Approximating freeness under constraints

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Sendai, August 2016

HONORING 60 YEARS FROM SAKAI'S THEOREM

General theme of lectures

Given a II_1 factor M, a subalgebra $B \subset M$, a finite set of "special" elements $F \subset M$, construct elements $u \in \mathcal{U}(B)$ that are "as independent as possible" with respect to F, i.e., words with alternating letters in $\{u^k \mid k \in \mathbb{Z}, k \neq 0\}$ and in F, should have moments close to zero, $\tau(x_0 u^{k_1} x_1 u^{k_2} \ldots) \approx 0$, $x_i \in F$, $k_i \neq 0$.

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But the fact that x_i are "constraint" to be in F and u in B may force that only part of this is possible, resulting in a set $\mathcal{R}$ of "achievable goals" (e.g., "all k_i must be positive", or "length of words must be ≤ 6", or "all letters x_i from F within a given word must be distinct")

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As it turns out, the solution to some special cases of this problem has important applications to various areas of von Neumann algebras, such as cohomology theory, subfactor theory, orbit equivalence, or paving problems.

In each one of the cases when we can solve this problem, the element $u \in \mathcal{U}(B)$ is constructed via **incremental patching**, a technique that's emblematic of "II_1 factor analysis".

Plan of lectures

1. The case B diffuse abelian: 1-independence and semigroup freeness
2. The case B an arbitrary MASA: 3-independence
3. The case B a singular MASA: complete freeness
4. The case B Cartan and $F \subset \mathcal{N}(B)$: Bernoulli-freeness
5. The case B a II_1 factor: freeness relative to $B' \cap M$
6. Applications

S. Popa: *A II_1 factor approach to the Kadison-Singer problem*, Comm. Math. Physics. **332** (2014), 379-414 (math.OA/1303.1424).

S. Popa: *Independence properties in subalgebras of ultraproduct II_1 factors*, JFA **266** (2014), 5818-5846 (math.OA/1308.3982)

1. Semigroup freeness when B is abelian diffuse

Lemma (local quantization)

Let M II_1 factor and $B \subset M$ a von Neumann subalgebra. For any finite F in unit ball of $M \ominus B \vee (B' \cap M)$, $Y_0 \subset B$ and any $\varepsilon > 0$, $\exists q \in \mathcal{P}(B)$ such that $\|qxq\|_1 < \varepsilon\tau(q)$, $|\tau(qy) - \tau(q)\tau(y)| < \varepsilon\tau(q)$, $\forall x \in F$, $y \in Y_0$. If in addition B is type II_1, then same holds true $\forall F \subset M \ominus (B' \cap M)$, and q can be taken to have scalar central trace in B.

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Proof. We'll only prove the first part. Sufficient to show the following

Fact. For any $y \in M \ominus (B \vee B' \cap M)$ there exists a finite partition with projections $q_1, ..., q_m \in B$ such that $\|\Sigma_i q_i y q_i\|_2^2 \leq 3/4\|y\|_2^2$.

Indeed, because $y \perp B \vee (B' \cap M)$ and $q_i \in B$ implies $\Sigma_i q_i y q_i \perp B \vee (B' \cap M)$ so we get the conclusion by applying the *Fact* recursively $n|F|$ times, to the elements $y \in F$ and to $\Sigma_i q_i B q_i$ in lieu of B, with n taken so that $(3/4)^n \leq \varepsilon/|F|$. Thus, in the end we get a partition $p_i \in \mathcal{P}(B)$ such that $\Sigma_{x\in F}\|\Sigma_i p_i x p_i\|_2^2 \leq \varepsilon^2 = \varepsilon^2\|\Sigma_i p_i\|_2^2$. By Pythagora, for "most of the i" we get $\|p_i x p_i\|_2^2. \leq \varepsilon\|p_i\|_2^2$, $\forall x \in F$, implying $\|p_i x p_i\|_1 \leq \varepsilon\tau(p_i)$, $\forall x \in F$, as well.

To prove the *Fact*, assume on the contrary that for any finite partition of 1 with projections $q_i \in B$ we have $\|\Sigma_i q_i y q_i\|_2^2 > 3/4\|y\|_2^2$. Thus, if $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| = 1$, then the unitary $u = \Sigma_i \lambda_i q_i \in B$ satisfies

$$\|uyu^* - y\|_2^2 = \|\Sigma_{i\neq j}(\lambda_i \overline{\lambda_j} - 1) q_i y q_j\|_2^2$$

$$= \Sigma_{i\neq j}|\lambda_i \overline{\lambda_j} - 1|^2 \|q_i y q_j\|_2^2 \leq 4\Sigma_{i\neq j}\|q_i y q_j\|_2^2$$

$$= 4\|\Sigma_{i\neq j} q_i y q_j\|_2^2 = 4\|y - \Sigma_i q_i y q_i\|_2^2 = 4\|y\|_2^2 - 4\|\Sigma_i q_i y q_i\|_2^2 < \|y\|_2^2.$$

Since unitaries with finite spectrum are norm dense in $\mathcal{U}(B)$, from first and last term we get $\|uyu^* - y\|_2^2 \leq \|y\|_2^2$, $\forall u \in \mathcal{U}(B)$. Thus $2\|y\|_2^2 - 2\Re\tau(uyu^*y^*) \leq \|y\|_2^2$, implying that $2\Re\tau(uyu^*y^*) \geq \|y\|_2^2$, $\forall u \in \mathcal{U}(B)$. By taking convex combinations of the elements of the form uyu^* over $u \in \mathcal{U}(B)$ and then taking into account that 0 belongs to the weak closure of such elements (because $L^2(M \ominus (B \vee B' \cap M))$ contains no non-zero points fixed by $\mathrm{Ad}\mathcal{U}(B)$), it follows that $0 \geq \|y\|_2^2$, a contradiction.

