Classification of gapped Hamiltonians in quantum spin chains

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Recall : Bulk classification

Definition (Bulk Classification)

We say that the Hamiltonians $H(h_0)$, $H(h_1)$ given by h_0, h_1 are bulk equivalent ($H(h_0) \simeq_B H(h_1)$) if the followings hold.

1. *There exist an $m \in \mathbb{N}$ and a continuous path of self-adjoint elements $h : [0,1] \to \mathcal{A}_{[0,m-1]}$ such that $h(0) = h_0$, and $h(1) = h_1$.*
2. *There is a constant $\gamma > 0$, such that*

$$\sigma\left(H_{\varphi_s,\alpha_{h(s)}}\right) \setminus \{0\} \subset [\gamma, \infty),$$

for any $s \in [0,1]$ and $\alpha_{h(s)}$-ground state φ_s. Furthermore, for any $\alpha_{h(s)}$-ground state φ_s, 0 is a non-degenerate eigenvalue of $H_{\varphi_s,\alpha_{h(s)}}$.

We want to classify all the Hamiltonians in the world.

To show that two Hamiltonians are equivalent, we have to construct a path of gapped Hamiltonians connecting them.

To do that, we need to know how to guarantee the existence of the gap.

But actually, even constructing an example of gapped Hamiltonian is a highly non-trivial problem. If local terms $\tau_x(h)$ and $\tau_y(h)$ in

$$H_I(h) = \sum_{x:[x,x+m-1]\subset I} \tau_x(h),$$

mutually commute, we can consider the joint distribution. But we are interested in quantum systems, where local terms do not commute in general.

Fortunately, for quantum spin chains, there is a great recipe by Fannes-Nachtergaele-Werner (1992) to construct gapped Hamiltonians.

This talk:

1. The recipe by Fannes-Nachtergaele-Werner.
2. The injectivity condition
3. The class of FNW is the backbone of our problem.

The recipe by Fannes-Nachtergaele-Werner

Parent Hamiltonian

Parent Hamiltonian

The Hamiltonians given by FNW is so called Parent Hamiltonians. A parent Hamiltonian is made out of a sequence of subspaces of $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$, $l \in \mathbb{N}$, which satisfies the condition called the intersection property.

Definition

We say that a sequence of subspaces $\{\mathcal{D}_N\}_{N\in\mathbb{N}}$, $\mathcal{D}_N \subset \bigotimes_{i=0}^{N-1} \mathbb{C}^n$, $N \in \mathbb{N}$, satisfies the intersection property, if there exists an $m \in \mathbb{N}$, such that the relation

$$\mathcal{D}_N = \bigcap_{x=0}^{N-m} (\mathbb{C}^n)^{\otimes x} \otimes \mathcal{D}_m \otimes (\mathbb{C}^n)^{\otimes N-m-x},$$

holds for all $N \geq m$.

(Recall that our quantum spin chain is $\bigotimes_{\mathbb{Z}} \mathrm{M}_n$.)

Parent Hamiltonian

Let $\{\mathcal{D}_N\}$ be a sequence of non-zero spaces satisfying the intersection property, i.e., there exists an $m \in \mathbb{N}$, such that the relation

$$\mathcal{D}_N = \bigcap_{x=0}^{N-m} (\mathbb{C}^n)^{\otimes x} \otimes \mathcal{D}_m \otimes (\mathbb{C}^n)^{\otimes N-m-x},$$

holds for all $N \geq m$.

Let h_m be the orthogonal projection onto the orthogonal complement of $\mathcal{D}_m$ in $\otimes_{i=0}^{m-1} \mathbb{C}^n$.

Then, by the intersection property, we see that

$$\ker\left(H_{[0,N-1]}(h_m)\right) = \mathcal{D}_N$$

for all $N \geq m$.

Hamiltonians given in this way are called Parent Hamiltonians.

