Classification of gapped Hamiltonians in quantum spin chains

Yoshiko Ogata

Graduate School of Mathematical Sciences, The University of Tokyo

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Recall : Bulk classification

Definition (Bulk Classification)

We say that the Hamiltonians $H(h_0)$, $H(h_1)$ given by h_0, h_1 are bulk equivalent ($H(h_0) \simeq_B H(h_1)$) if the following holds.

1. *There exist an $m \in \mathbb{N}$ and a continuous path of self-adjoint elements $h : [0,1] \to \mathcal{A}_{[0,m-1]}$ such that $h(0) = h_0$, and $h(1) = h_1$.*
2. *There is a constant $\gamma > 0$, such that*

$$\sigma\left(H_{\varphi_s, \alpha_{h(s)}}\right) \setminus \{0\} \subset [\gamma, \infty),$$

for any $s \in [0,1]$ and $\alpha_{h(s)}$-ground state φ_s. Furthermore, for any $\alpha_{h(s)}$-ground state φ_s, 0 is a non-degenerate eigenvalue of $H_{\varphi_s, \alpha_{h(s)}}$.

We denote by $\mathcal{J}_{FB}$, the set of h satisfying the followings.

1. $H(h)$ is gapped in the bulk.
2. There exists a unique α_h-ground state ω on $\mathcal{A}_{\mathbb{Z}}$.
3. There exists a constant $d \in \mathbb{N}$ such that

$$1 \leq \dim \ker \left(H_{[0,N-1]}(h) \right) \leq d,$$

   for all $N \in \mathbb{N}$.

We saw

Lemma

The bulk-classification problem of $\mathcal{J}_{FB}$ can be reduced to the bulk-classification problem of FNW-Hamiltonians.

Prepare an n-tuple of $k \times k$ matrices $\mathbf{v} := (v_1, \ldots, v_n)$ and $m \in \mathbb{N}$.

Fix some orthonormal basis of $\mathbb{C}^n$, $\{\psi_\mu\}_{\mu=1}^n$.

Define a subspace $\mathcal{G}_{m,\mathbf{v}}$ of $\bigotimes_{i=0}^{m-1} \mathbb{C}^n$ by the range of the following map $\Gamma_{m,\mathbf{v}} : \mathrm{M}_k \to \bigotimes_{i=0}^{m-1} \mathbb{C}^n$,

$$\Gamma_{m,\mathbf{v}}(X) = \sum_{\mu_0,\ldots,\mu_{m-1} \in \{1,\cdots,n\}} \left(\mathrm{Tr}\, X \left(v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{m-1}} \right)^* \right) \bigotimes_{i=0}^{m-1} \psi_{\mu_i}, \quad X \in \mathrm{M}_k\,.$$

Let $h_{m,\mathbf{v}}$ be the orthogonal projection onto $\mathcal{G}_{m,\mathbf{v}}^{\perp}$ in $\otimes_{i=0}^{m-1} \mathbb{C}^n$.

We consider the Hamiltonian $H(h_{m,\mathbf{v}})$.

We require that $\mathbf{v}$ to be primitive, to guarantee the spectral gap.

We would like to show that all the FNW-Hamiltonians are bulk-equivalent.

Our strategy is to carry this out

via type I/type II-classification with open boundary conditions.

However, there is an obstacle in carrying this out.