Theorem 1

Let M be a II_1 factor and $A \subset M$ abelian diffuse. Given any $F \subset M$ finite, any $\varepsilon > 0$ and $n \geq 1$, there exists a Haar unitary $u \in \mathcal{U}(A)$ such that $|\tau(x_0 \Pi_{i=1}^{k}(u^{j_i} x_i))| \leq \varepsilon$, $\forall\ x_i \in F$, $0 < j_i \leq n$, $1 \leq k \leq n$.

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Proof. Let $\mathcal{W}$ denote the set of partial isometries $v \in A$ satisfying $\tau(v^k) = 0$, $\forall k \neq 0$, and $|\tau(x_0 \Pi_{i=1}^{k}(v^{j_i} x_i))| \leq \varepsilon \tau(v^* v)$, $\forall\ x_i \in F$, $0 < j_i \leq n$, $0k \leq n$. We endow $\mathcal{W}$ with the order $v_1 \leq v_2$ if $v_1 = v_1 v_1^* v_2$. $(\mathcal{W}, \leq)$ is clearly inductively ordered. Let $v \in \mathcal{W}$ be a maximal element. Assume $v^* v \neq 1$ and let $p = 1 - v^* v \in A$. If w is a partial isometry in Ap and we denote $u = v + w$, then $u^{j_i} = v^{j_i} + w^{j_i}$ and we have

$$|\tau(x_0 \Pi_{i=1}^{k}(u^{j_i} x_i))| \leq |\tau(x_0 \Pi_{i=1}^{k}(v^{j_i} x_i))| + \Sigma |\tau(\ldots x_{i-1} w^{j_i} x_i \ldots.)|$$

where the sum is taken over all terms that have at least one occurrence of w^{j_i}. Since $v \in \mathcal{W}$, we have $|\tau(x_0 \Pi_{i=1}^{k} v^{j_i} x_i)| \leq \varepsilon \tau(v v^*)$. We will prove that we can choose $w \neq 0$ so that the summation on the right hand side is majorized by $\varepsilon \tau(w w^*)$, giving

$$|\tau(x_0 \Pi_{i=1}^{k}(u^{j_i} x_i))| \leq \varepsilon \tau(v v^*) + \varepsilon \tau(w w^*) = \varepsilon \tau(u u^*)$$

This will contradict the maximality of v, thus showing that $vv^* = 1$, i.e v is a Haar unitary in A. We construct w by first making a choice for its support projection $q = ww^*$, then choosing w as an appropriate Haar unitary in Aq. By applying the first part the Lemma to a sufficiently small $\delta > 0$ and the finite set X of all elements of the form $pzp - E_{(A' \cap M)p}(pzp) \in p(M \ominus A' \cap M)p$, where $z = x_0 \Pi(v^{j_i} x_i)$ with x_i, $j_i > 0$ as before, it follows that

$$\Sigma|\tau(...x_{i-1} w^{j_i} x_i....)| \leq \delta K \tau(q) + \Sigma_l |\tau(w^{m_l} z')|$$

for some $z' \in A' \cap M$ in a finite set (depending on F and n), some $K > 0$ and $m_l = \Sigma_{i=1}^{l} j_i$, with l the number (≥ 1) of occurrences of w. We take δ sufficiently small, so that $\delta K \leq \varepsilon/2$. Finally, since Aq contains Haar unitaries w converging weakly to 0, we can make $\Sigma_l |\tau(w^{m_l} z')| \leq (\varepsilon/2)\tau(q)$ as well.

2. Approximate 3-independence in arbitrary MASAs

Terminology

Two sets $V, W \subset M \ominus \mathbb{C}$ are ***n*-independent** if any alternating word $x_1 y_1 \ldots . x_k y_k$, with $k \leq n$ and $x_1 \in V \cup \{1\}$, $x_2, .., x_k \in V$, $y_1, ..., y_{k-1} \in W$, $y_k \in W \cup \{1\}$, has trace 0 (unless $k = 1$ and $x_1 = y_1 = 1$). An algebra $B_0 \subset M$ is n-independent to V if V and $B_0 \ominus \mathbb{C}$ are n-independent.

Note that 1-independence amounts to what one usually calls τ-independence.

More generally, if $P \subset M$ is a von Neumann subalgebra, then two sets $V \subset M \ominus P$, $W \subset M \ominus P$ are ***n*-independent relative to** P if $E_P(\Pi_{i=1}^k x_i y_i) = 0$, for all $1 \leq k \leq n$, all $x_1 \in V \cup \{1\}$, $x_i \in V$, $y_k \in W \cup \{1\}$, $y_i \in W$.