Parent Hamiltonians are nice in the sense that

- we know that the lowest eigenvalues of local Hamiltonians are zero, and
- we know how the ground state spaces of local Hamiltonians, i.e., $\ker\left(H_{[0,N-1]}(h_m)\right) = \mathcal{D}_N$ look like.

Now we would like to construct a sequence of spaces

1. satisfying the intersection property, and
2. the corresponding parent Hamiltonian is gapped with respect to the open boundary conditions.

This is carried out by a linear map determined by an n-tuple of matrices.

Construction of subspaces

Construction of subspaces

Prepare an n-tuple of $k \times k$ matrices $\mathbf{v} := (v_1, \ldots, v_n)$ and $m \in \mathbb{N}$.

Fix some orthonormal basis of $\mathbb{C}^n$, $\{\psi_\mu\}_{\mu=1}^n$.

Define a subspace $\mathcal{G}_{m,\mathbf{v}}$ of $\bigotimes_{i=0}^{m-1} \mathbb{C}^n$ by the range of the following map $\Gamma_{m,\mathbf{v}} : \mathrm{M}_k \to \bigotimes_{i=0}^{m-1} \mathbb{C}^n$,

$$\Gamma_{m,\mathbf{v}}(X) = \sum_{\mu_0,\ldots,\mu_{m-1} \in \{1,\cdots,n\}} \left(\mathrm{Tr}\, X \left(v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{m-1}} \right)^* \right) \bigotimes_{i=0}^{m-1} \psi_{\mu_i}, \quad X \in \mathrm{M}_k .$$

Let $h_{m,\mathbf{v}}$ be the orthogonal projection onto $\mathcal{G}_{m,\mathbf{v}}^{\perp}$ in $\otimes_{i=0}^{m-1}\mathbb{C}^n$.

We consider the Hamiltonian $H(h_{m,\mathbf{v}})$.

The injectivity condition

For a random choice of $\mathbf{v}$,

- $\{\mathcal{G}_{N,\mathbf{v}}\}_N$ does not satisfy the intersection property, and
- the Hamiltonian given by $h_{m,\mathbf{v}}$ does not have a gap.

Some sufficient condition for $\mathbf{v}$ to satisfy these properties?

Injectivity condition

Assume that $\Gamma_{m-1,\mathbf{v}}$ is injective.

Theorem (Fannes-Nachtergaele-Werner '92)

Assume that $\Gamma_{m-1,\mathbf{v}}$ is injective. Then the followings hold.

1. *The sequence of subspaces $\{\mathcal{G}_{N,\mathbf{v}}\}_N$ satisfies the intersection property. In particular, for N large enough, we have*

$$1 \leq \dim\ker\left(H_{[1,N]}(h_{m,\mathbf{v}})\right) \leq k^2$$

2. *The Hamiltonian $H(h_{m,\mathbf{v}})$ is gapped with respect to the open boundary conditions.*

How the injectivity implies these properties?

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Injectivity ⇒

- Intersection property
- Spectral gap (⇒ Exponential decay of correlation functions)

The proof of the spectral gap is due to the Martingale method. [Nachtergaele '96]

I would like to give a brief sketch of the proof for the intersection property and the exponential decay of correlation functions.

What does the injectivity mean?

What does the injectivity mean?

For each $l \in \mathbb{N}$, set

$$\mathcal{K}_l(\mathbf{v}) := \mathrm{span}\left\{ v_{\mu_1} v_{\mu_2} \cdots v_{\mu_l} \mid (\mu_1, \mu_2, \ldots, \mu_l) \subset \{1, \ldots, n\}^{\times l} \right\}.$$

Recall the definition

$$\Gamma_{l,\mathbf{v}}(X) = \sum_{\mu_0,\ldots,\mu_{l-1} \in \{1,\cdots,n\}} \left(\mathrm{Tr}\, X \left(v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{l-1}} \right)^* \right) \bigotimes_{i=0}^{l-1} \psi_{\mu_i}.$$

We have

$$\Gamma_{m-1,\mathbf{v}} \text{ is injective} \Rightarrow \mathcal{K}_{m-1}(\mathbf{v}) = \mathrm{M}_k.$$

This last condition has several characterizations.