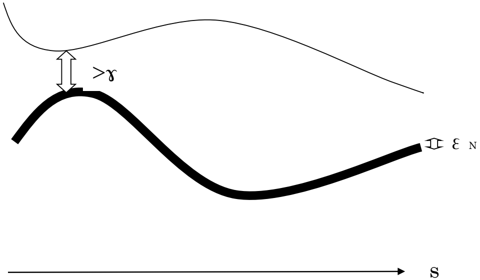
Invariant of typeII-classification

Recall the following theorem

Theorem (Bachmann, Michalakis, Nachtergaele, Sims '11)

Let $H(h_0), H(h_1)$ be Hamiltonians gapped with respect to the open boundary conditions. Suppose that $H(h_0)$ and $H(h_1)$ are type II-equivalent. Furthermore, assume that there are finite intervals $I(s)$ with smooth end points such that $\inf \sigma\left(H_{[1,N]}(h)\right) \in I(s)$ for all $s \in [0,1]$ and $N \in \mathbb{N}$. Let $\Gamma = \mathbb{Z}, (-\infty, -1] \cap \mathbb{Z}, [0, \infty) \cap \mathbb{Z}$. Then, there exist a quasi-local automorphism β_Γ, of $\mathcal{A}_\Gamma$ such that

$$\mathcal{S}_\Gamma(h_0) \circ \beta_\Gamma = \mathcal{S}_\Gamma(h_1).$$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For type II-classification of FNW-Hamiltonians, the previous theorem says that the degeneracies of edge ground states are invariant.

What are the edge degeneracies of a FNW-Hamiltonian $H(h_{m,\mathbf{v}})$, given by $\mathbf{v} \in \mathrm{M}_k$?

It turns out both of right/left degeneracies are k.

The type II-classification gives us an answer that if

$$\mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}, \quad \text{and} \quad \mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n},$$

with $k_0 \neq k_1$, then "they are not equivalent".

This is because the degeneracies of the edge ground states, which are k_0 and k_1, are different.

This sounds against to the Theorem we are trying to prove, i.e., all the $H(h)$ with $h \in \mathcal{J}_{FB}$ are the same in the bulk.

This is actually ok. It is because bulk forget about boundary in this model.

The boundary effect disappears exponentially.

This talk :

1. Exponential decay of boundary effect in FNW-model
2. Derivation of matrix product structure - edge version-
3. Type II classification and bulk classification

Exponential decay of boundary effect in FNW-model

Recall that we defined a subspace $\mathcal{G}_{N,\mathbf{v}}$ of $\bigotimes_{i=1}^{N} \mathbb{C}^n$ by the range of the following map $\Gamma_{N,\mathbf{v}} : \mathrm{M}_k \to \bigotimes_{i=1}^{N} \mathbb{C}^n$,

$$\Gamma_{N,\mathbf{v}}(X) = \sum_{\mu_1,\ldots,\mu_N \in \{1,\cdots,n\}} \left(\mathrm{Tr}\, X \left(v_{\mu_1} v_{\mu_2} \cdots v_{\mu_N} \right)^* \right) \bigotimes_{i=0}^{N-1} \psi_{\mu_i}, \quad X \in \mathrm{M}_k .$$

The orthogonal projection onto the subspace $\mathcal{G}_{N,\mathbf{v}}$ is the spectral projection of $H_{[1,N]}(h_{m,\mathbf{v}})$, corresponding to the lowest eigenvalue, i.e., 0.

We know the ground state space of the local Hamiltonian explicitly.

This let us show the following property.

Exponential decay of boundary effects

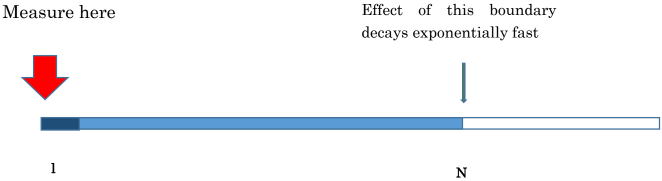
Let $G_{N,\mathbf{v}}$ be the spectral projection of $H_{[1,N]}(h_{m,\mathbf{v}})$ onto the lowest eigenvalue 0. There exist $0 < C_1$, $0 < s_1 < 1$, $N_1 \in \mathbb{N}$ and a factor state ω_R on the right infinite chain, such that

$$\left| \frac{\mathrm{Tr}_{[1,N]}\left(G_{N,\mathbf{v}} A\right)}{\mathrm{Tr}_{[1,N]}\left(G_{N,\mathbf{v}}\right)} - \omega_R(A) \right| \leq C_1 s_1^{N-l} \|A\|$$

for all $l \in \mathbb{N}$, $A \in \mathcal{A}_{[1,l]}$, and $N \geq \max\{l, N_1\}$, and

$$\inf \left\{ \sigma\left(\omega_R|_{\mathcal{A}_{[1,l]}} \right) \setminus \{0\} \mid l \in \mathbb{N} \right\} > 0.$$

