Quantum groups with projections and braided quantum groups

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Outline

Motivation

- Semidirect products build groups out of simpler pieces.

Example

Real and complex $ax + b$ groups	$\mathbb{R} \rtimes \mathbb{R}_{>0}$, $\mathbb{C} \rtimes \mathbb{C}^{\times}$
Isometries of the plane	$\mathrm{E}(2) = \mathbb{R}^2 \times \mathbb{T}$
Poincaré group	$\mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

Goal

1. Semidirect product construction for quantum groups
2. Analysis which quantum groups are semidirect products

- This should simplify the construction of quantum group deformations of semidirect product groups.

Semidirect product data

G a group

H a group

β an action $\beta\colon G \times H \to H$

- multiplication $H \times H \to H$ is G-equivariant

(A, Δ_A) a quantum group: C*-algebra A with comultiplication $\Delta_A\colon A \to A \otimes A$

(B, Δ_B) a quantum group

Δ_{BA} a coaction $\Delta_{BA}\colon B \to B \otimes A$

- Δ_B is (A, Δ_A)-equivariant

Question

Why is this ~~wrong~~ nonsense?

Algebras in braided tensor categories

- Quantum group representations form a monoidal category. But algebra tensor products need a braided monoidal category to permute b_1, a_2 in

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2).$$

- There is no canonical induced coaction of (A, Δ_A) on $B \otimes B$ unless (A, Δ_A) is a group.
  So no notion of (A, Δ_A)-equivariance for $\Delta_B \colon B \to B \otimes B$.

No-Go Theorem

Tensor products of algebras do not inherit quantum group coactions except group actions.

Quantum E(2) groups by Woronowicz

Quantum E(2) as a Hopf *-algebra

- A_μ universal *-algebra generated by unitary v, normal n with $v^*nv = \mu n$
- Δ maps $v \mapsto v \otimes v$, $n \mapsto v \otimes n + n \otimes v^*$
- projection to $\mathbb{C}[\mathbb{T}] = \mathbb{C}[v, v^*]$ maps $v \mapsto v$, $n \mapsto 0$

First problem: C*-algebras generated by unbounded operators

This is often hard, but easy here: Get $\mathrm{C}_0(\mathbb{C}) \rtimes_\alpha \mathbb{Z}$ with $\alpha f(n) = f(\mu n)$.

Second problem: existence of coproduct

The coproduct does not exist on $\mathrm{C}_0(\mathbb{C}) \rtimes \mathbb{Z}$!
Must add relation $\mathrm{Spectrum}(n) \subseteq \{z \in \mathbb{C} \mid |z| \in \{0\} \cup \mu^{\mathbb{Z}}\}$.

Twisted tensor products and semidirect products

- The second piece (B, Δ_B) in a semidirect product is a braided quantum group:
  $\Delta_B \colon B \to B \boxtimes B$ with a "twisted" tensor product $\boxtimes$.
- We proceed as follows
  - easy example of $\boxtimes$
  - $\boxtimes$ for two linked quantum groups
  - aside: crossed products and dual actions
  - define semidirect product data
  - build semidirect product

Tensor product for circle actions

A, B	C*-algebras with $\mathbb{T}$-actions α, β
A_n, B_m	subspaces where $\alpha_z(a_n) = z^n a_n$, $\beta_z(b_m) = z^m b_m$
ζ	scalar $\zeta \in \mathbb{T}$
$\alpha \times \beta$	induced action of $\mathbb{T} \times \mathbb{T}$ on $A \otimes B$
$A \boxtimes_\zeta B$	Rieffel deformation of $A \otimes B$ with parameter ζ

$$b_m a_n = \zeta^{mn} a_n b_m \qquad b_m \in B_m,\ a_n \in A_n$$

Theorem

$\mathbb{T}$-C-Algebras with $\boxtimes_\zeta$ form a monoidal category.*
$\boxtimes_\zeta$ associative and unital with unit $\mathbb{C}$.

