Character rigidity for lattices in higher-rank groups

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Characters

Definition

Let Γ be a discrete group.
A ***character*** *on Γ is a function $\tau : \Gamma \to \mathbb{C}$ such that*

- $\tau(e) = 1$.
- $\tau(ghg^{-1}) = \tau(h)$.
- $[\tau(g_j^{-1}g_i)]$ *is non-negative definite for* $g_1, \ldots, g_n \in \Gamma$.

τ is ***extremal*** *if it is an extreme point in the convex space of characters.*

Examples

- $\pi : \Gamma \to U(n)$ irreducible, then $\tau(g) = \frac{1}{n}\mathrm{Tr}(\pi(g))$ is an extremal character. (These are almost periodic, i.e., $\{L_g(\tau) \mid g \in G\}$ is uniformly pre-compact in $\ell^\infty\Gamma$).
- Γ virtually abelian iff every extremal character is almost periodic. (Thoma '64).
- $\pi : \Gamma \to \mathcal{U}(M)$, M finite factor, $\pi(\Gamma)'' = M$, $g \mapsto \tau(\pi(g))$ is extremal.

Classification of characters

- Segal-von Neumann '50: s.s. $\mathbb{R}$-Lie groups w/o compact factors.
- Kadison-Singer '52: connected groups.
- Thoma '64-'67: S_∞.
- Kirillov '65: GL_n, $n \geq 2$, SL_n, $n \geq 3$.
- Ovcinikov '71: Chevalley groups excluding SL_2 and Sp_4.
- Skudlarek '76: $GL_\infty(\mathbb{F})$.
- Voiculescu '76: $U(\infty)$.
- Dudko-Nessonov '05-'08: Wreath products.
- Bekka '07: $SL_3(\mathbb{Z})$
- Dudko '11: Full groups.
- Dudko-Medynets '12: Thompson's groups.
- Enomoto-Izumi '13: Unitary groups.
- P-Thom '13: $SL_2(\mathbb{Z}[\sqrt{2}])$
- Creutz-P '13.

Operator Algebraic superrigidity

Theorem (P; Conjectured by Connes, early 1980's)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then $\Gamma < \mathcal{U}(L\Gamma)$ is Operator Algebraic Superrigid:

- *If M is a finite factor;*
- *$\pi : \Gamma \to \mathcal{U}(M)$ a homomorphism such that $\pi(\Gamma)'' = M$,*

then either

- *$\overline{\pi(\Gamma)}$ is compact (and hence M is finite dimensional);*
- *or π extends to an isomorphism $\tilde{\pi} : L\Gamma \to M$.*

Theorem (Equivalent formulation)

G a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then every extremal character is either almost periodic or else equals δ_e.

Theorem (Margulis '77)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then $\Gamma < G$ is superrigid:

- *If H is a simple Lie group;*
- *$\pi : \Gamma \to H$ is a homomorphism such that $\pi(\Gamma)$ is Zariski dense,*

then either

- *$\overline{\pi(\Gamma)} = H$ is compact;*
- *or π extends to a homomorphism $\tilde{\pi} : G \to H$.*

Just infinite groups

OA-superrigidity

$\Downarrow$

just infinite (non-trivial normal subgroups are finite index).

Proof.

- If $\Sigma \lhd \Gamma$, consider $\tau(g) = 1_\Sigma(g) = \begin{cases} 1 & \text{if } g \in \Sigma; \\ 0 & \text{otherwise.} \end{cases}$
- Or consider $\lambda_\Sigma : \Gamma \to \mathcal{U}(L(\Gamma/\Sigma))$. $\square$

Theorem (Margulis normal subgroup theorem '79, '80; Kazhdan '67)

Irreducible lattices in higher rank groups are just infinite.

Theorem (Bader-Shalom '06, Shalom '00)

Most irreducible lattices in products of simple groups are just infinite.

Free actions

OA-superrigidity

$\Downarrow$

Every ergodic p.m.p. action on a diffuse space is free.

Proof.