(Similar property holds for the left infinite chain.)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Effect of this boundary decays exponentially fast
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The effect of boundary decays exponentially fast.

As a result, the degeneracies of the right/left edge ground states are smaller than the degeneracy of ground states of local Hamiltonians on finite intervals.

The degeneracy of ground states of local Hamiltonians

Recall the injectivity conditions, i.e., $\Gamma_{m-1,\mathbf{v}}$ is injective.

This implies injectivity of $\Gamma_{l,\mathbf{v}}$ for $l \geq m-1$.

From this, we have

$$\begin{aligned}&\text{the ground state degeneracy of } H_{[1,N]}(h_{m,\mathbf{v}})\\ &= \dim \mathcal{G}_{N,\mathbf{v}} = \dim \mathrm{M}_k = k^2.\end{aligned}$$

Degeneracy of edge ground states

Lemma (Bachmann-O '15)

There exist affine bijections

$$\Xi_L : \mathfrak{E}_k \to \mathcal{S}_{(-\infty,-1]}(H(h_{m,\mathbf{v}})), \quad \Xi_R : \mathfrak{E}_k \to \mathcal{S}_{[0,\infty)}(H(h_{m,\mathbf{v}})).$$

Here $\mathfrak{E}_k$ is the state space over M_k.

This means both of the left/right edge ground state degeneracies are k.

Furthermore, bulk ground state degeneracy of $H(h_{m,\mathbf{v}})$ is 1.

Derivation of matrix product structure - edge version-

So far we have seen several properties of FNW-Hamiltonians.

A1 $H(h_{m,\mathbf{v}})$ is gapped with respect to the open boundary conditions

A2 There exists a $d \in \mathbb{N}$ such that

$$1 \leq \dim \ker H_{[1,N]}(h_{m,\mathbf{v}}) \leq d,$$

for all $N \in \mathbb{N}$.

A3 The bulk ground state is unique

A4 The boundary effect decays exponentially fast.

Furthermore, we can also show that any edge ground state has an overlap with some of its space translation.

[A5] For any ground state ψ on the right infinite chain, there exists an $l_\psi \in \mathbb{N}$ such that

$$\left\| \psi - \psi \circ \tau_{l_\psi} \right\| < 2$$

Here, τ_l is the space translation.

(Similar property holds for the left infinite chain.)

Hence we have

$$\text{primitive } \mathbf{v} \Rightarrow \text{[A1]-[A5]}.$$

Recall that

$$\text{primitive } \mathbf{v} \Rightarrow h_{m,\mathbf{v}} \in \mathcal{J}_{FB},$$

and

$$h \in \mathcal{J}_{FB} \Rightarrow H(h) \simeq_B H(h_{m,\mathbf{v}}), \text{ with primitive } \mathbf{v}.$$

This is about bulk classification.

Is there corresponding type II-classification version?

Primitive $\mathbf{v} \Rightarrow$ [A1]-[A5]

Q. $H(h)$ satisfies [A1]-[A5]

$\Rightarrow H(h) \simeq_{II} H(h_{m,\mathbf{v}})$ for primitive $\mathbf{v}$?

Primitive $\mathbf{v} \Rightarrow$ [A1]-[A5]

Q. $H(h)$ satisfies [A1]-[A5]

$\Rightarrow H(h) \simeq_{II} H(h_{m,\mathbf{v}})$ for primitive $\mathbf{v}$?