Theorem 2

Let M be a finite von Neumann algebra and $A \subset M$ a MASA. Given any finite sets $F \subset M \ominus A$, $Y_0 \subset A \setminus \mathbb{C}1$, any $n \geq 1$ and $\varepsilon > 0$, there exists a Haar unitary $u \in A$ such that $|\tau(u^{j_0} y_0)| \leq \varepsilon$, $|\tau(\Pi_{j=1}^{k} u^{j_i} x_i)| \leq \varepsilon$, $\forall 0 < |j_i| \leq n$, $1 \leq k \leq 3$, $x_i \in F$, $y_0 \in Y_0$.

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Equivalently: given any $\| \|_2$-separable subsets $X \subset M^\omega \setminus A^\omega$, $Y \subset A^\omega \ominus \mathbb{C}1$, there exists a separable diffuse von Neumann subalgebra $A_0 \subset A^\omega$ such that A_0 is 3-independent to X and τ-independent to Y.

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Proof. Denote $\mathcal{W} = \{v \in A \mid vv^* \in \mathcal{P}(A), |\tau(\Pi_{i=1}^{k} v^{j_i} x_i)| \leq \delta\tau(v^* v), \forall 1 \leq k \leq 3, |j_i| \leq 3, \tau(v^m) = 0, \forall m \neq 0\}$. Endow $\mathcal{W}$ with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^* v) < 1$ and denote $p = 1 - v^* v$. If w is a partial isometry in Ap and $u = v + w$, then by using that $u^{j_i} = v^{j_i} + w^{j_i}$ and expanding $x = u^{j_1} x_1 u^{j_2} x_2 ... u^{j_k} x_k$, $k = 1, 2, 3$, as a binomial product, we get

$$|\tau(x)| \leq |\tau(\Pi_{j=1}^{k} v^{j_i} x_j)| + \Sigma |\tau(... x_{i-1} w^{j_i} x_i ....)|,$$

Note that for each summand for which we have 2 or 3 appearances of non-zero powers of w in the above sums (one term for $k = 2$ and four terms for $k = 3$), such appearances can be brought to be consecutive, i.e. they will be of the form $|\tau(....w^i y w^j ...)|$, for some $i, j \neq 0$, $y \in F \subset M \ominus A$.

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If $q = ww^*$, then for each one of these terms we have $|\tau(....w^i y w^j ...)| \leq \|qyq\|_1$. By Lemma, one can choose $q \in Ap$ such that $\|qyq\|_1 \leq 2^{-3}\varepsilon\tau(q)$, $\forall y \in pFp$. It thus follows that the sum of terms having two or more appearances of powers of w are majorized by $2^{-1}\varepsilon\tau(q)$.

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All remaining terms and the case $k=1$ have just one occurrence of w^j, $j \neq 0$, i.e are of the form $|\tau(y_1 w^j y_2)| = |\tau(w^j E_A(qy_2y_1q))|$, for some $y_1, y_2 \in M$, $1 \leq |j| \leq n$. There are k many such terms for each $k = 1, 2, 3$.

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Since $\{w_0^m\}_m$ tends to 0 in the weak operator topology and $Y \subset A$ is a finite set, there exists $n_0 \geq n$ such that $|\tau(w_0^m y)| \leq 2^{-4}\delta\tau(q)$, for all $y \in Y$ and $|m| \geq n_0$. But then $w = w_0^{n_0}$ is still a Haar unitary and it satisfies all the required conditions.

3. Approximate freeness in singular MASAs

Theorem 3

Let M be a II_1 factor and $B = A \subset M$ a singular MASA. Given any $F \subset M \ominus A$, $Y_0 \subset A \ominus \mathbb{C}1$ finite, any $\varepsilon > 0$, $n \geq 1$, there exists a Haar unitary $u \in A$ s.t. $\|E_A(x_0 \Pi_{j=1}^k u^{j_i} x_i)\|_1 \leq \varepsilon$, $|\tau(y_0 u^{j_0})| \leq \varepsilon$, for any $1 \leq k \leq n$, $x_0 \in F \cup \{1\}$, $x_i \in F$, $0 < |j_i| \leq n$, $y_0 \in Y_0$.

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Equivalently: given any $\| \, \|_2$-separable subsets $X \subset M^\omega \setminus A^\omega$ and $Y \subset A^\omega \ominus \mathbb{C}1$, there exists a separable diffuse von Neumann subalgebra $A_0 \subset A^\omega$ such that A_0 is free-independent to X and τ-independent to Y.

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Consequently: Given any $P \subset M^\omega$ separable vN subalgebra making a commuting square with A^ω and any $A_0 \subset A^\omega$ separable, there exists a diffuse vN subalgebra $A_1 \subset A^\omega$ such that if we denote by $A_1 = P \cap A^\omega$, then A_1 and A_0 are in tensor product and $P \vee A_1 = P *_{A_0} (A_1 \otimes A_0)$. In particular, if $P \perp A^\omega$ then $P \vee A_1 = P * A_1$.