Tensor product for $\mathbb{Z}/2$-graded C*-algebras

A, B C*-algebras with $\mathbb{Z}/2$-gradings

$A_\pm, B_\pm$ subspaces where $\alpha_z(a_n) = \pm a_n$, $\beta_z(b_m) = \pm b_m$

induced $\mathbb{Z}/2 \times \mathbb{Z}/2$-grading on $A \otimes B$

$A \boxtimes B$ Rieffel deformation of $A \otimes B$ for unique non-trivial bicharacter on $\mathbb{Z}/2 \times \mathbb{Z}/2$ (Koszul sign rule)

Theorem

$\mathbb{Z}/2$-graded C-algebras with $\boxtimes$ form a symmetric monoidal category.*

General twisted tensor product construction

A, Δ_A quantum group

C, γ C*-algebra, A-coaction

B, Δ_B quantum group

D, δ C*-algebra, B-coaction

χ unitary bicharacter in $\widehat{A} \otimes \widehat{B}$ compare case $\mathbb{T}$ and $\mathbb{Z}/2$

$C \boxtimes_\chi D$ twisted tensor product

Idea of construction

1. represent (C, γ, A, Δ_A) on Hilbert space $\mathcal{H}$
2. represent (D, δ, B, Δ_B) on Hilbert space $\mathcal{K}$
3. χ gives unitary on $\mathcal{H} \otimes \mathcal{K}$
4. $C \boxtimes_\chi D$ is the closed linear span of $(C \otimes 1) \cdot \chi(1 \otimes D)\chi^*$
5. difficult: proof that this is a C*-algebra

Quantum group crossed product

B, Δ_B dual quantum group $(\widehat{A}, \widehat{\Delta}_A)$

D, Δ_D also $(\widehat{A}, \widehat{\Delta}_A)$

χ reduced multiplicative unitary $W^A \in \mathcal{U}(\widehat{A} \otimes A)$

$C \boxtimes \widehat{A}$ reduced crossed product for (A, Δ_A)-coaction (C, γ)

- $\boxtimes$ is functorial for equivariant morphisms
- $\widehat{\Delta}_A \colon \widehat{A} \to \widehat{A} \otimes \widehat{A}$ is $\widehat{A}$-equivariant if $\widehat{A}$ coacts on $\widehat{A} \otimes \widehat{A}$ on second factor only
- $C \boxtimes (\widehat{A} \otimes \widehat{A}) \cong \widehat{A} \otimes (C \boxtimes \widehat{A})$

$\Rightarrow$ $\widehat{\Delta}_A$ induces $C \boxtimes \widehat{A} \to \widehat{A} \otimes (C \boxtimes \widehat{A})$.

- This is the dual action.
  It is coassociative by abstract nonsense.

Yetter–Drinfeld algebra

- We must iterate $\boxtimes$ to speak about $B \boxtimes B \boxtimes B$.
- Assume that B carries coactions of both A and $\widehat{A}$.
- Assume compatibility condition between the coactions of A and $\widehat{A}$ to make $\boxtimes$ associative:
  Yetter–Drinfeld algebra

Theorem (Vaes, Nest–Voigt)

Yetter–Drinfeld algebras over (A, Δ_A) with $\boxtimes$ form a monoidal category.

Semidirect product data

A, B C^*-algebras

β coaction $B \to B \otimes A$

$\widehat{\beta}$ coaction $B \to B \otimes \widehat{A}$

Δ_A comultiplication $A \to A \otimes A$

Δ_B comultiplication $B \to B \boxtimes B$

Assumptions

1. Δ_A coassociative
2. β coaction
3. $\widehat{\beta}$ coaction
4. Yetter–Drinfeld compatibility between β and $\widehat{\beta}$
5. Δ_B equivariant for β
6. Δ_B equivariant for $\widehat{\beta}$
7. Δ_B coassociative

Circle-braided case

$A = \mathrm{C}(\mathbb{T})$

Δ_A dual of usual group structure on $\mathbb{T}$

B C^*-algebra

β coaction $B \to B \otimes A$

$\widehat{\beta}$ compose β with $A = \mathrm{C}(\mathbb{T}) \to \mathrm{C}_0(\mathbb{Z}) = \widehat{A}$ from a bicharacter