This is not true.
There is a model satisfying [A1]-[A5], such that

left ground state degeneracy $\neq$ right ground state degeneracy.

PVBS-model [Bachmann-Nachtergaele '14]

Recall that for primitive $\mathbf{v}$,

left ground state degeneracy $=$ right ground state degeneracy$=k$.

Let us relax the requirement.

Q. $H(h)$ satisfies [A1]-[A5]

$\Rightarrow H(h) \simeq_{II} H(h_{m,\mathbf{v}})$ for larger (than primitive) class of $\mathbf{v}$?

It turns out that this is possible.

A generalization of the injectivity condition : ClassA

ClassA [O '16]

ClassA is a set of n-tuples of matrices $\mathbb{B} = (B_\mu)_{\mu=1}^n$ which satisfies

$$\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_{n_\mathbb{B}} \otimes \mathcal{D}_\mathbb{B} \Lambda_\mathbb{B}^l, \quad \text{for } l \text{ large enough},$$

where

- $n_\mathbb{B} \in \mathbb{N}$ and $k_{R,\mathbb{B}}, k_{L,\mathbb{B}} \in \mathbb{N} \cup \{0\}$,
- $\Lambda_\mathbb{B} = \mathrm{diag}(\lambda_{\mathbb{B},i})_{i=-k_{R,\mathbb{B}}}^{k_{L,\mathbb{B}}} \in \mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$, with $\lambda_{\mathbb{B},0} = 1$ and $0 < |\lambda_{\mathbb{B},i}| < 1$, for $i \neq 0$,
- $\mathcal{D}_\mathbb{B}$ is a subalgebra of upper triangular matrices (in $\mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}$) with $\mathbb{1} \in \mathcal{D}_\mathbb{B}$, satisfying some additional conditions,
- B_μ is an element of $\mathrm{M}_{n_\mathbb{B}} \otimes \mathcal{D}_\mathbb{B} \Lambda_\mathbb{B}$.

Remark

Recall the injectivity condition: $\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_k(\mathbb{C})$*,for l large enough.*
$k_{R,\mathbb{B}} = k_{L,\mathbb{B}} = 0$ *corresponds to the injectivity condition.*

Theorem (O '16)

For $\mathbb{B} \in \mathrm{ClassA}$, $H(h_{m,\mathbb{B}})$ satisfies [A1]-[A5].

Remark

We still have

$$\lim_{N\to\infty} T_{\mathbb{B}}^{N}(X) = \varphi_{\mathbb{B}}(X) e_{\mathbb{B}}.$$

But the support of $\varphi_{\mathbb{B}}$ is $\mathbb{1}_{n_{\mathbb{B}}} \otimes \left(\sum_{i=0}^{k_{L\mathbb{B}}} E_{ii}\right)$ and the support of $e_{\mathbb{B}}$ is $\mathbb{1}_{n_{\mathbb{B}}} \otimes \left(\sum_{i=-k_{R\mathbb{B}}}^{0} E_{ii}\right)$.
E_{ij} : matrix units of $\mathrm{M}_{k_{L\mathbb{B}}+k_{R\mathbb{B}}+1}$.

Characterization of ClassA

Theorem (O '16)

Suppose that $H(h)$ satisfies the properties [A1]-[A5]. Then there exist a $\mathbb{B} \in \mathrm{ClassA}$ and an $m \in \mathbb{N}$ satisfying the followings.

1. *The ground states of $H(h)$ and $H(h_{m,\mathbb{B}})$ on infinite intervals coincide.*
2. *There exist some $0 < s < 1$ and $C > 0$ such that*

$$\left\| G_{h,N} - G_{h_{m,\mathbb{B}},N} \right\| \leq C \cdot s^N, \quad N \in \mathbb{N}.$$

Remark

$G_{h,N}$, $G_{h_{m,\mathbb{B}},N}$ are the projections onto the ground state spaces of $H_{[1,N]}(h)$ and $H_{[1,N]}(h_{m,\mathbb{B}})$.

