Classification of gapped Hamiltonians in quantum spin chains

Yoshiko Ogata

Graduate School of Mathematical Sciences, The University of Tokyo

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Classification problem

But from a physical point of view.

Ground state, Hamiltonian, Quantum spin chain...

Goal of this first talk is to explain the problem.

Ground state

Ground state

Definition

Let $\mathfrak{A}$ be a C^*-algebra and α a strongly continuous one parameter group of automorphisms on $\mathfrak{A}$. Let δ be the generator of α. A state ω on $\mathfrak{A}$ is called an α-ground state if the inequality

$$-i\omega\left(A^*\delta\left(A\right)\right) \geq 0$$

holds for any element A in the domain $\mathcal{D}(\delta)$ of δ.

Ground state : $\mathfrak{A} = \mathrm{M}_n$ case

Let $\mathfrak{A} = \mathrm{M}_n$ and

$$\alpha_t(A) = e^{itH} A e^{-itH}, \quad t \in \mathbb{R}, \quad A \in \mathfrak{A},$$

with a self-adjoint element H in M_n. Let P be the spectral projection of H corresponding to the lowest eigenvalue.

Then a state ω is an α-ground state if and only if the support $s(\omega)$ of ω satisfies $s(\omega) \leq P$.

The definition in the previous slide is a generalization of finite dimensional case.

Hamiltonians associated to ground states

Hamiltonians associated to ground states

Proposition

Let ω be an α-ground state. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of ω. Then ω is α-invariant and there exists a unique positive operator $H_{\omega,\alpha}$ on $\mathcal{H}$ such that $e^{itH_{\omega,\alpha}}\pi(A)\Omega = \pi(\alpha_t(A))\Omega$, for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$.

Note : Ω is an eigenvector of $H_{\omega,\alpha}$ with eigenvalue 0.
In this talk, we call this $H_{\omega,\alpha}$, the Hamiltonian associated to ω.

Gapped Hamiltonian

Definition

We say that $H_{\omega,\alpha}$ is gapped if 0 is a non-degenerate eigenvalue of $H_{\omega,\alpha}$ and there exists a constant $\gamma > 0$ such that

$$\sigma(H_{\omega,\alpha}) \setminus \{0\} \subset [\gamma, \infty).$$

(Here, $\sigma(H_{\omega,\alpha})$ denotes the spectrum of $H_{\omega,\alpha}$.)
We also say that $H_{\omega,\alpha}$ has a gap γ to specify the γ.

Quantum spin chain

Quantum spin chain

Let $n \in \mathbb{N}$ be fixed. A quantum spin chain is the C^*-algebra

$$\mathcal{A}_{\mathbb{Z}} := \bigotimes_{\mathbb{Z}} \mathrm{M}_n \,.$$

There is an obvious action τ of $\mathbb{Z}$ on $\mathcal{A}_{\mathbb{Z}}$ i.e.,

$$\tau_x\left(\cdots \otimes \mathbb{1}_{\{y-1\}} \otimes A \otimes \mathbb{1}_{\{y+1\}} \cdots\right) = \cdots \otimes \mathbb{1}_{\{x+y-1\}} \otimes A \otimes \mathbb{1}_{\{x+y+1\}} \cdots$$

for $A \in \mathrm{M}_n$ and $x, y \in \mathbb{Z}$.

For each $\Lambda \subset \mathbb{Z}$, $\mathcal{A}_\Lambda := \bigotimes_\Lambda \mathrm{M}_n$ is naturally regarded as a subalgebra of $\mathcal{A}_{\mathbb{Z}}$. We use the notation

$$\mathcal{A}_R := \mathcal{A}_{[0,\infty)\cap\mathbb{Z}}, \quad \mathcal{A}_L := \mathcal{A}_{(-\infty,-1]\cap\mathbb{Z}}.$$

Dynamics on quantum spin chains

Our favorite dynamics is the one given by translation invariant finite range interactions.

