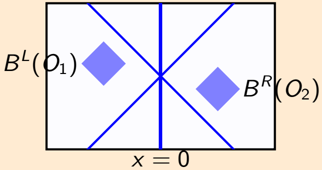
There is a "universal construction" = a nonlocal extension $C \supset A$ (along with a vacuum representation) such that

$$\boxed{\begin{array}{ccccc} & & B^L & & \\ A & \begin{array}{c}\nearrow \\ \searrow\end{array} & & \begin{array}{c}\searrow \\ \nearrow\end{array} & C \\ & & B^R & & \end{array}} ,$$

and the embedded B^L is left-local w.r.t. the embedded B^R, and both generate C.

Left-local means that $B^L(O_1)$ commutes with $B^R(O_2)$ whenever O_1 is spacelike-left of O_2.

Left-locality ensures that the following prescription ("boundary") is physically meaningful: Choose a timelike boundary, e.g., the time axis $x = 0$ in the two-dimensional spacetime. For regions O left of the boundary ($x < 0$) assign the local observables B^L, and for regions right of the boundary assign $B^R(O)$. Namely, Einstein causality holds also "across the boundary".



The above scenario is also necessary if the QFTs on both sides of the boundary share the same stress-energy tensor, e.g., because energy and momentum are conserved at the boundary.

$$\boxed{\begin{array}{ccccc} & & B^L & & \\ A & \nearrow\!\!\!\!\hookrightarrow & & \searrow\!\!\!\!\hookrightarrow & C\,, \\ & \searrow\!\!\!\!\hookrightarrow & & \nearrow\!\!\!\!\hookrightarrow & \\ & & B^R & & \end{array}}$$

We denote the embeddings as

$$\iota^X : A \hookrightarrow B^X, \qquad \jmath^X : B^X \hookrightarrow C \qquad (X = L, R)$$

such that

$$\jmath^R \circ \iota^R = \jmath^L \circ \iota^L.$$

Even if $B^L = B^R (=: B)$, there are **two copies of B** in C, embedded as $\jmath^L(B) \neq \jmath^R(B)$, but only one copy of A.

The universal construction is neither irreducible nor factorial. We have proven (exploiting that B^L, B^R are local):

Proposition. (BKLR 2015)

$A' \cap C = C' \cap C$, hence the central decomposition of C equals the decomposition into irreducible extensions.

Every irreducible summand of the universal construction is a **boundary condition** between B^L and B^R, and every boundary condition arises in this way.

The boundary conditions can be characterized in terms of relations between charged fields, as follows:

$A \subset B$ is generated by A and a finite number of charged fields $\Psi_\rho \in H_\rho$. Thus, C is generated by A and $\jmath^L(H^L_\rho)$ and $\jmath^R(H^R_\rho)$.

The center of C is spanned by the "neutral" products $\Psi^{L*}_{\rho,j}\Psi^R_{\rho,i}$.

Hence these are linear combinations of the minimal central projections of C:

$$\Psi^{L*}_{\rho,j}\Psi^R_{\rho,i} = \sum\nolimits_a S_{\rho,ji;a} \cdot E_a.$$

The boundary condition specified by the range of E_a is therefore characterized by the **sesquilinear relations** among the charged fields

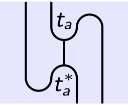
$$\boxed{\pi_a(\Psi^{L*}_{\rho,j}\Psi^R_{\rho,i}) = S_{\rho,ji;a} \cdot 1.}$$

The practical problem is to compute the coefficients $S_{\rho,ji;a}$ from the given pair of Frobenius algebras, e.g., by computing E_a as linear combinations of $\Psi^{L*}_{\rho,j}\Psi^R_{\rho,i}$ and solving for the latter.

Proposition. (BKLR 2015)

The minimal central projections E_a (hence the irreducible boundary conditions) are in 1:1 correspondence with the **irreducible subsectors** $\alpha_a \prec \iota^L\bar\iota^R : B^R \to B^L$.
Locality of B^L, B^R ensures that $\iota^L\bar\iota^R$ is multiplicity-free.