Theorem

For $\mathbf{v} := (v_1, \ldots, v_n) \in \mathrm{M}_k^{\times n}$, *let* $T_{\mathbf{v}} : \mathrm{M}_k \to \mathrm{M}_k$ *be the completely positive map given by*

$$T_{\mathbf{v}}(A) = \sum_{i=1}^{n} v_i A v_i^*, \quad A \in \mathrm{M}_k .$$

Assume spectral radius $r_{\mathbf{v}} = 1$. *The following properties are equivalent.*

1. *There exist a unique faithful state* $\varphi_{\mathbf{v}}$ *and a strictly positive element* $e_{\mathbf{v}} \in \mathrm{M}_k$ *satisfying*

$$\lim_{N\to\infty} T_{\mathbf{v}}^N(A) = \varphi_{\mathbf{v}}(A) e_{\mathbf{v}}, \quad A \in \mathrm{M}_k .$$

2. *The spectrum* $\sigma(T_{\mathbf{v}})$ *of* $T_{\mathbf{v}}$ *satisfies* $\sigma(T_{\mathbf{v}}) \cap \mathbb{T} = \{1\}$. 1 *is a non degenerate eigenvalue of* $T_{\mathbf{v}}$. *There exist a faithful* $T_{\mathbf{v}}$*-invariant state* $\varphi_{\mathbf{v}}$ *and a strictly positive* $T_{\mathbf{v}}$*-invariant element* $e_{\mathbf{v}} \in \mathrm{M}_k$.
3. *There exists an* $m \in \mathbb{N}$ *such that* $\mathcal{K}_m(\mathbf{v}) = \mathrm{M}_k$.

Remark

We say $\mathbf{v}$ is primitive when it satisfies the (equivalent) conditions above.

Remark

If $\mathcal{K}_m(\mathbf{v}) = \mathrm{M}_k$, then we have $\mathcal{K}_l(\mathbf{v}) = \mathrm{M}_k$, for all $l \geq m$.

Remark

We may assume $r_{\mathbf{v}} = 1$ for our $\mathbf{v}$. From now on, we assume that this condition holds.

Injectivity $\Rightarrow$ Intersection property

Intersection property

We want to prove

$$\mathcal{G}_{N,\mathbf{v}} = \bigcap_{x=0}^{N-m} (\mathbb{C}^n)^{\otimes x} \otimes \mathcal{G}_{m,\mathbf{v}} \otimes (\mathbb{C}^n)^{\otimes N-m-x}.$$

To prove this, it suffices to show

$$(\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}}) = \mathcal{G}_{N,\mathbf{v}},$$

for all $N \geq m+1$. The inclusion

$$\mathcal{G}_{N,\mathbf{v}} \subset (\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}})$$

is clear from the definition.

We prove

$$(\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}}) \subset \mathcal{G}_{N,\mathbf{v}}.$$

The intersection property [FNW '92]

We show the inclusion $(\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}}) \subset \mathcal{G}_{N,\mathbf{v}}$, for all $N \geq m+1$.
Let $\Phi \in (\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}})$. Then by some sets of $k \times k$ matrices, $\{C_\mu\}_{\mu=1}^n$ and $\{D_\nu\}_{\nu=1}^n$, we can write Φ as

$$\Phi = \sum_{\mu=1}^{n} \psi_\mu \otimes \Gamma_{N-1,\mathbf{v}}(C_\mu) = \sum_{\nu=1}^{n} \Gamma_{N-1,\mathbf{v}}(D_\nu) \otimes \psi_\nu.$$