We define a C^*-dynamics on $\mathcal{A}_{\mathbb{Z}}$ from

$$m \in \mathbb{N} \quad \text{and} \quad h \in \mathcal{A}_{[0,m-1],\mathrm{sa}}.$$

Local Hamiltonians

Fix some $m \in \mathbb{N}$ and a self-adjoint element $h \in \mathcal{A}_{[0,m-1]}$.

The local Hamiltonian associated with h on a finite interval $I \subset \mathbb{Z}$ is defined by

$$H_I(h) = \sum_{x:[x,x+m-1]\subset I} \tau_x(h).$$

We denote the net of local Hamiltonians by

$$H(h) := (H_I(h))_I\,,$$

and call it the Hamiltonian given by h.

Dynamics given by h

Proposition

Let h be a self-adjoint element in $\mathcal{A}_{[0,m-1]}$. Then for any $A \in \mathcal{A}_{\mathbb{Z}}$ and $t \in \mathbb{R}$, the limit

$$\alpha_{t,h}(A) := \lim_{I \nearrow \mathbb{Z}} e^{itH_I(h)} A e^{-itH_I(h)}$$

exists and defines a strongly continuous one parameter group of automorphisms α_h on $\mathcal{A}_{\mathbb{Z}}$.

The dynamics we consider in this talk is the dynamics of this type. We call it a dynamics given by translation invariant finite range interactions.

Dynamics given by h

Remark

Similarly, we can define $\alpha_{h,R}/\alpha_{h,L}$ on $\mathcal{A}_R/\mathcal{A}_L$ by

$$\alpha_{t,h,R}(A_R) = \lim_{I \nearrow [0,\infty)\cap\mathbb{Z}} e^{itH_I(h)} A_R e^{-itH_I(h)}, \quad A_R \in \mathcal{A}_R,$$

$$\alpha_{t,h,L}(A_L) = \lim_{I \nearrow (-\infty,-1]\cap\mathbb{Z}} e^{itH_I(h)} A_L e^{-itH_I(h)}, \quad A_L \in \mathcal{A}_L.$$

Ground states on quantum spin chains

Given dynamics α_h, $\alpha_{h,R}$, $\alpha_{h,L}$, we can consider ground states.

We call an α_h-ground state, a bulk ground state.

We also call an $\alpha_{h,R}/\alpha_{h,L}$-ground state on $\mathcal{A}_R/\mathcal{A}_L$, a right/left edge ground state.

Thermodynamic limit

For each finite interval I, let $\alpha_{h,I}$ be the dynamics on $\mathcal{A}_I$ given by $\alpha_{t,h,I}(A) := e^{itH_I(h)}Ae^{-itH_I(h)}$.

Lemma

Let $\{\omega_I\}_I$ be a net of states on $\mathcal{A}_{\mathbb{Z}}$ labeled by finite intervals I in $\mathbb{Z}$. Assume that for each I, the restriction of ω_I to $\mathcal{A}_I$ is an $\alpha_{h,I}$-ground state. Then any of the wk*-accumulation point of $\{\omega_I\}_I$ is an α_h-ground state. In particular, there exists an α_h-ground state.

Remark

Similar statement holds for $\Gamma = [0,\infty)\cap\mathbb{Z}, (-\infty,-1]\cap\mathbb{Z}$. We denote by $\mathcal{S}_\Gamma(h)$ the set of all ground states which are wk*-accumulation points as above.

Gapped in the bulk

Definition

We say a Hamiltonian $H(h)$ is gapped in the bulk if there exists some $\gamma > 0$ such that H_{φ,α_h} has the gap γ, for any α_h-ground state φ.

Bulk classification

Bulk classification

Definition (Bulk Classification)

We say that the Hamiltonians $H(h_0)$, $H(h_1)$ gapped in the bulk, given by h_0, h_1 are bulk equivalent ($H(h_0) \simeq_B H(h_1)$) if the followings hold.

