

Poly- \mathbb{Z} group actions on Kirchberg algebras III

Hiroki Matui

Chiba University

August, 2016

Operator Algebras and Mathematical Physics

Tohoku University

Goal

Classify outer actions of poly- \mathbb{Z} groups on Kirchberg algebras up to KK -trivial cocycle conjugacy (as much as possible).

- A C^* -algebra A is called a **Kirchberg algebra** if A is separable, nuclear, simple and purely infinite.
- A **poly- \mathbb{Z} group** is a group of the form $(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \dots) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$.
- $\alpha, \beta : G \curvearrowright A$ are said to be **KK -trivially cocycle conjugate** if $\exists \alpha$ -cocycle $(u_g)_g$, $\exists \theta \in \text{Aut}(A)$ such that $KK(\theta) = 1$ and $\text{Ad } u_g \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ holds for all $g \in G$.

- ✓① Conjecture and partial answers
- ✓② Equivariant version of Nakamura's theorem
- ✓③ Uniqueness of outer G -actions on \mathcal{O}_∞
- ✓④ Absorption of outer G -actions on \mathcal{O}_∞
- ⑤ Stability
- ⑥ Classification

Stability

Theorem (poly- \mathbb{Z} stability)

Let $\alpha : G \curvearrowright A$ be an action of a poly- \mathbb{Z} group G on a unital separable C^* -algebra A , which accepts $(\mathcal{O}_\infty, \beta)$.

Let $I \subset A$ be an ideal. Suppose that a family $(x_g)_{g \in G}$ of continuous maps from $[0, 1] \times [0, \infty)$ to $U(I)$ satisfies

$$x_g(0, t) = 1, \quad \lim_{t \rightarrow \infty} \max_{s \in [0, 1]} \|[x_g(s, t), a]\| = 0 \quad \forall g \in G, a \in A,$$

$$\lim_{t \rightarrow \infty} \max_{s \in [0, 1]} \|x_g(s, t)\alpha_g(x_h(s, t)) - x_{gh}(s, t)\| = 0 \quad \forall g, h \in G.$$

Then there exists a continuous map $y : [0, \infty) \rightarrow U(I)$ such that

$$\lim_{t \rightarrow \infty} \|[y(t), a]\| = 0 \quad \forall a \in A,$$

$$\lim_{t \rightarrow \infty} \|x_g(1, t) - y(t)\alpha_g(y(t)^*)\| = 0 \quad \forall g \in G.$$

Cocycle actions (1/2)

Let A be a unital C^* -algebra and let G be a group.

A pair (α, u) of a map $\alpha : G \rightarrow \text{Aut}(A)$ and a map $u : G \times G \rightarrow U(A)$ is called a **cocycle action** of G on A if

$$\alpha_g \circ \alpha_h = \text{Ad } u(g, h) \circ \alpha_{gh}$$

and

$$u(g, h)u(gh, k) = \alpha_g(u(h, k))u(g, hk)$$

hold for any $g, h, k \in G$. We write $(\alpha, u) : G \curvearrowright A$.

We always assume $\alpha_1 = \text{id}$, $u(g, 1) = u(1, g) = 1$ for all $g \in G$.

When α_g is not inner for any $g \in G \setminus \{1\}$, (α, u) is said to be **outer**.

When $u \equiv 1$, $\alpha : G \curvearrowright A$ is a genuine action.

Cocycle actions (2/2)

Two cocycle actions $(\alpha, u) : G \curvearrowright A$ and $(\beta, v) : G \curvearrowright B$ are said to be **cocycle conjugate** if there exist a family of unitaries $(w_g)_{g \in G}$ in B and an isomorphism $\theta : A \rightarrow B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \text{Ad } w_g \circ \beta_g$$

and

$$\theta(u(g, h)) = w_g \beta_g(w_h) v(g, h) w_{gh}^*$$

hold for every $g, h \in G$.

Our eventual goal is

- to classify (α, u) up to (KK -trivial) cocycle conjugacy and
- to determine when (α, u) is cocycle conjugate to a genuine action.

Stability

Let G be a poly- \mathbb{Z} group and fix a finite generating set $S \subset G$.
Let $\beta : G \curvearrowright \mathcal{O}_\infty$ be an outer action.