This means

$$\mathrm{Tr}\left(\left(C_{\mu_1} v_{\mu_N}^* - v_{\mu_1}^* D_{\mu_N}\right) v_{\mu_{N-1}}^* \cdots v_{\mu_2}^*\right) = 0,$$

for all $\mu_1, \ldots, \mu_N \in \{1, \ldots, n\}$.
As $\mathcal{K}_{N-2}(\mathbf{v}) = \mathrm{span}\left\{v_{\mu_2} v_{\mu_3} \ldots v_{\mu_{N-1}}\right\} = \mathrm{M}_k$, this implies

$$C_{\mu_1} v_{\mu_N}^* - v_{\mu_1}^* D_{\mu_N} = 0, \text{ for all } \mu_1, \mu_N \in \{1, \ldots, n\}.$$

We have

$$C_\mu v_\nu^* - v_\mu^* D_\nu = 0, \text{ for all } \mu, \nu \in \{1, \dots, n\}.$$

Recall $\varphi_{\mathbf{v}} \in \mathrm{M}_k$ from

$$\lim_{N \to \infty} T_{\mathbf{v}}^N(A) = \varphi_{\mathbf{v}}(A) e_{\mathbf{v}}, \quad A \in \mathrm{M}_k .$$

As $\varphi_{\mathbf{v}}$ is a faithful state on M_k, there exists an invertible density matrix $\rho_{\mathbf{v}}$ such that $\varphi_{\mathbf{v}} = \mathrm{Tr}\,(\rho_{\mathbf{v}} \cdot)$. By the $T_{\mathbf{v}}$-invariance of $\varphi_{\mathbf{v}}$, we have

$$\sum_\nu v_\nu^* \rho_{\mathbf{v}} v_\nu = \rho_{\mathbf{v}}.$$

From this, we get

$$C_\mu = C_\mu \rho_{\mathbf{v}} \rho_{\mathbf{v}}^{-1} = C_\mu \cdot \left(\sum_\nu v_\nu^* \rho_{\mathbf{v}} v_\nu \right) \cdot \rho_{\mathbf{v}}^{-1}$$

$$= v_\mu^* \left(\sum_\nu D_\nu \rho_{\mathbf{v}} v_\nu \right) \cdot \rho_{\mathbf{v}}^{-1} =: v_\mu^* X.$$

Hence we obtain $C_\mu = v_\mu^* X$, for all $\mu = 1, \dots, n$.
Substituting this, we obtain

$$\Phi = \sum_{\mu=1}^{n} \psi_\mu \otimes \Gamma_{N-1,\mathbf{v}}(C_\mu) = \Gamma_{N,\mathbf{v}}(X) \in \mathcal{G}_{N,\mathbf{v}}.$$

Hence we obtain

$$(\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}}) \subset \mathcal{G}_{N,\mathbf{v}}.$$

Bulk-ground state and

the exponential decay of its correlation functions

Finitely correlated state

The bulk ground state of $H(h_{m,\mathbf{v}})$ is given by a finitely correlated state.

Definition (Finitely correlated state)

Let $(\mathfrak{B}, \mathbb{E}, \varphi, e)$ *be a quadruplet given by a finite dimensional* C^**-algebra* $\mathfrak{B}$, *a CP map* $\mathbb{E} : \mathrm{M}_n \otimes \mathfrak{B} \to \mathfrak{B}$, *a state* φ *on* $\mathfrak{B}$, *and* $e \in \mathfrak{B}_+$ *with* $\varphi(e) \neq 0$ *such that*

$$\mathbb{E}(\mathbb{1} \otimes e) = e, \quad \varphi \circ \mathbb{E}(\mathbb{1} \otimes X) = \varphi(X), \quad X \in \mathfrak{B}.$$

For each $A \in \mathrm{M}_n$, *we define a map* $\mathbb{E}_A : \mathfrak{B} \to \mathfrak{B}$ *by*

$$\mathbb{E}_A(X) = \mathbb{E}\left(A \otimes X\right), \quad X \in \mathfrak{B}.$$

A state ω *on* $\bigotimes_{\mathbb{Z}} \mathrm{M}_n$ *is generated by* $(\mathfrak{B}, \mathbb{E}, \varphi, e)$ *if*

$$\omega\left(\bigotimes_{i=a}^{a+l-1} A_i\right) = \varphi(e)^{-1}\varphi \circ \mathbb{E}_{A_a} \circ \mathbb{E}_{A_{a+1}} \circ \cdots \circ \mathbb{E}_{A_{a+l-1}}\left(e\right),$$