Corollary (O '16)

If h satisfies [A1]-[A5], then there exist $\mathbb{B} \in \mathrm{ClassA}$ and $m \in \mathbb{N}$ such that $H(h) \simeq_{II} H(h_{m,\mathbb{B}})$.

$\mathbb{B} \in \mathrm{ClassA} \Rightarrow H(h_{m,\mathbb{B}})$ satisfies [A1]-[A5]

$H(h)$ satisfies [A1]-[A5]

$\Rightarrow H(h) \simeq_{II} H(h_{m,\mathbb{B}})$ for $\mathbb{B} \in \mathrm{ClassA}$.

Edge ground states

Theorem (O '16)

For the Hamiltonian given by $\mathbb{B} \in \mathrm{ClassA}$, the ground state space on the right/left infinite chain is isomorphic to the state space over $\mathrm{M}_{n_{\mathbb{B}}(k_{R,\mathbb{B}}+1)} / \mathrm{M}_{n_{\mathbb{B}}(k_{L,\mathbb{B}}+1)}$.

Remark

If $k_{L,\mathbb{B}} \neq k_{R,\mathbb{B}}$, then the ground state structure is asymmetric. (Injective case : $k_{L,\mathbb{B}} = k_{R,\mathbb{B}} = 0$ symmetric)

Type II classification and bulk classification

What was our task?

Given

$$\mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}, \quad \text{and} \quad \mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n}, \quad \text{primitive}$$

we have to construct a path of Hamiltonians gapped in the bulk connecting $H(h_{m_0,\mathbf{v}_0})$ and $H(h_{m_1,\mathbf{v}_1})$.

$k_0 = k_1$ case

Suppose that $k_0 = k_1$, i.e.,

$$\mathbf{v}_0, \mathbf{v}_1 \in \mathrm{M}_k^{\times n},$$

for some $k \in \mathbb{N}$. Then it suffices to show that there exists a path of n-tuples $\mathbf{v}(s) \in \mathrm{M}_k^{\times n}$, such that $\mathbf{v}(s)$ is primitive for all $s \in [0, 1]$.

Theorem (Bachmann-O '15, Szehr-Wolf '15 preprint)

The set of n-tuples

$$\left\{ \mathbf{v} \in \mathrm{M}_k^{\times n} \mid \mathbf{v} : \textit{primitive} \right\}$$

is arcwise connected.

Corollary (Bachmann-O '15)

Let $\mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}$, $\mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n}$ be primitive. If $k_0 = k_1$ holds, then we have

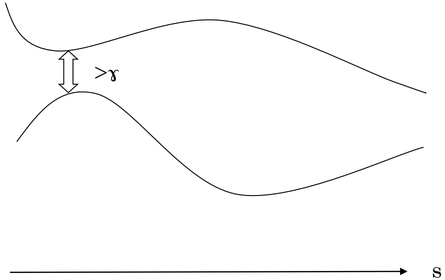
$$H(h_{m_0,\mathbf{v}_0}) \simeq_I H(h_{m_1,\mathbf{v}_1}).$$

In particular, if $k_0 = k_1$, we have

$$H(h_{m_0,\mathbf{v}_0}) \simeq_B H(h_{m_1,\mathbf{v}_1}).$$

Remark

Recall that $\simeq_I$ means the type I C^1-equivalence with open boundary conditions.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$k_0 \neq k_1$ case

If $k_0 \neq k_1$, $\mathbf{v}_0$ and $\mathbf{v}_1$ are not living in the same world.

Therefore, it is no longer sufficient to think of primitive $\mathbf{v}$.

Note that ClassA includes primitive $\mathbf{v} \in \mathrm{M}_k^{\times n}$ for any $k \in \mathbb{N}$.