Proof. Let $\delta = 2^{-(n+1)^2}\varepsilon$. Denote $\varepsilon_0 = \delta,\ \varepsilon_k = 2^k\varepsilon_{k-1}, k = 1, 2, ..., n$. Let

$$\mathcal{W} = \{v \in A \mid vv^* \in \mathcal{P}(A), \|E_A(x_0\Pi_{i=1}^k v^{j_i}x_i)\|_1 \leq \varepsilon_k\tau(v^*v),$$

$$|\tau(yv^{j_0})| \leq \varepsilon\tau(vv^*), \forall 1 \leq k \leq n, 0 < |j_i| \leq n, y \in Y\}.$$

Endow $\mathcal{W}$ with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 = w_2w_1^*w_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$.

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If w is a partial isometry in Ap and $u = v + w$, then

$$x_0 \Pi_{i=1}^k u_s x_s = x_0 \Pi_{i=1}^k v_s x_s + \Sigma_\ell \Sigma_i z_{0,i} \Pi_{j=1}^\ell w_{i_j} z_{j,i},$$

where the second sum is taken over all $\ell = 1, 2, \ldots, k$ and all $i = (i_1, \ldots, i_\ell)$, with $1 \leq i_1 < \cdots < i_\ell \leq k$, and where $w_{i_j} = w^t$ whenever $v_{i_j} = v^t$, $z_{0,i} = x_0 v_1 x_1 \cdots x_{i_1 - 1} p$, $z_{j,i} = p x_{i_j} v_{i_j + 1} \cdots v_{i_{j+1}} x_{i_{j+1}} p$, for $1 \leq j < \ell$, and $z_{\ell,i} = p x_{i_\ell} v_{i_\ell + 1} \cdots v_k x_k$.

Thus we get

$$\|E_A(x_0\Pi_{i=1}^k u_s x_s\|_1 \leq \|E_A(x_0\Pi_{s=1}^k v_s x_s\|_1 + \Sigma_\ell \Sigma_i z\|E_A(_{0,i}\Pi_{j=1}^\ell w_{i_j} z_{j,i}\|_1,$$

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By applying Lemma (or even better Thm. 2) to the finite set X of all elements of the form $pzp - E_{Ap}(pzp) \in pMp \ominus Ap$, where z is of the form $z_{j,i}$, for some $i = (i_1, \ldots, i_\ell)$, $1 \leq j \leq \ell - 1$, $\ell \geq 2$, as well as to the set Y of elements $|E_{Ap}(pzp)|$ for such z, it follows that $\forall \alpha > 0$, $\exists q \in \mathcal{P}(Ap)$ such that

$$\|qzq - E_{Ap}(pzp)q\|_{1,pMp} < \alpha \tau_{pMp}(q)$$

with q "almost" τ-independent to the elements $E_{Ap}(pzp)$.
Arguing like in the proofs of Thm 1 and 2, this is used to take care of terms in the sum with $l \geq 2$ appearances of w.

Let us now estimate the terms with $\ell = 1$, i.e., of the form $z = z_{0,i} w_i z_{1,i}$, where $i = 1, 2, ..., k$, $z_{0,i} = x_0 v_1 x_1 ... v_{i-1} x_{i-1} p$, $z_{1,i} = p x_i v_{i+1} ... v_k x_k$ and $w_i = w^t$ if $v_i = v^t$.

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Note that in the above estimates we only used the fact that $w^* w = w w^* = q$ and that A is a MASA, not the actual form of w, nor the fact that A is singular. It is due to the singularity of A that we can choose $w \in \mathcal{U}(Aq)$ so that, at the same time, we have

$$\|E_A(((x_0 v_1 x_1 \ldots v_{j-1} x_{j-1} - E_A(x_0 v_1 x_1 \ldots v_{j-1} x_{j-1} p) w_j x_j v_{j+1} \ldots v_k x_k)\|_1$$

$$\leq \varepsilon_{k-1} \tau(q)/2k,$$

$$\|E_A(x_0 v_1 x_1 \ldots v_{j-1} x_{j-1} w_j (x_j v_{j+1} \ldots v_k x_k - E_A(p x_j v_{j+1} \ldots v_k x_k)))\|_1$$

$$\leq \varepsilon_{k-1} \tau(q)/2k.$$

as well as w_j (almost) τ-independent to a finite set.

Indeed, this amounts to to choosing w so that $\|E_A(z_1 w^s z_2)\|_1$ can be made arbitrarily small relative to size $\tau(q)$, for all y_1, y_2 in a finite set $\perp A$ and all $0 < |s| \leq n$. But this is indeed possible because A singular means it has no "self-intertwiners" that are orthogonal to A.

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Combined with the inequalties corresponding to $\ell \geq 2$, this shows that $\|E_A(x_0 \Pi_{i=1}^k u_i x_i)\|_1 \leq \varepsilon_k \tau(uu^*)$, $1 \leq k \leq n$, contradicting the maximality of v, thus showing that v is a unitary.

4. The case $F \subset \mathcal{N}(B)$: Bernoulli-freeness

Theorem 4

Given any free pmp action $\Gamma \curvearrowright X$, one can "simulate" the Bernoulli Γ-action $\Gamma \curvearrowright \mathbb{T}^\Gamma$ inside it. More precisely:

If $B = L^\infty(X)$, then for any separable Γ-invariant $A_1 \subset B^\omega$, there exists $A_0 \subset B^\omega$ separable diffuse such that $A_1, \{g(A_0)\}_{g \in \Gamma}$, are multi τ-independent. Thus, if we denote $\tilde{A} = A_1 \vee \vee_{g \in \Gamma} g(A_0) \subset B^\omega$, then $\tilde{A} \simeq A_1 \otimes A_0^{\otimes \Gamma}$ is Γ-invariant and its action on it is same as the product action $\Gamma \curvearrowright A_1 \otimes A_0^{\otimes \Gamma}$.