We have given a formula for E_a in terms of intertwiners $I_a \in \mathrm{Hom}(\theta^R, \theta^L)$, that "diagonalize the $*$-product". Because the projections $t_a t_a^* \in \mathrm{Hom}(\iota^L\bar\iota^R, \iota^L\bar\iota^R)$ (where $t_a \in \mathrm{Hom}(\alpha_a^{LR}, \iota^L\bar\iota^R)$) diagonalize the $\circ$-product, I_a **are their Frobenius-Fourier transforms**.

$I_a =$ 

In particular: In the case $B^L = B^R =: B$, the boundary conditions are in 1:1 correspondence with the elements of the hypergroup K such that $A = B^K$.

Proposition. (KHR 2016)

If $B^L = B^R = B$, then one has

$$\boxed{E_a = \sum\nolimits_\rho \frac{d_a d_\rho}{d_\theta} \sum\nolimits_i \jmath^L \circ \phi_a(\Psi^*_{\rho,i})\jmath^R(\Psi_{\rho,i})}$$

where the sum over i runs over any orthonormal basis of H_ρ.

Since ϕ_a acts on H_ρ as a matrix representation of K, the coefficients $S_{\rho,ji;a}$ are **determined by the representation theory of K**, by solving this equation for $\Psi^{L*}_{\rho,j}\Psi^R_{\rho,i}$.

Special case: $B^L = B^R = B$ and $A = B^G$ the fixed points under the action of a finite group G, i.e., $K = G$.

In this case, $\alpha_a \prec \iota\bar\iota = \bigoplus_g \alpha_g$ are the **gauge automorphisms**, and $\phi_g = \alpha_g$. Thus, we have $E_g = \sum_\rho \frac{d_\rho}{|G|} \sum_{ji} \overline{u^\rho(g)_{ji}} \cdot \Psi^{L*}_{\rho,j} \Psi^R_{\rho,i}$. By **Peter-Weyl**, one can solve:

$$\Psi^{L*}_{\rho,j} \Psi^R_{\rho,i} = \sum\nolimits_g u^\rho(g)_{ji} \cdot E_g,$$

and because $\Psi_{\rho,i}$ are isometries and u^ρ **unitary** matrices, the **Cauchy-Schwarz** inequality implies the **linear relations**

$$\pi_g(\Psi^R_{\rho,i}) = \sum\nolimits_j \pi_g(\Psi^L_{\rho,j}) u_\rho(g)_{ji}.$$

Thus, $H^L_\rho = H^R_\rho$ coincide as spaces in each irreducible representation π_g of the universal construction C, but the identification of the charged fields depends on g, acting by the gauge transformation α_g.

In the case of hypergroups, the sesquilinear relations in general do **not** imply linear relations, hence **in general** $H^L_\rho \neq H^R_\rho$.

An interesting picture emerges in the case $B^L = B^R = B$ and $A = B^G$:

The local algebras $B^L(O)$ and $B^R(O)$ coincide for every region, but their generators on the left and on the right of the boundary differ by a gauge transformation. The embeddings into C are related by

$$\boxed{\jmath^R = \sum_{g\in G} E_g \cdot \jmath^L \circ \alpha_g.}$$

Thus, boundary conditions are **"local gauge transformations"** with only two values.

By subdividing spacetime into (infinitely) many cells with many boundaries = edges of the dual graph, and a boundary condition = gauge transformation attached to each edge, one obtains the local lattice gauge group!

What is the analogue of the **composition of boundary conditions** upon juxtaposition of two boundaries in the general (low-D) case?

THE HYPERGROUPOID of BOUNDARY CONDITIONS

Back to the general case: fix a local QFT A and two local extensions, and consider

$$\begin{array}{ccccc} & \nearrow & B^L & \searrow & \\ A & & & & C \\ & \searrow & B^R & \nearrow & \end{array} ,$$

where C is the "universal construction".

Notice that a completely rational QFT A admits only finitely many inequivalent irreducible local extensions.

We want to make more precise the statement

"The subsectors α_a^{LR} of $\iota^L\bar{\iota}^R : B^R \to B^L$ classify the boundary conditions."