Theorem (H^2 -stability)

*For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds:
Let $(\sigma, w) : G \curvearrowright A$ be a cocycle action which accepts $(\mathcal{O}_\infty, \beta)$.
If $\|w(g, h) - 1\| < \delta$ for all $g, h \in S$, then there exist unitaries $(v_g)_g$ in A such that $\|v_g - 1\| < \varepsilon$ for all $g \in S$ and*

$$w(g, h) = \sigma_g(v_h^*)v_g^*v_{gh} \quad \forall g, h \in G,$$

that is, w is a coboundary.

We can prove H^1 -stability and H^2 -stability by induction on the Hirsch length:

$$\begin{aligned} H^2\text{-stability for } G &\implies H^1\text{-stability for } G \\ \implies H^2\text{-stability for } G \rtimes \mathbb{Z} &\implies H^1\text{-stability for } G \rtimes \mathbb{Z} \implies \dots \end{aligned}$$

- ✓① Conjecture and partial answers
- ✓② Equivariant version of Nakamura's theorem
- ✓③ Uniqueness of outer G -actions on \mathcal{O}_∞
- ✓④ Absorption of outer G -actions on \mathcal{O}_∞
- ✓⑤ Stability
- ⑥ Classification

Conjecture

Let us recall our conjecture.

Conjecture (Izumi 2010)

Let A be a unital Kirchberg algebra and let G be a poly- \mathbb{Z} group. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright A$ be outer actions.

The following are equivalent.

- 1 α and β are KK -trivially cocycle conjugate.
- 2 There exists a base point preserving isomorphism between \mathcal{P}_α^s and \mathcal{P}_β^s .

The second condition is equivalent to the following:
there exists a continuous map $\Phi : EG \rightarrow \text{Aut}(A \otimes \mathbb{K})_0$
such that $\Phi(x_0) = \text{id}$ and $\Phi(g.x) \circ \alpha_g^s = \beta_g^s \circ \Phi(x)$.

Intertwining argument

For a poly- \mathbb{Z} group G , we let $\mu^G : G \curvearrowright \mathcal{O}_\infty$ be an outer action.

Theorem (Intertwining argument)

Let G be a poly- \mathbb{Z} group and let A be a unital separable C^* -algebra. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, v) : G \curvearrowright A$ be cocycle actions which accept $(\mathcal{O}_\infty, \mu^G)$.

Suppose that there exists a family $(x_g)_g$ of unitaries in A^b such that

$$(\text{Ad } x_g \circ \alpha_g)(a) = \beta_g(a) \quad \forall a \in A,$$

$$x_g \alpha_g(x_h) u(g, h) x_{gh}^* = v(g, h) \quad \forall g, h \in G.$$

Then (α, u) and (β, v) are cocycle conjugate via an asymptotically inner automorphism.

Asymptotically representable actions

As an immediate corollary, we get the following.

Theorem (Izumi-M)

Let A be a unital Kirchberg algebra and let G be a poly- \mathbb{Z} group. All asymptotically representable outer actions of G on A are mutually KK -trivially cocycle conjugate.

Proof.

Let $\alpha : G \curvearrowright A$ be an asymptotically representable outer action. By definition, there exist $x_g : [0, \infty) \rightarrow U(A)$ ($g \in G$) such that

$$x_g(t)x_h(t) - x_{gh}(t) \rightarrow 0, \quad x_g(t) a x_g(t)^* \rightarrow \alpha_g(a).$$

Let $\mu^G : G \curvearrowright \mathcal{O}_\infty$ be an outer action. Then $(x_g \otimes 1)_{g \in G}$ is thought of as a family of unitaries in $(A \otimes \mathcal{O}_\infty)^b$. Thanks to the previous theorem, we can conclude that $\alpha \otimes \mu^G$ is KK -trivially cocycle conjugate to $\text{id} \otimes \mu^G$. □

Obstruction (1/4)

Let $\alpha, \beta : G \curvearrowright A$ be outer actions of a poly- \mathbb{Z} group G on a unital Kirchberg algebra A such that $KK(\alpha_g) = KK(\beta_g)$ for all $g \in G$.