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Proof. We assume for simplicity that $H = 1$. We need to prove that given any $n \geq 1$, any finite $F \subset G$, $Y_0 \in A$ and any $\delta > 0$, there exists a Haar unitary $v \in A$ such that $|\tau(y_0 \Pi_{i=1}^k g_i(v^{j_i}))| \leq \delta$, $\forall y_0 \in Y_0$, $1 \leq k \leq n$, $1 \leq |j_i| \leq n$ and any distinct elements $g_1, ..., g_k \in F$.

To prove this, let $\mathcal{W} := \{v \in A \mid |\tau(y_0\Pi_{i=1}^k g_i(v^{j_i}))| \leq \delta\tau(v^*v), y_0 \in Y_0, g_i \in F \text{distinct}, 1 \leq |j_i| \leq n, \tau(v^m) = 0, \forall m \neq 0\}$, endowed with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$.

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Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$. If $w \in Ap$ is a partial isometry satisfying $ww^* = w^*w$, $\tau(w^m) = 0$, $\forall m \neq 0$, and we denote $u = v + w$, then by noticing that $(v+w)^{j_i} = v^{j_i} + w^{j_i}$, we obtain:

$$y_0 \Pi_{i=1}^k u_{g_i} u^{j_i} u_{g_i}^* = y_0 \Pi_{i=1}^k u_{g_i} v^{j_i} u_{g_i}^* + \Sigma y_0 \Pi_{i=1}^k u_{g_i} z_i^{j_i} u_{g_i}^*,$$

where $z_i \in \{v, w\}$ and the sum is taken over all possible choices for $z_i = v$ or $z_i = w$, with at least one occurrence of $z_i = w$ (thus, there are $2^{k+1} - 1$ many terms in the summation).

We thus get the estimate

$$|\tau(y_0\Pi_{i=1}^{k}u_{g_i}u^{j_i}u_{g_i}^*)|$$

$$\leq |\tau(y_0\Pi_{i=1}^{k}u_{g_i}v^{j_i}u_{g_i}^*)| + \Sigma|\tau(y_0\Pi_{i=1}^{k}u_{g_i}z_i^{j_i}u_{g_i}^*)|$$

$$\leq \delta\tau(vv^*) + \Sigma'|\tau(y_0\Pi_{i=1}^{k}u_{g_i}z_i^{j_i}u_{g_i}^*)| + \Sigma''|\tau(y_0\Pi_{i=1}^{k}u_{g_i}z_i^{j_i}u_{g_i}^*)|$$

where the summation Σ' contains the terms with just one occurrence of $z_j = w$ and Σ'' is the summation of the terms that have at least 2 occurrences of $z_j = w$. Since A is abelian, the terms $u_{g_i}z_i^{j_i}u_{g_i}^*$ in a product can be permuted arbitrarily. Thus, in each summand of Σ'' we can bring two of the occurrences of w so that to be adjacent, i.e., of the form $y_1u_{g_i}w^{j_i}u_{g_i}^*u_{g_l}w^{j_l}u_{g_l}^*y_2$ with $i \neq l$.

Since $g_i \neq g_l$ for all $i \neq l$, by applying the Lemma to $Q = Ap$ and the finite set $F = \{pu_{g_i}^* u_{g_l} p \mid i \neq j\} \perp A = A' \cap M$, it follows that for any $\alpha > 0$, there exists a non-zero $q \in \mathcal{P}(Ap)$ such that

$$\|qu_{g_i}^* u_{g_l} q\|_1 < \alpha\tau(q), \forall g_i \neq g_l \in F.$$

Since there are $2^{k+1} - (k+1) - 1$ terms in the summation Σ'', this shows that $\Sigma'' < (2^{k+1} - (k+1) - 1)\alpha\tau(q)$, for any choice of w that has support q satisfying above condition. Thus, if we choose $\alpha \leq 2^{-n-2}\delta$, then we get $\Sigma'' \leq \delta\tau(q)/2$.

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Then we estimate Σ' (one occurence of w) by taking Haar unitaries w in Aq that tend weekly to 0, as in the proof of Th 1, Th 2, to get that $\Sigma' \leq \delta\tau(q)/2$ as well. Alltogether this gives $|\tau(y_0 \Pi_{i=1}^k g_i(u^{j_i}))| \leq \delta\tau(u^* u)$, contradicting the maximality of v, thus showing that v must be a unitary.

5. B a II_1 factor: freeness relative to $B' \cap M$

Theorem 5

Let M be a II_1 factor and $B = N \subset M$ a subfactor with $N' \cap M = \mathbb{C}$. Given any $F \subset M \ominus \mathbb{C}$ finite, $n \geq 1$, $\varepsilon > 0$, there exists $u \in N$ Haar unitary such that $|\tau(x_0 \Pi_{i=1}^{k} u^{j_i} x_i)| \leq \varepsilon$, for any $1 \leq k \leq n$, $x_0 \in F \cup \{1\}$, $x_i \in F$, $0 < |j_i| \leq n$, $1 \leq i \leq k$.

