Poly-$\mathbb{Z}$ group actions on Kirchberg algebras II

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Goal

Classify outer actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras up to KK-trivial cocycle conjugacy (as much as possible).

- A C^*-algebra A is called a Kirchberg algebra if A is separable, nuclear, simple and purely infinite.
- A poly-$\mathbb{Z}$ group is a group of the form $(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \dots) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$.
- $\alpha, \beta : G \curvearrowright A$ are said to be KK-trivially cocycle conjugate if $\exists$ α-cocycle $(u_g)_g$, $\exists$ $\theta \in \mathrm{Aut}(A)$ such that $KK(\theta) = 1$ and $\mathrm{Ad}\, u_g \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ holds for all $g \in G$.

Overview

✓① Conjecture and partial answers

✓② Equivariant version of Nakamura's theorem

③ Uniqueness of outer G-actions on $\mathcal{O}_\infty$

④ Absorption of outer G-actions on $\mathcal{O}_\infty$

⑤ Stability

⑥ Classification

Equivariant Nakamura's theorem

Let us recall what we have done !

- $G \cong N \rtimes \langle \xi \rangle$
- $\alpha, \beta : G \curvearrowright A$ are outer actions on a Kirchberg algebra A.
- $\alpha|N$, $\beta|N$ are asymptotically representable.
- $\beta|N$ is a cocycle perturbation of $\alpha|N$.
- $\alpha_\xi, \beta_\xi \in \mathrm{Aut}(A)$ extend to the crossed products.

Theorem (Equivariant Nakamura's theorem)

If $KK(\tilde{\alpha}_\xi) = KK(\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})$, then $\alpha : G \curvearrowright A$ is KK-trivially cocycle conjugate to $\beta : G \curvearrowright A$.

Uniqueness of actions on $\mathcal{O}_\infty$ (1/4)

By using equivariant Nakamura's theorem repeatedly,
we want to prove the following.

Theorem (Uniqueness of $G \curvearrowright \mathcal{O}_\infty$)

Let G be a poly-$\mathbb{Z}$ group.
Any outer actions $G \curvearrowright \mathcal{O}_\infty$ are cocycle conjugate to each other.
In particular, they are asymptotically representable.

The proof is by induction on the Hirsch length of G. When $G = \mathbb{Z}$, this is a corollary of Nakamura's theorem. Assume we have known the theorem for poly-$\mathbb{Z}$ groups with Hirsch length less than l.

Let G be a poly-$\mathbb{Z}$ group with Hirsch length l.
Then G is of the form $N \rtimes \langle \xi \rangle$, where $N \lhd G$ is a poly-$\mathbb{Z}$ group with Hirsch length $l-1$ and $\xi \in G$ is of infinite order.
Suppose that outer actions $\alpha, \beta : G \curvearrowright \mathcal{O}_\infty$ are given.
We want to prove that α is cocycle conjugate to β.

Uniqueness of actions on $\mathcal{O}_\infty$ (2/4)

Let $\alpha, \beta : G \curvearrowright \mathcal{O}_\infty$ be outer actions.
By the induction hypothesis, outer actions $N \curvearrowright \mathcal{O}_\infty$ are unique up to cocycle conjugacy. Hence $\alpha|N$ and $\beta|N$ are cocycle conjugate, and (in particular) they are asymptotically representable.
There exists an isomorphism $\theta : \mathcal{O}_\infty \rtimes_\beta N \to \mathcal{O}_\infty \rtimes_\alpha N$ such that

$$\theta(\mathcal{O}_\infty) = \mathcal{O}_\infty, \quad \theta(\lambda_g^\beta) = u_g \lambda_g^\alpha \quad \forall g \in N,$$

where $(u_g)_g$ is an α-cocycle.

Let $\iota_\alpha : C^*(N) \to \mathcal{O}_\infty \rtimes_\alpha N$ and $\iota_\beta : C^*(N) \to \mathcal{O}_\infty \rtimes_\beta N$ be the canonical inclusions.
By induction, we can prove that $KK(\iota_\alpha)$ and $KK(\iota_\beta)$ are invertible. (Note: $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$.)