Δ_B comultiplication $B \to B \boxtimes B$

Assumptions

- β coaction
- Δ_B equivariant for β
- Δ_B coassociative
- Podleś/cancellation conditions for $\Delta_A, \beta, \Delta_B$

Construction of the semidirect product

- The C*-algebra of the semidirect product is $C = A \boxtimes B$.
  This is the crossed product of B for the $\widehat{A}$-coaction.
- Comultiplication (cheating a bit)
  $A\boxtimes B \xrightarrow{\mathrm{id}_A\boxtimes\Delta_B} A\boxtimes B\boxtimes B \xrightarrow{j_{124}} A\boxtimes B\boxtimes A\boxtimes B \cong (A\boxtimes B)\otimes(A\boxtimes B)$
- Above formula uses a Yetter–Drinfeld structure on A,
  which only exists under "regularity" assumptions on A.
- This assumption is avoided in our paper
  by a different formula.

Example: Braided SU(2) and braided free orthogonal quantum groups

- There is a braided variant of Woronowicz' quantum SU(2) for complex q, using $\boxtimes_{q/\bar{q}}$ with respect to a circle action (Kasprzak–M–R–W).
- resulting semidirect product is quantum U(2) of Zhang–Zhao.
- $\mathbb{T}$-braided free orthogonal quantum groups
  are universal $\mathbb{T}$-braided quantum groups
  for a finite-dimensional representation u
  with a given invariant vector in $u \otimes u(d)$
  (d): shift of $\mathbb{T}$-action.

Question

How can we see that a C^*-quantum group is a semidirect product? And if it is, how to construct the two pieces?

- We first answer this question for groups.
- This leads to the notion of a C^*-quantum group with projection.
- We go back and forth between braided quantum groups and quantum groups with projection on the level of multiplicative unitaries.

Detecting semidirect product groups

Proposition

semidirect product decomposition of a group $L \equiv$ idempotent group homomorphism $p\colon L \to L$ (projection)

Proof.

- if $L = H \rtimes_\alpha G$, let $p\colon L \twoheadrightarrow G \hookrightarrow L$
- $G = p(L)$
- $H = \ker p$
- $H \times G \to L$, $(h, g) \mapsto h \cdot g$, is homeomorphism
- read $\alpha\colon G \to \mathrm{Aut}(H)$ from product in $H \times G \cong L$ $\square$

Quantum Groups with Projection

Definition (quantum group with projection)

quantum group L with idempotent homomorphism $p\colon L \to L$

Example

Many deformations of semidirect product groups have a projection:

- $\mathrm{U}_q(2)$ by Zhang–Zhao
- $\mathrm{E}_q(2)$ by Woronowicz
- quantum $az + b$ by Woronowicz and Sołtan
- quantum $ax + b$ by Woronowicz–Zakrzewski

Theorem

Semidirect products admit a projection.

Quantum Group Extensions

Quantum group extensions are another construction of quantum groups out of two smaller pieces inspired by group extensions.

Example

Most quantum group extensions have no projection:

- quantum $az + b$ groups by Baaj and Skandalis
- quantum $ax + b$ group by Stachura
- κ-Poincaré group by Stachura

The Factors of a Quantum Group with Projection

Let $p\colon L \to L$ be an idempotent quantum group homomorphism.

Proposition

There is an image quantum group G:
There are quantum group homomorphisms $L \xrightarrow{\pi} G \xrightarrow{\iota} L$
with $p = \iota \circ \pi$ and $\pi \circ \iota = \mathrm{id}_G$.

- The kernel of p should be a braided quantum group H over G.
- We expect $\mathrm{C}^*(L) = \mathrm{C}^*(H) \boxtimes \mathrm{C}^*(G)$.
- This is a reduced crossed product.
  Landstad theory reconstructs the coefficients from a crossed product. It only works if G is "regular."
- We shall use another approach to quantum groups.