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More generally, let $N \subset M$ be a subfactor with $N \not\prec_M N' \cap M$. For any $F \subset M \ominus (N' \cap M)$ finite, any $\varepsilon > 0$, $n \geq 1$, there exists a Haar unitary $u \in N$ s.t. $\|E_{N' \cap M}(x_0 \Pi_{i=1}^{k} u^{j_i} x_i)\|_2 \leq \varepsilon$, $\forall 1 \leq k \leq n$, $x_0 \in F \cup \{1\}$, $x_i \in F$, $0 < |j_i| \leq n$, $1 \leq i \leq k$.

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In particular: if $P \subset M^\omega$ is a separable von Neumann subalgebra making a commuting square with $N^{\omega\prime} \cap M^\omega$ and one denotes $B_1 = P \cap (N^{\omega\prime} \cap M^\omega)$, then there exists an abelian diffuse von Neumann subalgebra $B_0 \subset N^\omega$ such that $P \vee B_0 \simeq P *_{B_1} (B_1 \overline{\otimes} B_0)$.

Proof. One formally proceeds exactly as in the proof of Thm 3 (when $B = A$ singular MASA). Thus, we let $\delta = 2^{-(n+1)^2}\varepsilon$ and denote $\varepsilon_0 = \delta, \varepsilon_k = 2^{k+1}\varepsilon_{k-1}, k \geq 1$. Denote $\mathcal{W} = \{v \in N \mid vv^* = v^*v \in \mathcal{P}(N)$, $\|E_{N'\cap M}(x_0\Pi_{i=1}^k u^{j_i} x_i)\|_1 \leq \varepsilon_k \tau(v^*v)$, $\forall 1 \leq k \leq n$, $x_0, x_k \in F \cup \{1\}$, $x_1, ..., x_{k-1} \in F, 0 < |j_i| \leq n\}$.

Endow $\mathcal{W}$ with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 = w_2 w_1^* w_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) < 1$ and denote $p = 1 - v^*v$.

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$$x_0\Pi_{s=1}^k u^{j_s}x_s = x_0\Pi_{s=1}^k v^{j_s}x_s + \Sigma_\ell\Sigma_i z_{0,i}\Pi_{j=1}^\ell w_{i_j}z_{j,i}$$

where the sum is taken over all $\ell = 1, 2, \ldots, k$ and all $i = (i_1, \ldots, i_\ell)$, with $1 \leq i_1 < \cdots < i_\ell \leq k$, and where $w_{i_j} = w^t$ whenever $v_{i_j} = v^t$, $z_{0,i} = x_0 v_1 x_1 \cdots x_{i_1-1}p$, $z_{j,i} = px_{i_j}v_{i_j+1}\cdots v_{i_{j+1}-1}x_{i_{j+1}-1}p$, for $1 \leq j < \ell$, and $z_{\ell,i} = px_{i_\ell}v_{i_\ell+1}\cdots v_k x_k$.

Then the terms with $\ell \geq 2$ occurrences of w are dealt with exactly as in the proof of Thm 3, but using the last part of the LQ Lemma instead of its first part (i.e., $\exists q \in \mathcal{P}(N)$ s.t. $\|qzq - E_{N'\cap M}(z)q\|_1 \leq \varepsilon\tau(q)$, for all z in a prescribed finite subset of M). Like in all proofs, this part only uses the choice of support $q = ww^*$ of the partial isometry.

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Dealing with the terms having $\ell = 1$ occurrences of w means choosing the "phase" $w \in \mathcal{U}(qNq)$ such that $\|E_{N' \cap M}(y_1 w^s y_2)\|_1$ is small relative to the size of q, for y_i in a prescribed finite set of $M \ominus N' \cap M$. But this is exactly the condition that $N \not\prec N' \cap M$.

6.1. Applications to vanishing cohomology results

(a) Vanishing of smooth, operatorial cohomology

- Popa 1984: If M is a II_1 factor (any von Neumann algebra for that matter) normally represented on $\mathcal{H}$, then any derivation $\delta : M \to \mathcal{K}(\mathcal{H})$ is implemented by a compact operator.

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- Galatan-Popa: More generally, let M be a vN algebra acting normally on $\mathcal{H}$ and $\mathcal{X}$ a norm closed M-submodule of $s^*_M(\mathcal{B}(\mathcal{H})) := \{T \in \mathcal{B}(\mathcal{H}) \mid (M)_1 \mapsto xT, Tx \text{ are } \| \|_2 - \| \| \text{ continuous}\}$ (the space of operators that are "smooth relative to M"). Then any derivation $\delta : M \to \mathcal{X}$ is inner.

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- Galatan-P 2014, answering a question of Pisier: Let M_0 be a C^*-algebra with a faithful trace τ and $M_0 \subset \mathcal{B}(\mathcal{H})$ a faithful representation of M_0. Let $\delta : M_0 \to \mathcal{B}(\mathcal{H})$ be a derivation. Assume δ is continuous from the unit ball of M_0 with the topology given by the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M_0$, to $\mathcal{B}(\mathcal{H})$ with the operator norm topology. Then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\delta = \mathrm{ad}\, T$ and $\|T\| \leq \|\delta\|$.