For each $g \in G$, there exists $u_g : [0, \infty) \rightarrow U(A)$ such that $\text{Ad } u_g(t) \circ \alpha_g \rightarrow \beta_g$ as $t \rightarrow \infty$. Then the unitaries

$$w(g, h) = u_g \alpha_g(u_h) u_{gh}^*$$

belong to $A_b = A^b \cap A'$. Define $\sigma_g \in \text{Aut}(A_b)$ by $\sigma_g = \text{Ad } u_g \circ \alpha_g$ for each $g \in G$. Then it is easy to check that (σ, w) is a cocycle action of G on A_b .

Lemma

If w is a coboundary, α and β are KK -trivially cocycle conjugate.

Proof.

If $\exists v_g \in U(A_b)$ such that $w(g, h) = v_g \sigma_g(v_h) v_{g,h}^*$, then $(v_g u_g)_g$ is an α -cocycle satisfying $(\text{Ad } v_g u_g \circ \alpha_g)(a) = \beta_g(a)$ for $a \in A$. \square

Obstruction (2/4)

For outer actions $\alpha, \beta : G \curvearrowright A$ satisfying $KK(\alpha_g) = KK(\beta_g)$, we have chosen $u_g \in A^b$ so that $(\text{Ad } u_g \circ \alpha_g)|_A = \beta_g|_A$ and defined a cocycle action $(\sigma, w) : G \curvearrowright A_b$ by

$$\sigma_g = (\text{Ad } u_g \circ \alpha_g)|_{A_b}, \quad w(g, h) = u_g \alpha_g(u_h) u_{gh}^*.$$

We denote the cohomology class of the 2-cocycle $(g, h) \mapsto K_1(w(g, h))$ by

$$o^2(\alpha, \beta) \in H^2(G, K_1(A_b)) \cong H^2(G, KK^1(A, A)),$$

which does not depend on the choice of $(u_g)_g$.

Obstruction (3/4)

Suppose $o^2(\alpha, \beta) = 0$.

By replacing $(u_g)_g$ if necessary, we may assume $K_1(w(g, h)) = 0$ for all $g, h \in G$.

Choose a continuous path $\tilde{w}(g, h) : [0, 1] \rightarrow U(A_b)$ from 1 to $w(g, h)$. Then

$$\sigma_g(\tilde{w}(h, k))\tilde{w}(g, hk) (\tilde{w}(g, h)\tilde{w}(gh, k))^* \quad g, h, k \in G$$

are unitaries in $S(A_b)$, and their K_1 -classes form a 3-cocycle in $K_1(S(A_b)) = K_0(A_b)$.

We denote its cohomology class by

$$o^3(\alpha, \beta, u) \in H^3(G, K_0(A_b)) \cong H^3(G, KK(A, A)),$$

which does not depend on the choice of the path $(\tilde{w}(g, h))_{g, h}$, but may (a priori) depend on the choice of $(u_g)_g \subset U(A^b)$.

Obstruction (4/4)

For $\alpha, \beta : G \curvearrowright A$,

- if $KK(\alpha_g) = KK(\beta_g)$, then $o^2(\alpha, \beta) \in H^2(G, KK^1(A, A))$ is defined,
- if $o^2(\alpha, \beta) = 0$, then $o^3(\alpha, \beta, u) \in H^3(G, KK(A, A))$ is defined.

Lemma

If there exists a base point preserving isomorphism between \mathcal{P}_α^s and \mathcal{P}_β^s , then $KK(\alpha_g) = KK(\beta_g)$, $o^2(\alpha, \beta) = 0$ and $o^3(\alpha, \beta, u) = 0$ for some $(u_g)_g$.

Hirsch length two (1/2)

Let G be a poly- \mathbb{Z} group with **Hirsch length two** and let $\alpha, \beta : G \curvearrowright A$ be outer actions on a unital Kirchberg algebra. Assume $KK(\alpha_g) = KK(\beta_g)$ and $o^2(\alpha, \beta) = 0$. We want to show that α and β are KK -trivially cocycle conjugate.

Let $\sigma_g = \text{Ad } u_g \circ \alpha_g \in \text{Aut}(A_b)$ and $w(g, h) \in U(A_b)$ be as before. By replacing $(u_g)_g$, we may assume $K_1(w(g, h)) = 0 \ \forall g, h \in G$. Let $G = N \rtimes \langle \xi \rangle$. By assumption, N is \mathbb{Z} , and so we may assume that $\sigma|_N$ is a genuine action. Then, there exists a $\sigma|_N$ -cocycle $(v_g)_g$ in A_b such that

$$\sigma_\xi \circ \sigma_{\xi^{-1}g\xi} \circ \sigma_\xi^{-1} = \text{Ad } v_g \circ \sigma_g \quad \forall g \in N,$$

and $K_1(v_g) = 0$. Since $N = \mathbb{Z}$, the $\sigma|_N$ -cocycle $(v_g)_g$ can be approximated by coboundaries. By the H^2 -stability, we can conclude that the 2-cocycle w is a coboundary.