Quantum groups and homomorphisms

Definition (Woronowicz, Sołtan)

A C^*-quantum group is a C^*-bialgebra (A, Δ) coming from a manageable multiplicative unitary $W \in \mathrm{U}(\mathcal{H} \otimes \mathcal{H})$.
Then $W \in \mathrm{U}(\widehat{A} \otimes A)$.

Problem

C^*-quantum group theory prefers
reduced over full group C^*-algebras.
A group homomorphism $G \to H$ need not induce a Hopf *-homomorphism $\mathrm{C}^*_\mathrm{r}(G) \to \mathrm{C}^*_\mathrm{r}(H)$.

Definition (Ng, M–R–W)

quantum group homomorphism $(A, \Delta_A) \to (B, \Delta_B)$
$\equiv$ unitary bicharacter in $\widehat{A} \otimes B$

Characterisations of quantum group homomorphisms

1. unitary bicharacter in $\widehat{A} \otimes B$
2. unitary operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ with two pentagon equations
3. Hopf *-homomorphism $A^{\mathrm{u}} \to B^{\mathrm{u}}$ on universal quantum groups
4. functor from A-coactions to B-coactions on C^*-algebras not changing the underlying C^*-algebra
5. tensor functor from A-corepresentations to B-corepresentations not changing the underlying Hilbert space
6. coaction of B on A with a comultiplicativity property

Theorem (Tannaka–Krein)

A C^*-quantum group is determined uniquely up to isomorphism by its tensor category of representations with its fibre functor.

Quantum group with projection

Lemma

A quantum group with projection is equivalent to unitaries $W, P \in \mathrm{U}(\mathcal{H} \otimes \mathcal{H})$ satisfying the pentagon equations

$$
\begin{aligned}
W_{23} W_{12} &= W_{12} W_{13} W_{23}, \\
P_{23} W_{12} &= W_{12} P_{13} P_{23}, \\
W_{23} P_{12} &= P_{12} P_{13} W_{23}, \\
P_{23} P_{12} &= P_{12} P_{13} P_{23},
\end{aligned}
$$

such that W is manageable.
Then P is also manageable.

Braided multiplicative unitary

Definition

Let $W \in \mathrm{U}(\mathcal{H} \otimes \mathcal{H})$ be a manageable, multiplicative unitary.
A braided multiplicative unitary over W consists of unitaries

$$U \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H}), \quad \widehat{V} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L}), \quad F \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$$

2. U right corepresentation of W
3. $\widehat{V}$ left corepresentation of W
4. $U, \widehat{V}$ Drinfeld compatible
5. F is $U \otimes U$-invariant
6. F is $\widehat{V} \otimes \widehat{V}$-invariant
7. F satisfies braided pentagon equation

Semidirect product for braided multiplicative unitaries

Theorem

Let $(U, \widehat{V}, F)$ be a manageable braided multiplicative unitary over W. Define

$$X = W_{13} U_{23} \widehat{V}_{34}^{*} F_{24} \widehat{V}_{34},$$
$$P = W_{13} U_{23}$$

on $\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}$. This is a C^**-quantum group with projection.*

Proof.

Long direct checking, manageability is always technical.
Pentagon equation for X uses all 7 conditions on $(W, U, \widehat{V}, F)$.
Better argument uses regular objects by Pinzari–Roberts. □

Braided multiplicative unitary from projection

Theorem

A C^*-quantum group (X, P) with projection on $\mathcal{H}$ gives a manageable braided multiplicative unitary $(U, \widehat{V}, F)$ on $\overline{\mathcal{H}} \otimes \mathcal{H}$. The C^*-quantum group of $(P, U, \widehat{V}, F)$ is isomorphic to (X, P).

Proof.

The Hilbert space is enlarged to $\overline{\mathcal{H}} \otimes \mathcal{H}$ to make room for a Drinfeld compatible pair.
There are explicit formulas for $U, \widehat{V}, F$ in terms of certain canonical representations of X on $\overline{\mathcal{H}} \otimes \mathcal{H}$. □

Summary

- The quantum group analogue of the semidirect product construction for groups starts with braided quantum groups.
- A semidirect product decomposition of a quantum group is the same as a projection on that quantum group.