(b) P-Vaes: Vanishing of Connes-Shlyakhtenko-Thom 1st L^2-cohomology

Let M be a finite vN algebra and $\delta : M \to \mathcal{E} = Aff(M \overline{\otimes} M^{op})$ a derivation, where $\mathcal{E}$ is given the bimodule structure $x \cdot \xi \cdot y = (x \otimes y^{op})\xi$. If δ is continuous from M with its norm topology to $\mathcal{E}$ with the measure topology, then δ is inner.

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Proof of (a) uses Theorem 1. Let us only prove that any derivation $\delta : M \to \mathcal{K}(L^2M)$ is inner. Then δ is automatically $\| \, \|_2$-$\| \, \|$ continuous on $(M)_1$ (Popa 1983). Assume first that $M = L(\mathbb{F}_2)$. Denote $u, v \in M$ the canonical generators and let $A = \{u\}''$. By (Johnson-Parrott 1974), $\exists K \in \mathcal{K}(L^2M)$ such that $\delta(a) = Ka - aK$, $\forall a \in A$. We'll show that this implies $\delta = \mathrm{ad}K$ on all M?

If not, then may assume $\delta(u) = 0$, $\langle \delta(v)(\hat{1}), \hat{v} \rangle = 1$. Denote $A_m := \{u^m v\}''$ and let $\delta_m : A_m \to \mathcal{K}(L^2(A_m))$, by

$$\delta_m(x) = p_{L^2 A_m} \delta(x)_{|L^2 A_m}, x \in A_m.$$

By spatiality, δ_m can all be viewed as $\delta_m : L(\mathbb{Z}) \to \mathcal{K}(\ell^2 \mathbb{Z})$, which are uniformaly $\| \|_2$-$\| \|$ continuous on $(L(\mathbb{Z}))_1$. Let $\Delta(x) := \mathrm{Lim}_m \delta_m(x)$.

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Then Δ is a derivation and it is easy to see that $\Delta(x) = \mathrm{ad} p_{\ell^2 \mathbb{Z}_+}(x)$ for $x \in \lambda(\mathbb{Z})$. In particular, $\Delta(\mathbb{C}\mathbb{Z}) \subset \mathcal{K}$ so by continuity $\Delta(L(\mathbb{Z})) \subset \mathcal{K}$ and also $\Delta = \mathrm{ad} p_{\ell^2 \mathbb{Z}_+}$. A contradiction.

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All we used is that $\delta(A) = 0$, $\langle \delta(v)\hat{1}, \hat{v} \rangle = 1$ and that $\exists U_m \in A$ s.t. $U_m v$ Haar unitary with $\lim_m (U_m v)^k = 0$ in *wo*, $\forall k \neq 0$. But this can be done in any M, with respect to any diffuse abelian A by Thm 1 ("approximate semigroup freeness").

For the proof of Part (b) (vanishing of the CST L^2-cohomology), we use Thm 5. To see this, let $\delta : M \to \mathcal{A}ff(M\overline{\otimes}M^{op})$ be a continuous derivation. Let us first prove that if $\{u_n\}_n \cup \{v\} \subset M$ are free independent and $\delta(u_n) = 0$, $\forall n$, then $\delta(v) = 0$. Indeed, if $w_n = u_1 v u_2 v ... u_n v$, then

$$\delta(w_n) = (\Sigma_{k=1}^n u_1 v ... u_{k-1} v u_k \otimes u_{k+1} v .... u_n v)\delta(v)$$

with the n elements in the sum Σ being free independent. So $\delta(n^{-1/4} w_n)$ $= (n^{-1/4}\Sigma)\delta(v)$ with $\lim_n \|n^{-1/4} w_n\| = 0$ while $n^{-1/4}\Sigma = n^{1/4}(\Sigma/\sqrt{n})$ is "large" on a projection close to 1, forcing $\delta(v) = 0$ by continuity of δ.

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$$\delta(w_n) = (\Sigma_{k=1}^{n} u_1 v ... u_{k-1} v u_k \otimes u_{k+1} v .... u_n v)\delta(v)$$

with the n elements in the sum Σ being free independent. So $\delta(n^{-1/4} w_n)$ $= (n^{-1/4}\Sigma)\delta(v)$ with $\lim_n \|n^{-1/4} w_n\| = 0$ while $n^{-1/4}\Sigma = n^{1/4}(\Sigma/\sqrt{n})$ is "large" on a projection close to 1, forcing $\delta(v) = 0$ by continuity of δ.

In general: take $R \subset M$ hyperfinite with $R' \cap M = \mathbb{C}$ (Popa 81). May assume $\delta = 0$ on R (trivial). If $v \in \mathcal{U}(M)$, then $\forall n$, $\exists u_1, ..., u_n \in \mathcal{U}(R)$ s.t. $u_1, ..., u_n, v$ "simulate" $L(\mathbb{F}_{n+1})$ (by Thm 5). By the previous argument, $\delta(v) = 0$.