Theorem (Fannes-Nachtergaele-Werner '92)

Suppose that the injectivity condition holds. Then there exists a unique $\alpha_{h_{m,\mathbf{v}}}$-ground state $\omega_{h_{m,\mathbf{v}}}$. The state $\omega_{h_{m,\mathbf{v}}}$ is a finitely correlated state generated by $(\mathrm{M}_k, \mathbb{E}^{(\mathbf{v})}, \varphi_{\mathbf{v}}, e_{\mathbf{v}})$, where

$$\mathbb{E}^{(\mathbf{v})}\left(e_{\mu\nu}^{(n)} \otimes X\right) := v_\mu X v_\nu^*, \quad X \in \mathrm{M}_k.$$

$e_{\mu\nu}^{(n)}$: *matrix units of* M_n.

We would like to show the exponential decay of correlation functions in $\omega_{h_{m,\mathbf{v}}}$.

Namely, we want to show the function

$$\mathbb{N} \ni N \mapsto \omega_{h_{m,\mathbf{v}}}\left(A\tau_{N+l}\left(B\right)\right) - \omega_{h_{m,\mathbf{v}}}\left(A\right)\omega_{h_{m,\mathbf{v}}}\left(\tau_{N+l}\left(B\right)\right)$$

decays exponentially fast for each local A, B.

Exponential decay of correlation functions in $\omega_{h_{m,\mathbf{v}}}$

Let $A = \bigotimes_{i=0}^{l-1} A_i$, $D = \bigotimes_{i=0}^{l-1} D_i \in \mathcal{A}_{[0,l-1]}$, for some $l \in \mathbb{N}$. By the definition of the finitely correlated state, for any $N \in \mathbb{N}$,

$$\omega_{h_{m,\mathbf{v}}} \left(A \tau_{N+l} \left(D \right) \right) = \varphi_{\mathbf{v}} \circ \mathbb{E}_{A_0} \circ \cdots \circ \cdots \circ \mathbb{E}_{A_{l-1}} \circ T_{\mathbf{v}}^{N} \circ \mathbb{E}_{D_0} \circ \cdots \circ \cdots \circ \mathbb{E}_{D_{l-1}} \left(e_{\mathbf{v}} \right).$$

and

$$\begin{aligned} & \omega_{h_{m,\mathbf{v}}} \left(A \right) \omega_{h_{m,\mathbf{v}}} \left(\tau_{N+l} \left(D \right) \right) \\ & = \varphi_{\mathbf{v}} \circ \mathbb{E}_{A_0} \circ \cdots \circ \cdots \circ \mathbb{E}_{A_{l-1}} (e_{\mathbf{v}}) \cdot \varphi_{\mathbf{v}} \left(\mathbb{E}_{D_0} \circ \cdots \circ \cdots \circ \mathbb{E}_{D_{l-1}} \left(e_{\mathbf{v}} \right) \right). \end{aligned}$$

Recall that the spectrum of $T_{\mathbf{v}}$ satisfies $\sigma(T_{\mathbf{v}}) \cap \mathbb{T} = \{1\}$, and 1 is a non degenerate eigenvalue of $T_{\mathbf{v}}$. Hence there exist constants $C > 0$ and $0 < s < 1$ such that

$$\left\| T_{\mathbf{v}}^{N}(X) - \varphi_{\mathbf{v}}(X) e_{\mathbf{v}} \right\| \le C s^{N} \left\| X \right\|, \quad X \in \mathrm{M}_k.$$

From this spectral property of $T_{\mathbf{v}}$, we obtain the exponential decay of correlation functions.

$$\left|\omega_{h_{m,\mathbf{v}}}\left(A\tau_{N+l}\left(B\right)\right)-\omega_{h_{m,\mathbf{v}}}\left(A\right)\omega_{h_{m,\mathbf{v}}}\left(\tau_{N+l}\left(B\right)\right)\right|\leq C's^{N}.$$

The FNW-Hamiltonian class is the backbone of our problem

Anyhow, we got a class of gapped Hamiltonians by the FNW-recipe.