- Suppose $\Gamma \curvearrowright (X, \mu)$ ergodic p.m.p.
- Stab: $X \to \mathrm{Sub}(\Gamma)$, $\nu = \mathrm{Stab}_* \mu$ gives an invariant random subgroup.
- Consider $\tau(g) = \mathbb{P}(g \in \nu) = \int 1_\Sigma(g)\, \mathrm{d}\nu(\Sigma)$. (Vershik character)
- Or consider $\Gamma \to [\mathcal{R}_{\Gamma \curvearrowright X}] \subset \mathcal{U}(L(\mathcal{R}_{\Gamma \curvearrowright X}))$. $\square$

Theorem (Stuck-Zimmer '94, Creutz-P '12)

For irreducible lattices in G where every factor of G is higher-rank, then every ergodic p.m.p. action on a diffuse space is free.

Kazhdan's property (T)

Definition

- Γ *has* *property (T)* *if almost invariant vectors* $\implies$ *invariant vectors.*

- If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\xi_n \in \mathcal{H}$, $\|\xi_n\| = 1$, $\|\pi(g)\xi_n - \xi_n\| \to 0$, for $g \in G$.
- Then there exists $\eta \in \mathcal{H}$, $\eta \neq 0$, such that $\pi(g)\eta = \eta$ for $g \in \Gamma$.

Kazhdan '67

- Lattices in higher-rank simple groups have property (T).
- Property (T) passes to quotients.

Amenability (Von Neumann '29)

Definition

- Γ *is* amenable *if there is an invariant state on* $\ell^\infty\Gamma$.
- *Equivalently (Følner '55) there exists* $F_n \subset \Gamma$ *finite such that* $\frac{|F_n \Delta g F_n|}{|F_n|} \to 0$ *for all* $g \in \Gamma$.

Note

- Γ is finite iff Γ is both amenable and has property (T).

Proof.

- Amenable implies $\ell^2\Gamma$ has almost invariant vectors.
- Property (T) then implies $\ell^2\Gamma$ has a non-zero invariant vector. $\square$

Margulis' strategy

Theorem (Margulis normal subgroup theorem '79; Kazhdan '67)

Lattices in higher rank simple groups with trivial center are just infinite.

Outline.

- Suppose $\Sigma \lhd \Gamma$, is a non-trivial normal subgroup.
- Γ/Σ has property (T). (Kazhdan '67)
- Take P the minimal parabolic subgroup. Then P is amenable and so $\Gamma \curvearrowright G/P$ is amenable. (Zimmer '77)

  I.e., there exists an invariant conditional expectation

  $$E : L^\infty((G/P) \times (\Gamma/\Sigma)) \to L^\infty(G/P).$$
- Σ acts trivially on the range $E(\ell^\infty(\Gamma/\Sigma))$.
  But $\Sigma \curvearrowright G/P$ is ergodic (Margulis factor theorem), hence $E_{|\ell^\infty(\Gamma/\Sigma)}$ is an invariant mean, and so Γ/Σ is amenable. $\square$

Amenable von Neumann algebras

Definition

A von Neumann algebra $B \subset \mathcal{B}(\mathcal{H})$ is amenable (or injective) if there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to B$.

Theorem (Schwartz '63)

If H is an amenable group and $\sigma : H \to \mathrm{Aut}(B)$ with B amenable, then $B^H := \{x \in B \mid \sigma_h(x) = x, h \in H\}$ is amenable.

Corollary (Zimmer '77)

If $P < G$ is an amenable subgroup, $\Gamma < G$ a lattice, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, then $\mathcal{B} = L^\infty(G/P; \mathcal{B}(\mathcal{H}))^\Gamma$ is amenable. ($\Gamma \curvearrowright \mathcal{B}(\mathcal{H})$ by conjugation).

Proof.

$\mathcal{B} \cong L^\infty(G/\Gamma; \mathcal{B}(\mathcal{H}))^P$ for an induced action of P. □

Von Neumann algebras with property (T)

Definition

A finite factor M has property (T) if every Hilbert bimodule having almost central vectors has a non-zero central vector.

Theorem (Connes-Jones '85)

If Γ has property (T), M is a finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ such that $\pi(\Gamma)'' = M$, then M has property (T).

(If π is the left-regular representation then also the converse holds.)

Note

- A finite factor M is finite dimensional iff M is both amenable and has property (T).