1. *There exist an $m \in \mathbb{N}$ and a continuous path of self-adjoint elements $h : [0,1] \to \mathcal{A}_{[0,m-1]}$ such that $h(0) = h_0$, and $h(1) = h_1$.*
2. *There is a constant $\gamma > 0$, such that*

$$\sigma\left(H_{\varphi_s, \alpha_{h(s)}}\right) \setminus \{0\} \subset [\gamma, \infty),$$

for any $s \in [0,1]$ and $\alpha_{h(s)}$-ground state φ_s. Furthermore, for any $\alpha_{h(s)}$-ground state φ_s, 0 is a non-degenerate eigenvalue of $H_{\varphi_s, \alpha_{h(s)}}$.

What does this classification mean?

When we consider some classification, we regard two elements which are equivalent are essentially same.

In which sense we regard equivalent gapped Hamiltonians are essentially same?

In order to see this, we have to think what the existence of gap means.

What does the existence of gap mean?

1. Stability under perturbation
2. Exponential decay of correlation functions

Stability

Stability : Regular perturbation theory

Suppose that H_{ω,α_h} has a spectral gap $\gamma > 0$. Let $(\mathcal{H}, \pi, \Omega)$ be the GNS triple of ω.

Then there exists an $\varepsilon > 0$ satisfying the followings.

1. For any $V = V^* \in \mathcal{A}_{\mathbb{Z}}$ with $\|V\| < \varepsilon$, the spectral projection
$$\mathrm{Proj}\left[H_{\omega,\alpha_h} + \pi\left(V\right) \in \left(-\frac{\gamma}{2}, \frac{\gamma}{2}\right)\right]$$
of $H_{\omega,\alpha_h} + \pi\left(V\right)$ for $(-\frac{\gamma}{2}, \frac{\gamma}{2})$ is one rank.
2. The map
$$V \mapsto \mathrm{Proj}\left[H_{\omega,\alpha_h} + \pi\left(V\right) \in \left(-\frac{\gamma}{2}, \frac{\gamma}{2}\right)]\right]$$
is continuous.
3. For a unit vector $\xi \in \mathrm{Proj}\left[H_{\omega,\alpha_h} + \pi\left(V\right) \in \left(-\frac{\gamma}{2}, \frac{\gamma}{2}\right)]\right]\mathcal{H}$, $\langle \xi, \pi\left(\cdot\right)\xi \rangle$ is an α_V-ground state. (Here, α_V is the C^*-dynamics given as the perturbation of α by V.)

Stability under shallow perturbation

Theorem (Michalakis-Zwolak '13)

Assume some additional conditions on h. Then for any $V \in \mathcal{A}_{[0,m-1],\mathrm{sa}}$, there exists an $\varepsilon_0 > 0$ such that $H(h + sV)$ is gapped in the bulk, for all $|s| < \varepsilon_0$.

Exponential decay of correlation functions

Exponential decay of correlation functions

Two random variables X, Y are independent if and only if

$$\mathbb{E}\left(\left(F(X) - \mathbb{E}\left(F(X)\right)\right) \cdot \left(G(Y) - \mathbb{E}\left(G(Y)\right)\right)\right) = 0,$$

for any bounded continuous functions F, G on $\mathbb{R}$.

The covariance

$$\mathbb{E}\left(F(X)G(Y)\right) - \mathbb{E}\left(F(X)\right) \cdot \mathbb{E}\left(G(Y)\right)$$

indicates the correlation of two variables.

Exponential decay of correlation functions

One important question is

Q:Let us consider two observables far away from each other.
How much correlation do they have?

Namely, we are interested in the decay of the following function

$$\mathbb{Z} \ni x \mapsto \omega\left(A\tau_x\left(B\right)\right) - \omega\left(A\right)\omega\left(\tau_x\left(B\right)\right),$$

for each $A \in \mathcal{A}_{[a_1,a_2]}$ and $B \in \mathcal{A}_{[b_1,b_2]}$.

Does the correlation decay as a function of the distance $|x|$?

If it does, how fast? Exponentially fast, or just polynomially fast?

If it is just polynomial decay, then we regard the state has strong correlation.