Let us interpolate them in ClassA

Classification of the non-degenerate part of ClassA

Recall ClassA is a set of n-tuples of matrices $\mathbb{B}$ which satisfies

$$\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_{n_{\mathbb{B}}}(\mathbb{C}) \otimes \mathcal{D}_{\mathbb{B}} \Lambda_{\mathbb{B}}^l, \quad \text{for } l \text{ large enough},$$

where

- $n_{\mathbb{B}} \in \mathbb{N}$ and $k_{R,\mathbb{B}}, k_{L,\mathbb{B}} \in \mathbb{N} \cup \{0\}$,
- $\mathbb{B}$ is an element of $\mathrm{M}_{n_{\mathbb{B}}}(\mathbb{C}) \otimes \mathrm{M}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}(\mathbb{C})$,
- $\Lambda_{\mathbb{B}} = \mathrm{diag}(\lambda_{\mathbb{B},i})_{i=-k_{R,\mathbb{B}}}^{k_{L,\mathbb{B}}}$, with $\lambda_{\mathbb{B},0} = 1$ and $0 < |\lambda_{\mathbb{B},i}| < 1$, for $i \neq 0$,
- $\mathcal{D}_{\mathbb{B}}$ is a subalgebra of upper triangular matrices satisfying some additional conditions.

We typeII-classify the subset ClassA', where we require $(\lambda_{\mathbb{B},i})_{i=-k_{R,\mathbb{B}}}^{k_{L,\mathbb{B}}}$ to be non-degenerate.

Type II-classification of ClassA'

Theorem (O '16 preprint)

Let $\mathbb{B}, \mathbb{B}' \in \mathrm{ClassA}'$. Then the Hamiltonians given by $\mathbb{B}$, $\mathbb{B}'$ are equivalent with respect to the type II-classification, if and only if

$$n_{\mathbb{B}}(k_{R\mathbb{B}}+1) = n_{\mathbb{B}'}(k_{R\mathbb{B}'}+1), \quad \textit{and} \quad n_{\mathbb{B}}(k_{L\mathbb{B}}+1) = n_{\mathbb{B}'}(k_{L\mathbb{B}'}+1).$$

Remark

Recall that the numbers $n_{\mathbb{B}}(k_{R\mathbb{B}}+1)/n_{\mathbb{B}}(k_{L\mathbb{B}}+1)$ are the ground state degeneracies of right/left ground states. The theorem says that the edge ground state degeneracies are the complete invariant.

Still, what we can do with type II-classification is to connect $H(h)$s with same edge ground state degeneracies.

How can we conclude bulk-equivalence?

Observation

Let $h_0, h_1 \in \mathcal{J}_{FB}$ and $G_{N,0}$, $G_{N,1}$ be the spectral projection corresponding to eigenvalue 0 of $H_{[1,N]}(h_0)$, $H_{[1,N]}(h_1)$.

Assume that

$$G_{N,0} \leq G_{N,1}, \quad N \in \mathbb{N}.$$

This implies that the unique bulk ground state of $H(h_0)$ and that of $H(h_1)$ are the same.

This implies the bulk equivalence of $H(h_0)$ and $H(h_1)$.

Bulk ground state of $H(h_{m,\mathbb{B}})$ with $\mathbb{B} \in \mathrm{ClassA}$

Recall

$$\mathcal{K}_l(\mathbb{B}) = \mathrm{M}_{n_\mathbb{B}}(\mathbb{C}) \otimes \mathcal{D}_\mathbb{B} \Lambda_\mathbb{B}^l, \quad \text{for } l \text{ large enough.} \tag{1}$$

Define $\boldsymbol{\omega}_\mathbb{B} = (\omega_1, \ldots, \omega_n) \in \mathrm{M}_{n_\mathbb{B}}$ by

$$\omega_\mu \otimes E_{00} = (\mathbb{1} \otimes E_{00})\, B_\mu\, (\mathbb{1} \otimes E_{00}), \quad \mu = 1, \ldots, n.$$

E_{ab} : matrix units of $\mathrm{M}_{k_{L\mathbb{B}}+k_{R\mathbb{B}}} + 1$.