Uniqueness of actions on $\mathcal{O}_\infty$ (3/4)

$$
\begin{tikzcd}
\mathcal{O}_\infty \rtimes_\beta N \arrow[rr, "\theta"] & & \mathcal{O}_\infty \rtimes_\alpha N \\
& C^*(N) \arrow[ul, "\iota_\beta"] \arrow[ur, "\iota_\alpha"'] &
\end{tikzcd}
$$

With some extra effort (using asymptotical representability), we can make $KK(\theta) = KK(\iota_\alpha) \cdot KK(\iota_\beta)^{-1}$.
Then

$$
\begin{aligned}
&KK(\theta \circ \tilde{\beta}_\xi \circ \theta^{-1}) \\
&= KK(\iota_\alpha) \cdot KK(\iota_\beta)^{-1} \cdot KK(\tilde{\beta}_\xi) \cdot KK(\iota_\beta) \cdot KK(\iota_\alpha)^{-1} \\
&= KK(\tilde{\alpha}_\xi).
\end{aligned}
$$

Thanks to equivariant Nakamura's theorem, we can conclude that $\alpha : G \curvearrowright \mathcal{O}_\infty$ is cocycle conjugate to $\beta : G \curvearrowright \mathcal{O}_\infty$.
This completes the induction.

Uniqueness of actions on $\mathcal{O}_\infty$ (4/4)

In the same way as $\mathcal{O}_\infty$, we can prove the following.

Theorem (Izumi-M)

Let G be a poly-$\mathbb{Z}$ group. If A is either $\mathcal{O}_2$, $\mathcal{O}_\infty$ or $\mathcal{O}_\infty \otimes B$ with B being a UHF algebra of infinite type, there exists a unique cocycle conjugacy class of outer G-actions on A.

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McDuff type theorem

Next, we want to prove that any outer action $G \curvearrowright A$ absorbs tensorially the outer action $G \curvearrowright \mathcal{O}_\infty$.
To this end, we need the following theorem.

Theorem (McDuff type theorem)

Let G be a countable discrete group.
Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on unital separable C^*-algebras. Suppose that the following conditions hold.

1. $\exists$ *unital homo.* $\pi : B \to A_\omega$ *s.t.* $\pi \circ \beta_g = \alpha_g \circ \pi \ \forall g \in G$.
2. $\exists$ *sequence* $(v_n)_n$ *of unitaries in* $U(B \otimes B)_0$ *such that*

$$v_n(b \otimes 1)v_n^* \to 1 \otimes b, \quad (\beta_g \otimes \beta_g)(v_n) - v_n \to 0.$$

Then $\alpha : G \curvearrowright A$ is cocycle conjugate to $\alpha \otimes \beta : G \curvearrowright A \otimes B$ via an isomorphism $\psi : A \to A \otimes B$ which is asymptotically unitarily equivalent to the embedding $A \ni a \mapsto a \otimes 1 \in A \otimes B$.

Absorption of actions on $\mathcal{O}_\infty$ (1/4)

Let G be an infinite countable group.
Suppose that $\mathcal{O}_\infty$ is generated by isometries $(s_g)_{g\in G}$.
Define $\beta : G \curvearrowright \mathcal{O}_\infty$ by $\beta_g(s_h) = s_{gh}$ for all $g, h \in G$.

Lemma

Let $\alpha : G \curvearrowright A$ be an outer action on a unital Kirchberg algebra. Then $(\mathcal{O}_\infty, \beta)$ (defined above) is embeddable into (A_ω, α).

Proof.

Since $\alpha : G \curvearrowright A_\omega$ is outer and A_ω is purely infinite simple, one can find a nonzero projection $e \in A_\omega$ such that $(\alpha_g(e))_g$ are mutually orthogonal. By replacing e with a smaller one if necessary, we may assume that e has the same K_0-class as $1 \in A_\omega$. Choose an isometry $t \in A_\omega$ such that $tt^* = e$. Define a homomorphism $\pi : \mathcal{O}_\infty \to A_\omega$ by $\pi(s_g) = \alpha_g(t)$, which meets the requirement. $\square$

Absorption of actions on $\mathcal{O}_\infty$ (2/4)

Let $\beta : G \curvearrowright \mathcal{O}_\infty$ be as before.

Lemma

Suppose that $\beta \otimes \beta : G \curvearrowright \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ is approximately representable. Then there exists a sequence $(v_n)_n$ of unitaries in $U(\mathcal{O}_\infty \otimes \mathcal{O}_\infty)_0$ such that

$$v_n(b \otimes 1)v_n^* \to 1 \otimes b, \quad (\beta_g \otimes \beta_g)(v_n) - v_n \to 0.$$

Proof.

Let ρ_l, ρ_r be the two embeddings

$$\mathcal{O}_\infty \rtimes_\beta G \to (\mathcal{O}_\infty \otimes \mathcal{O}_\infty) \rtimes_{\beta\otimes\beta} G.$$

Since $KK(\rho_l) = KK(\rho_r)$, $\exists\, u_n \in (\mathcal{O}_\infty \otimes \mathcal{O}_\infty) \rtimes_{\beta\otimes\beta} G$ such that $\mathrm{Ad}\, u_n \circ \rho_l \to \rho_r$. By using approximate representability of $\beta \otimes \beta$, we can replace u_n with $v_n \in \mathcal{O}_\infty \otimes \mathcal{O}_\infty$. $\square$

Absorption of actions on $\mathcal{O}_\infty$ (3/4)

Let G be a poly-$\mathbb{Z}$ group and let $\beta : G \curvearrowright \mathcal{O}_\infty$ be an outer action, which is unique up to cocycle conjugacy.
Let $\alpha : G \curvearrowright A$ be an outer action on a unital Kirchberg algebra.