Exactly as the subsectors of $\gamma = \iota\bar{\iota}$ decompose $\iota \circ \mu$ into stochastic maps (see above), the sector decomposition of $\iota^L\bar{\iota}^R$ gives rise to the convex decomposition

$$\boxed{\iota^L \circ \mu^R(\cdot) = \frac{1}{d^R}\iota^L(w^{R*}\bar{\iota}^R(\cdot)w^R) = \frac{1}{d^L d^R}\sum\nolimits_a d_a \cdot \phi_a^{LR}(\cdot)}$$

where $\phi_a^{LR}(\cdot) = \frac{d^L}{d_a} \cdot \iota^L(w^{R*})t_a\, \alpha_a^{LR}(\cdot)\, t_a^*\iota^L(w^R)$.

Notice that now α_a^{LR} are homomorphisms and ϕ_a^{LR} are stochastic maps : $B^R \to B^L$ between different algebras.

Consider now the juxtaposition of two boundaries, say between B^X and B^Y, and between B^Y and B^Z.

Their boundary conditions provide us with stochastic maps $\phi_a^{XY} : B^Y \to B^X$ and $\phi_b^{YZ} : B^Z \to B^Y$.

Proposition. (KHR 2016)

(i) The boundary conditions among local nets $B^i \supset A$ **define a hypergroupoid** by the $\circ$-product of $l_a \in \mathrm{Hom}(\theta^Y, \theta^X)$, or equivalently by the $*$-product of $t_a t_a^* \in \mathrm{Hom}(\iota^X \bar{\iota}^Y, \iota^X \bar{\iota}^Y)$.

(ii) The stochastic maps $\phi_a^{XY} : B^Y \to B^X$ **provide an action** of the hypergroupoid by homomorphisms among the algebras B^i.

The sources and ranges of the hypergroupoid correspond to the irreducible local extensions $B^i \supset A$ of the given local QFT A.

Proposition. (KHR 2016)

Again, ϕ_a^{LR} map charged fields $\Psi_\rho^R \in H_\rho^R$ linearly to H_ρ^L (giving a matrix representation of the hypergoupoid), and one has

$$\boxed{E_a = \sum\nolimits_\rho \frac{d_a d_\rho}{d^L d^R} \cdot \sum\nolimits_i \jmath^L \phi_a^{LR}\big(\Psi_{\rho,i}^{R*}\big)\jmath^R\big(\Psi_{\rho,i}^R\big),}$$

which can be solved for the numerical values of $\jmath^L(\Psi_{\rho,j}^{L*})\jmath^R(\Psi_{\rho,i}^R)$ in the range of the projections E_a.

Again, these sesquilinear relations in general do not imply linear relations among the charged fields. The subspaces H_ρ^L and H_ρ^R may have different dimensions, and stand "in skew angles" within C.

The hypergroupoid fusion by concatenation of intertwiners $I_a \in \mathrm{Hom}(\theta^Y, \theta^X)$ was presented in [KHR, ICMP 2015] (without recognizing the hypergroupoid structure, nor knowing the stochastic maps).

Bartels-Douglas-Henriques (2013) consider a different composition of boundary conditions, by composing the homomorphisms $\alpha_a^{XY} \circ \alpha_b^{YZ}$. The result is a subsector of $\iota^X \theta^Y \bar{\iota}^Z$ but in general not of $\iota^X \bar{\iota}^Z$.

Hence the BDH fusion does not close among the boundary conditions but among a larger set of **defects** (they violate the condition that C is generated by B^L and B^R). BDH fusion is a tensor two-category.

"Morally", the BDH fusion differs from the hypergroup fusion as follows: The former retains the intermediate charged fields Ψ^Y as additional degrees of freedom, in general not contained in $B^X \vee B^Z$; whereas the latter classifies the inequivalent ways to "eliminate" Ψ^Y from the specified numerical values of $\Psi^{X*}\Psi^Y$ and of $\Psi^{Y*}\Psi^Z$.

There are cases (apart from the group case), especially when modular invariance is available, where the two different fusions become the same, i.e., the hypergroupoid is the fusion algebra of a tensor two-category. Other cases are very distinct.

I want to suggest that the hypergroupoid fusion of boundary conditions is the closest analogue of "local gauge transformations", that one may hope to get in low-D QFT.