The condition (1) implies $\boldsymbol{\omega}_\mathbb{B}$ to be primitive.

By definition, we have

$$G_{\omega_{\mathbb{B}},N} \le G_{\mathbb{B},N}, \quad N \in \mathbb{N}.$$

The unique bulk ground state of $H(h_{m_0,\omega_{\mathbb{B}}})$ and that of $H(h_{m_1,\mathbb{B}})$ are the same. Therefore, we have $H(h_{m_0,\omega_{\mathbb{B}}}) \simeq_B H(h_{m_1,\mathbb{B}})$.

Now let us complete the bulk-classification. Recall

Theorem (O '16 preprint)

Let $\mathbb{B}, \mathbb{B}' \in \mathrm{ClassA}'$. Then the Hamiltonians given by $\mathbb{B}$, $\mathbb{B}'$ are equivalent with respect to the type II-classification, if and only if

$$n_{\mathbb{B}}(k_{R\mathbb{B}}+1) = n_{\mathbb{B}'}(k_{R\mathbb{B}'}+1), \quad \textit{and} \quad n_{\mathbb{B}}(k_{L\mathbb{B}}+1) = n_{\mathbb{B}'}(k_{L\mathbb{B}'}+1).$$

$$\text{Let} \quad \mathbf{v}_0 \in \mathrm{M}_{k_0}^{\times n}, \quad \text{and} \quad \mathbf{v}_1 \in \mathrm{M}_{k_1}^{\times n} \quad \text{primitive.}$$

Let $\mathbb{B}_0, \mathbb{B}_1 \in \mathrm{ClassA}$, such that

$n_{\mathbb{B}_0} = n_{\mathbb{B}_1} = 1, \quad k_{L\mathbb{B}_0} = k_{R\mathbb{B}_0} = k_0 - 1, \quad k_{L\mathbb{B}_1} = k_{R\mathbb{B}_1} = k_1 - 1.$

Then the edge ground state degeneracies of $H(h_{m_0,\mathbf{v}_0})$ and $H(h_{m_0',\mathbb{B}_0})$ are the same because

$$n_{\mathbb{B}_0}(k_{L\mathbb{B}_0} + 1) = k_0, \quad n_{\mathbb{B}_0}(k_{R\mathbb{B}_0} + 1) = k_0.$$

Therefore, we have

$$H(h_{m_0,\mathbf{v}_0}) \simeq_{II} H(h_{m_0',\mathbb{B}_0}),$$

hence

$$H(h_{m_0,\mathbf{v}_0}) \simeq_{B} H(h_{m_0',\mathbb{B}_0}).$$

Similarly, $H(h_{m_1,\mathbf{v}_1}) \simeq_{B} H(h_{m_1',\mathbb{B}_1})$.

We already know

$$H(h_{m_0',\mathbb{B}_0}) \simeq_{B} H(h_{m_0",\omega_{\mathbb{B}_0}}), \quad \text{and} \quad H(h_{m_1',\mathbb{B}_1}) \simeq_{B} H(h_{m_1",\omega_{\mathbb{B}_1}}).$$

Furthermore, as $\boldsymbol{\omega}_{\mathbb{B}_0}$ and $\boldsymbol{\omega}_{\mathbb{B}_1}$ are primitive elements of $\mathrm{M}_1^{\times n}$. This implies

$$H(h_{m_0",\omega_{\mathbb{B}_0}}) \simeq_{B} H(h_{m_1",\omega_{\mathbb{B}_1}}).$$

Hence we obtain the Theorem of our goal.

Theorem (O '16 preprint)

Let $h_0, h_1 \in \mathcal{J}_{FB}$. Then we have $H(h_0) \simeq_B H(h_1)$.

Thank you.