In statistical mechanics, we consider Macroscopic observables.

$$X_N(B) := \frac{1}{N}\sum_{i=0}^{N-1} \tau_x(B), \quad B \in \mathcal{A}_{[0,m-1]}.$$

The exponential decay of correlation functions implies that the distribution of macroscopic observables satisfy the central limit theorem in $N \to \infty$ limit [Matsui].

If the correlation functions show exponential decay, we regard the state to repersent a normal phase.

Exponential decay of correlation functions

Theorem (Hastings-Koma '06, Nachtergaele-Sims '09)

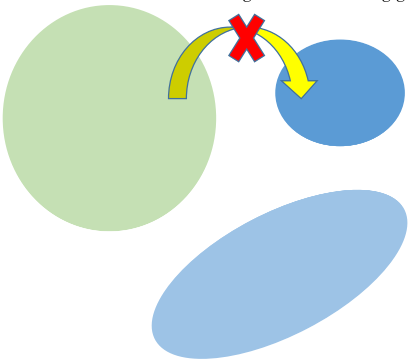
Suppose that ω is a unique α_h-ground state. If H_{ω,α_h} has a spectral gap, then the correlation functions decay exponentially fast.

Bulk classification in quantum spin systems

For two Hamiltonians $H(h_0)$ and $H(h_1)$, the equivalence $H(h_0) \simeq_B H(h_1)$ means they can be connected keeping these normal properties, i.e., stability and exponential decay of correlation functions. In other words, they can be connected without crossing any critical phenomena.

What we would like to do is to group the gapped Hamiltonians with this criterion.

Cannot go without closing gap.



Goal of my talk

We denote by $\mathcal{J}_{FB}$, the set of h satisfying the followings.

1. $H(h)$ is gapped in the bulk.
2. There exists a unique α_h-ground state ω on $\mathcal{A}_{\mathbb{Z}}$.
3. There exists a constant $d \in \mathbb{N}$ such that

$$1 \leq \dim \ker \left(H_{[1,N]}(h) \right) \leq d,$$

for all $N \in \mathbb{N}$.

Theorem (O '16 preprint)

For any $h_0, h_1 \in \mathcal{J}_{FB}$, we have $H(h_0) \simeq_B H(h_1)$.

Classification with open boundary conditions

Local version of classification

Gapped Hamiltonian

Definition

A Hamiltonian $H(h)$ associated with h is gapped with respect to the open boundary conditions if the distance between $\inf\left(\sigma\left(H_{[1,N]}(h)\right)\right)$ and the rest of the spectrum of $H_{[1,N]}(h)$ is uniformly bounded from below by some $\gamma > 0$.

Local gaps imply the gap in the bulk

Lemma

Assume that the Hamiltonian $H(h)$ is gapped with respect to the open boundary conditions. Let $\gamma > 0$ be a lower bound of the gap. Assume that there exists a unique α_h-ground state ω. Then, we have

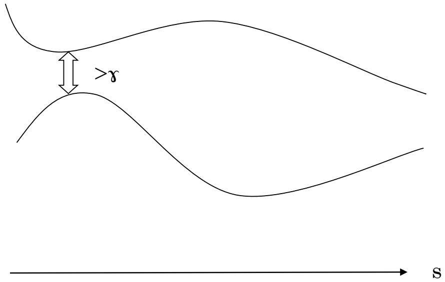
$$\sigma\left(H_{\omega,\alpha_h}\right) \setminus \{0\} \subset [\gamma, \infty).$$

Classification with respect to the open boundary conditions

Definition (Type I- classification)

We say that $H(h_0)$, $H(h_1)$ are equivalent in the type-I classification ($H(h_0) \simeq_I H(h_1)$) with respect to the open boundary conditions if there is a piecewise smooth path of interactions $h(s)$, $s \in [0,1]$ connecting h_0 and h_1, such that the gap of $H_{[1,N]}(h(s))$ is uniformly (in s and N) bounded from below by some $\gamma > 0$.