It is nice to have a systematic way of constructing gapped Hamiltonians.

...but we wanted to classify more general class of Hamiltonians, didn't we?

The class we were to classify is $\mathcal{J}_{FB}$, the set of h satisfying the followings.

1. $H(h)$ is gapped in the bulk.
2. There exists a unique α_h-ground state ω on $\mathcal{A}_{\mathbb{Z}}$.
3. There exists a constant $d \in \mathbb{N}$ such that

$$1 \leq \dim \ker \left(H_{[1,N]}(h) \right) \leq d,$$

for all $N \in \mathbb{N}$.

Recall our goal.

Theorem

For any $h_0, h_1 \in \mathcal{J}_{FB}$, we have $H(h_0) \simeq_B H(h_1)$.

FNW-Hamiltonians are given by a special recipe, and given by projections.

Therefore, they may look just examples of gapped Hamiltonians.

However, it turns out that this gives a backbone of our problem, thanks to the following theorem.

Theorem (T. Matsui '13)

Let $h \in \mathcal{J}_{FB}$ and ω its unique bulk ground state. Then ω is a finitely correlated state.

Theorem (Arveson '69)

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and $n \in \mathbb{N}$. Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital endomorphism of $B(\mathcal{H})$ such that $(\Phi(B(\mathcal{H})))'$ is isomorphic to M_n. Then there exist $S_i \in B(\mathcal{H})$, $i = 1, \ldots, n$ such that

$$S_i^* S_j = \delta_{ij}, \quad \sum_{j=1}^{n} S_j x S_j^* = \Phi(x), \quad x \in B(\mathcal{H}). \tag{1}$$

Remark

The n-tuple $\mathbf{v}$ generating the finitely correlated state ω, in the previous theorem is given as a restriction of S_is to a suitable subspace.

The condition $1 \leq \dim\ker\left(H_{[1,N]}(h)\right) \leq d$ is used to guarantee that this subspace is of finite dimension.

Reduction of the problem

Let $h \in \mathcal{J}_{FB}$. Let ω be the unique α_h-ground state, and $(\mathcal{H}, \pi, \Omega)$ its GNS triple. Let P_Ω be the orthogonal projection onto $\mathbb{C}\Omega$.

1. By the general theory of finitely correlated state in [FNW], the n-tuple $\mathbf{v}$ of elements in M_k generating ω can be taken to be primitive. From now on, we assume that this condition holds.
2. By the recipe of [FNW], for m large enough, $H(h_{m,\mathbf{v}})$ is gapped with respect to the open boundary conditions, and ω is the unique $\alpha_{h_{m,\mathbf{v}}}$-ground state. Therefore, for some $\gamma_1 > 0$, we have

$$\gamma_1 (1 - P_\Omega) \leq H_{\omega, \alpha_{h_{m,\mathbf{v}}}}.$$

3. By the assumption $h \in \mathcal{J}_{FB}$, for some $\gamma_0 > 0$, we have

$$\gamma_0 (1 - P_\Omega) \leq H_{\omega, \alpha_h}.$$

4 Therefore, we have

$$\min\{\gamma_0, \gamma_1\} (1 - P_\Omega) \leq H_{\omega, \alpha_{sh+(1-s)h_{m,\mathbf{v}}}}, \quad s \in [0,1].$$

We can show that ω is a unique $\alpha_{sh+(1-s)h_{m,\mathbf{v}}}$-ground state.

5 Hence we conclude $H(h) \simeq_B H(h_{m,\mathbf{v}})$ for a primitive $\mathbf{v}$.

The bulk-classification problem of $\mathcal{J}_{FB}$ is reduced to the bulk-classification problem of FNW-Hamiltonians.

On Thursday we carry out the classification of FNW-Hamiltonians.