(a) P 2013: Optimal L^2-paving over arbitrary MASAs

If M is a II_1 factor and $A \subset M$ is a MASA, then for any separable $X \subset M \ominus A$ and any $n \geq 1$, there exists $p_1, ..., p_n \in \mathcal{P}(A^\omega)$ partition of 1 with $\tau(p_i) = 1/n$, such that $\|\Sigma_i p_i x p_i\|_2 = \|x\|_2/\sqrt{n}$, $\forall x \in X$.

This is immediate by Thm 2 (3-independence in arbitrary MASAs): if $A_0 \subset A^\omega$ is merely 2-independent to $X \vee X^*$ then

$$\|p_i x p_i\|_2^2 = \tau(p_i x p_i x^*)$$

$$= \tau(p_i)\tau(p_i x x^*) + \tau(p_i x(p_i - \tau(p_i)1)x^*)$$
$$= \tau(p_i)\tau(p_i x x^*) + 0 = \tau(p_i)^2\|x\|_2^2.$$

(b) Popa 2013, P-Vaes 2015: Norm paving over singular MASAs

If M is a II_1 factor and $A \subset M$ is a singular MASA, then for any separable $X \subset M \ominus A$ and any $n \geq 1$, there exists $p_1, ..., p_n \in \mathcal{P}(A^\omega)$ partition of 1 with $\tau(p_i) = 1/n$, such that $\|\Sigma_i p_i x p_i\| \leq (2\sqrt{n-1}/n)\|x\|$, $\forall x \in X$.

This follows from Thm 3 (free independence in singular MASAs) and Kesten's 1959 theorem stating that if $u_1, ..., u_n$ are free independent Haar unitaries in a II_1 factor then $\|u_1 + ... + u_n\| = 2\sqrt{n-1}$.

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To see this, assume for simplicity that $X = X^*$ is a set of unitaries. By Thm 3 there exists a diffuse $A_0 \subset A^\omega$ free independent to X. Let $p_1, ..., p_n \in A_0$ be a partition of 1 with projections of trace $1/n$. Let $\lambda = e^{2\pi i/n}$ and $u = \Sigma_{k=1}^n \lambda^{k-1} p_k$. If $v \in X$, then any word with alternating letters in $\{v, v^*\}$ and respectively $\{u^k \mid 1 \leq k \leq n-1\}$ has trace 0. It is immediate to check that this implies the unitaries $\{v^* u^k v u^{-k} \mid 1 \leq k \leq n-1\}$ generate $\mathbb{F}_{n-1}$. Thus, by Kesten, we have $\|\Sigma_{k=1}^n u^{k-1} v u^{-k+1}\| = \|1 + \Sigma_{k=2}^{n-1} v^* u^{k-1} v u^{-k+1}\| = 2\sqrt{n-1}$. But a trivial calculation shows that $\|\Sigma_{k=1}^n u^{k-1} v u^{-k+1}\| = \|\Sigma_{k=1}^n p_k v p_k\|$.

(c) A general paving conjecture inspired by (b) (P-Vaes 2014)

Given any sequence of finite factors with MASAs $A_m \subset M_m$, s.t. $\dim M_m \to \infty$, the MASA $\Pi_\omega A_m = A \subset M = \Pi_\omega M_m$ has the norm-paving property with paving size $n(\varepsilon) \sim \varepsilon^{-2}$, i.e., there is some universal constant $C > 0$ such that $\forall x \in M \ominus A$, $\forall \varepsilon > 0$, $\exists p_1, ..., p_n \in \mathcal{P}(A)$, with $\Sigma_i p_i = 1$, $n \leq C\varepsilon^{-2}$ and $\|\Sigma_i p_i x p_i\| \leq \varepsilon \|x\|$. Moreover $C = 4$, i.e. $n(\varepsilon) \leq 4\varepsilon^{-2}$.

6.3. P 1989-1994: Subfactor and embedding problems

- Let M be a II_1 factor, N_1, N_2 separable finite vN algebras with a common amenable subalgebra $Q \subset N_i$. If $N_1, N_2 \hookrightarrow M^\omega$, then $\exists u \in \mathcal{U}(M^\omega)$ s.t. $N_1 \vee uN_2u^* \simeq N_1 *_Q N_2$. Thus, if N_1, N_2 are R^ω embeddable then $N_1 *_Q N_2$ is R^ω embeddable.

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• Let $N \subset M \subset M_1 \subset ... \nearrow M_\infty$ be a subfactor of finite index with its Jones tower. Given any $Q \subset M^\omega$ separable diffuse vN subalgebra, there exists $u \in \mathcal{U}(M^\omega)$ such that:

$\mathcal{M}_\infty := uQu^* \vee N' \cap M_\infty \simeq (Q \otimes M' \cap M_\infty) *_{M' \cap M_\infty} (N' \cap M_\infty)$ Also, $\mathcal{M}_k := M_k^\omega \cap \mathcal{M}_\infty$ forms a Jones tower with same higher relative commutants as the $N \subset M \subset M_1 ...$.

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• This immediately suggests a **reconstruction method** for subfactors $N_\alpha \subset M_\alpha$ with A_∞-graph (so TLJ standard invariant) and index $\alpha \in [4, \infty)$. More generally, it offers you "on a plate" the appropriate necessary and sufficient conditions (**axiomatization**) for a lattice of inclusions of finite dimensional algebras A_{ik} to be h.r.c. $A_{ik} = \mathcal{M}_i' \cap \mathcal{M}_k$