Type I classification $\sigma(H_{[1,N]}(h(s)))$



Classification with respect to the open boundary conditions

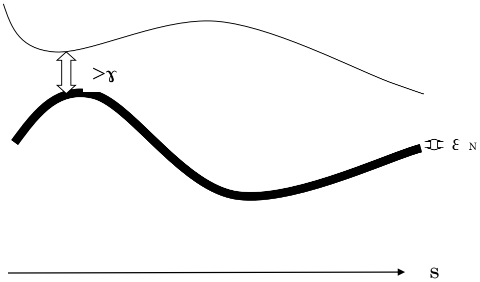
Definition (Type II Classification)

We say that $H(h_0)$, $H(h_1)$ are equivalent in the type II-classification ($H(h_0) \simeq_{II} H(h_1)$) with respect to the open boundary conditions if there are a piecewise smooth path of interactions $h(s)$, $s \in [0,1]$ connecting h_0 and h_1, and a sequence of positive numbers $\{\varepsilon_N\}_N$, $\varepsilon_N \to 0$, such that the gap between

$$[\lambda_{N,s}, \lambda_{N,s} + \varepsilon_N] \cap \sigma(H_{[1,N]}(h(s)))$$

and the rest of the spectrum of $H_{[1,N]}(h(s))$ is uniformly bounded from below by some $\gamma > 0$. (Here, $\lambda_{N,s}$ is the lowest eigenvalue of $H_{[1,N]}(h(s))$ and $\sigma(H_{[1,N]}(h(s)))$ is the spectrum of $H_{[1,N]}(h(s))$.)

Type II classification $\sigma(H_{[1,N]}(h(s)))$



Lemma

Let $H(h_0), H(h_1)$ be Hamiltonians gapped with respect to the open boundary conditions.
Suppose that $H(h_0)$ and $H(h_1)$ are type I-equivalent and that the bulk-ground state is unique along the path.
Then, we have $H(h_0) \simeq_B H(h_1)$.

Invariant of typeII-classification

Invariant of typeII-classification

Theorem (Bachmann, Michalakis, Nachtergaele, Sims '11)

Let $H(h_0), H(h_1)$ be Hamiltonians gapped with respect to the open boundary conditions. Suppose that $H(h_0)$ and $H(h_1)$ are type II-equivalent. Furthermore, assume that there are finite intervals $I(s)$ with smooth end points such that $\inf \sigma \left(H_{[1,N]}(h)\right) \in I(s)$ for all $s \in [0,1]$ and $N \in \mathbb{N}$. Let $\Gamma = \mathbb{Z}, (-\infty, -1] \cap \mathbb{Z}, [0, \infty) \cap \mathbb{Z}$. Then, there exist a quasi-local automorphism β_Γ, of $\mathcal{A}_\Gamma$ such that

$$\mathcal{S}_\Gamma(h_0) \circ \beta_\Gamma = \mathcal{S}_\Gamma(h_1).$$

Quasi-local automorphism

Let β be an automorphism of $\mathcal{A}_{\mathbb{Z}}$. In general,

$$\beta\left(\mathcal{A}_{[a,b]}\right) \subset \mathcal{A}_{[a',b']},$$

is not true. But if it is true for any $[a, b]$, we can regard β to be local.

Quasi-locality is a relaxed version of this. We say β is quasi-local if there exist a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, decaying faster than any polynomials, such that

$$\left\|\beta(A) - \mathbb{E}_{[a-N,b+N]}\left(\beta\left(A\right)\right)\right\| \leq f(N)\left\|A\right\|, \quad A \in \mathcal{A}_{[a,b]}, \quad N \in \mathbb{N}.$$

Here

$$\mathcal{A}_{\mathbb{Z}} \ni A \to \mathbb{E}_{[a-N,b+N]}\left(A\right) \in \mathcal{A}_{[a-N,b+N]}$$

is the conditional expectation with respect to the tracial state.

If two Hamiltonians are type II-equivalent, their ground state space can be translated to each other by some quasi-local automorphism.

This gives another justification about type II -classification.

The ground state structure is essentially same if two Hamiltonians are in the same class.

Tomorrow, I would like to start the classification.