Character rigidity strategy

Theorem

Lattices in higer rank simple groups with trivial center are OA superrigid.

Outline.

- Suppose $\pi : \Gamma \to \mathcal{U}(M)$ is a finite factor representation which does not extend to $L\Gamma$.
- M has property (T). (Kazhdan '67, Connes-Jones '85)
- Take P the minimal parabolic subgroup. Then $\mathcal{B} = L^\infty(G/P; \mathcal{B}(L^2M))^\Gamma$ is amenable, where $\Gamma \curvearrowright \mathcal{B}(L^2M)$ as conjugation by $J\pi(\gamma)J$.
- (If π were the left-regular representation then $\mathcal{B} \cong L^\infty(G/P) \rtimes \Gamma$.)
- We show that $M = \mathcal{B}$ (Ergodicity type result), and so M is amenable. $\square$

Lattices in products

Theorem (P)

- *G_1, G_2 compactly generated with trivial amenable radical, $G = G_1 \times G_2$,*
- *$\Gamma < G$ a lattice,*
- *If $H \lhd G$ is any proper normal subgroup then $H \cap \Gamma = \{e\}$.*

Then for any finite factor representation $\pi : \Gamma \to \mathcal{U}(M)$, $\pi(\Gamma)'' = M$,

- *either M is amenable, or π extends to an isomorphism $\tilde{\pi} : L\Gamma \to M$.*

Proof.

- Fix Poisson boundaries $G_i \curvearrowright (B_i, \eta_i)$.
- Use $\Gamma \curvearrowright (B_1, \eta_1) \times (B_2, \eta_2)$ as a replacement for $\Gamma \curvearrowright G/P$.
- Show that if π does not extend to an isomorphism $\tilde{\pi} : L\Gamma \to M$, then $\mathcal{B} := L^\infty(B_1 \times B_2; \mathcal{B}(L^2 M))^\Gamma = M$. $\square$

Lattices in products

Theorem (P)

- *G_1, G_2 non-compact, $G = G_1 \times G_2$,*
- *$\Gamma < G$ a lattice,*
- *Whenever $G \curvearrowright (N, \tau)$ is ergodic, then $\Gamma \curvearrowright (N, \tau)$ is properly outer.*

Then for any finite factor representation $\pi : \Gamma \to \mathcal{U}(M)$, $\pi(\Gamma)'' = M$,

- *either M is amenable, or π extends to an isomorphism $\tilde{\pi} : L\Gamma \to M$.*

Proof.

- Fix Poisson boundaries $G_i \curvearrowright (B_i, \eta_i)$.
- Use $\Gamma \curvearrowright (B_1, \eta_1) \times (B_2, \eta_2)$ as a replacement for $\Gamma \curvearrowright G/P$.
- Show that if π does not extend to an isomorphism $\tilde{\pi} : L\Gamma \to M$, then $\mathcal{B} := L^\infty(B_1 \times B_2; \mathcal{B}(L^2 M))^\Gamma = M$. $\square$

Lemma (Lebesgue density type property)

- $G = G_1 \times G_2$,
- $\Gamma < G$ *an irreducible* ($\overline{p_i(\Gamma)} = G_i$, p_i *the projection to* G_i) *lattice,*
- $G_1 \curvearrowright (B_1, \eta_1)$ *the Poisson boundary action.*

Then for all $E \subset B_1$, $\eta_1(E) > 0$, *there exist* $\gamma_n \in \Gamma$ *so that* $\eta_1(p_1(\gamma_n)E) \to 1$, *and* $p_2(\gamma_n) \to e$ *in* G_2.

Proof when $G_2 = \{e\}$, and G/Γ is compact.

- There exist $g_n \in G_1$ so that $\eta_1(g_n E) \to 1$. (This is easy when considering the identification $L^\infty(B_1, \eta_1) = \mathrm{H}^\infty(G, \mu_0)$.)
- We may assume $g_n = k_n \gamma_n$ where $k_n \to k$.
- $\eta_1(\gamma_n E) \to 1$. $\square$

For general lattices use Kakutani's random ergodic theorem.