Then, by the lemma mentioned before, α and β are KK -trivially cocycle conjugate.

Hirsch length two (2/2)

Theorem (Izumi-M)

The conjecture is true for any poly- \mathbb{Z} group with *Hirsch length two*.

Example

Let A be the Cuntz algebra \mathcal{O}_n . Then $KK^1(A, A) \cong \mathbb{Z}_{n-1}$.

- When $G = \mathbb{Z}^2$, $H^2(\mathbb{Z}^2, \mathbb{Z}_{n-1}) \cong \mathbb{Z}_{n-1}$, and so there exist $n-1$ outer actions $\mathbb{Z}^2 \curvearrowright \mathcal{O}_n$ up to KK -trivial cocycle conjugacy.
- When $G = \langle a, b \mid bab = a^{-1} \rangle$, $H^2(G, \mathbb{Z}_{n-1}) \cong \mathbb{Z}_{n-1} \otimes \mathbb{Z}_2$, and so

$$\#(\{\text{outer actions } G \curvearrowright \mathcal{O}_n\} / \sim) = \begin{cases} 1 & n \in 2\mathbb{N} \\ 2 & n \in 2\mathbb{N}+1. \end{cases}$$

Hirsch length three (1/2)

Now, let us consider a poly- \mathbb{Z} group G with **Hirsch length three**. Let $\alpha, \beta : G \curvearrowright A$ be outer actions on a unital Kirchberg algebra. Assume $KK(\alpha_g) = KK(\beta_g)$, $o^2(\alpha, \beta) = 0$ and $o^3(\alpha, \beta, u) = 0$ for some $(u_g)_g$.

We want to show that α and β are KK -trivially cocycle conjugate.

Let $\sigma_g = \text{Ad } u_g \circ \alpha_g \in \text{Aut}(A_b)$ and $w(g, h) \in U(A_b)$ be as before. By replacing $(u_g)_g$, we may assume $K_1(w(g, h)) = 0 \forall g, h \in G$. Let $\tilde{w}(g, h) : [0, 1] \rightarrow U(A_b)$ be a path from 1 to $w(g, h)$. Since $o^3(\alpha, \beta, u) = 0$, we may assume that the loop

$$\sigma_g(\tilde{w}(h, k))\tilde{w}(g, hk) (\tilde{w}(g, h)\tilde{w}(gh, k))^*$$

in $U(A_b)$ is homotopic to 1 for every $g, h, k \in G$. Let $G = N \rtimes \langle \xi \rangle$. By assumption, N is a poly- \mathbb{Z} group with Hirsch length two. As $K_1(w(g, h)) = 0$, we may assume that $\sigma|_N$ is a genuine action. Moreover, So, w is a coboundary!

Hirsch length three (2/2)

Theorem (Izumi-M)

The conjecture is true for any poly- \mathbb{Z} group with *Hirsch length three*.

Let G be a poly- \mathbb{Z} group with Hirsch length three.

Example

Let A be \mathcal{O}_n . Then $KK^i(A, A) \cong \mathbb{Z}_{n-1}$ for $i = 0, 1$, and

$$\{\text{outer actions } G \curvearrowright A\} / \sim \cong H^2(G, KK^1(A, A)).$$

Example

Let A be the Cuntz standard form of \mathcal{O}_n . Then, there exists a natural bijection between the set of KK -trivial cocycle conjugacy classes of outer actions $\alpha : G \curvearrowright A$ such that $KK(\alpha_g) = 1$ and

$$H^2(G, KK^1(A, A)) \oplus H^3(G, KK(A, A)).$$

- ✓① Conjecture and partial answers
- ✓② Equivariant version of Nakamura's theorem
- ✓③ Uniqueness of outer G -actions on \mathcal{O}_∞
- ✓④ Absorption of outer G -actions on \mathcal{O}_∞
- ✓⑤ Stability
- ??⑥ Classification