Lebesgue density/contractive automorphisms

Lemma

- *$\Gamma < G_1 \times G_2$ an irreducible lattice.*
- *$G_i \curvearrowright (B_i, \eta_i)$ Poisson boundary.*
- *$\mathcal{B} = L^\infty(B_1 \times B_2; \mathcal{B}(L^2M))^\Gamma$.*

If $f \in \mathcal{B} = L^\infty(B_1; L^\infty(B_2; \mathcal{B}(L^2M)))^\Gamma$, and f_0 is in the (SOT)-essential range of f, then there exists $\tilde{f} \in L^\infty(B_2; \mathcal{B}(L^2M))^\Gamma$, such that

$$P_{\hat{1}} f_0 P_{\hat{1}} = P_{\hat{1}} \tilde{f} P_{\hat{1}}.$$

Proof.

- Let $E_n \subset B_1$ be positive measure such that $f_{|E_n} \sim f_0$.
- Take $\gamma_n \in \Gamma$ so that $\eta_1(\gamma_n E_n) \to 1$ and $p_2(\gamma_n) \to e$.
- Take $\tilde{f}$ to be any wot-cluster point of $\{\pi(\gamma_n) f \pi(\gamma_n^{-1})\}$.
- Using that $\pi(\gamma_n) P_{\hat{1}} = J\pi(\gamma_n^{-1}) J P_{\hat{1}}$ check that $\tilde{f}$ works. $\square$

Theorem (P)

- G_1, G_2 *non-compact,* $G = G_1 \times G_2$,
- $\Gamma < G$ *a lattice,*
- *Whenever* $G \curvearrowright (N, \tau)$ *is ergodic, then* $\Gamma \curvearrowright (N, \tau)$ *is properly outer.*

Then for any finite factor representation $\pi : \Gamma \to \mathcal{U}(M)$, $\pi(\Gamma)'' = M$,

- *either* M *is amenable, or* π *extends to an isomorphism* $\tilde{\pi} : L\Gamma \to M$.

We've now reduced this theorem to showing:

Theorem (Ergodicity type result, Creutz-P '13)

$\Gamma < G$ *as above,* $G_2 \curvearrowright (B_2, \eta_2)$ *Poisson boundary. Then for any finite factor representation* $\pi : \Gamma \to \mathcal{U}(M)$, $\pi(\Gamma)'' = M$, *either* π *is the left regular representation or* $L^\infty(B_2; \mathcal{B}(L^2 M))^\Gamma = M$.

Proof sketch.

- Suppose $\gamma_0 \in \Gamma \setminus \{e\}$, such that $\tau(\pi(\gamma_0)) \neq 0$. We need to show $\mathcal{B}_2 := L^\infty(B_2; \mathcal{B}(L^2M))^\Gamma = M$.
- For each open set $O \subset G_1$ set $\mathcal{K}_O = \overline{\mathrm{co}}\{\pi(\lambda\gamma_0\lambda^{-1}) \mid \lambda \in p_1(\Gamma) \cap O\}$. Set $\mathcal{K} = \cap_{\{O \text{ nbhd of } e\}} \mathcal{K}_O$.
- Then $\alpha = \tau(\pi(\gamma_0))$ is the unique element of minimal $\|\cdot\|_2$ in $\mathcal{K}$. (Convexity argument à la Popa's intertwining, etc.). This is where we use the properly outer assumption.
- For $\sigma^0_{\gamma_0} \in \mathcal{U}(L^2(B_2))$ (the Koopman representation) we have

$$\begin{aligned}\sigma^0_{\gamma_0} \otimes \alpha &\sim \sum \sigma^0_{\gamma_0} \otimes \alpha_i J\pi(\lambda_i\gamma_0\lambda_i^{-1})J \\ &\sim \sum \alpha_i \sigma^0_{\lambda_i\gamma_0\lambda_i^{-1}} \otimes J\pi(\lambda_i\gamma_0\lambda_i^{-1})J \in (\mathcal{B}_2)'.\end{aligned}$$

- Hence, for any g in the closure of the normal subgroup of Γ generated by γ_0, we have $\sigma^0_g \otimes 1 \in \mathcal{B}_2'$. (By hypothesis this is all of G_2).
- By ergodicity of $G_2 \curvearrowright B_2$ we conclude that $\mathcal{B}_2 = M$. □