We have verified the following:

- $(\mathcal{O}_\infty, \beta)$ is embeddable to (A_ω, α).
- The flip on $O_\infty \otimes \mathcal{O}_\infty$ is 'β-equivariantly' approximately inner.

So, the McDuff type theorem applies and yields:

Theorem

*Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright \mathcal{O}_\infty$ be as above.
Then, $\alpha : G \curvearrowright A$ is cocycle conjugate to $\alpha \otimes \beta : G \curvearrowright A \otimes \mathcal{O}_\infty$ via an isomorphism $\psi : A \to A \otimes \mathcal{O}_\infty$ which is asymptotically unitarily equivalent to the embedding $A \ni a \mapsto a \otimes 1 \in A \otimes \mathcal{O}_\infty$.*

Absorption of actions on $\mathcal{O}_\infty$ (4/4)

Why is the $\mathcal{O}_\infty$-absorption useful?
Because good properties of $\beta : G \curvearrowright \mathcal{O}_\infty$ are transmitted to $\alpha \otimes \beta : G \curvearrowright A \otimes \mathcal{O}_\infty$, and hence to $\alpha : G \curvearrowright A$.

- Assume G is of the form $G = N \rtimes \langle \xi \rangle$. The automorphism β_ξ has 'Rohlin projections' in $(\mathcal{O}_\infty)_\omega^{\beta|N}$,
  i.e. $\forall m \in \mathbb{N}$, $\exists$ partition of unities consisting of projections $e_0, e_1, \ldots, e_{m-1}, f_0, f_1, \ldots, f_m$ in $(\mathcal{O}_\infty)_\omega^{\beta|N}$ such that
  $$\beta_\xi(e_i) = e_{i+1}, \quad \beta_\xi(f_j) = f_{j+1}.$$
- $(\mathcal{O}_\infty)_\omega^\beta$ and $(\mathcal{O}_\infty)_\flat^\beta$ contain unital copies of $\mathcal{O}_\infty$. So, one can use Nakamura's homotopy lemma in $(\mathcal{O}_\infty)_\omega^\beta$ and $(\mathcal{O}_\infty)_\flat^\beta$.

Thanks to the $\mathcal{O}_\infty$-absorption theorem, we can conclude that any outer action $\alpha : G \curvearrowright A$ also has these properties.

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Nakamura's stability theorem (1/2)

Let $\alpha : \mathbb{Z} \curvearrowright A$ be an outer action on a unital Kirchberg algebra A.

Theorem (Nakamura 2000)

For any sequence of unitaries $x_n \in U(C([0,1], A))$ satisfying

$$x_n(0) = 1, \quad \lim_{n\to\infty} \sup_{s\in[0,1]} \|[x_n(s), a]\| = 0 \quad \forall a \in A,$$

there exists a sequence of unitaries $y_n \in U(A)_0$ such that

$$\lim_{n\to\infty} \|[y_n, a]\| = 0 \quad \forall a \in A,$$

and

$$\lim_{n\to\infty} \|x_n(1) - y_n \alpha(y_n^*)\| = 0.$$

Indeed, we have already used this stability theorem (or its variant) in the proof of the equivariant Nakamura's theorem.

Nakamura's stability theorem (2/2)

Let us review a rough sketch of his proof.
The given unitaries $x_n \in C([0,1],A)$ are not necessarily equicontinuous, that is, the Lipschitz constant $\mathrm{Lip}(x_n)$ are not necessarily bounded. First, since $\mathcal{O}_\infty$ is embedded into A_ω, we may replace x_n with $x'_n \in C([0,1],A)$ satisfying

$$x'_n(0) = 1, \quad x'_n(1) = x_n(1), \quad \mathrm{Lip}(x'_n) < 6\pi,$$

$$\lim_{n\to\infty} \sup_{s\in[0,1]} \|[x'_n(s), a]\| = 0 \quad \forall a \in A.$$

Then, by using the Rohlin property of $\alpha \in \mathrm{Aut}(A)$, we can construct unitaries $y_n \in U(A)_0$ such that

$$\|[y_n, a]\| \to 0 \quad \forall a \in A, \quad \|x'_n(1) - y_n\alpha(y_n^*)\| \to 0.$$

We want to establish a 'poly-$\mathbb{Z}$ version' of this theorem.

Stability (1/2)

We want to establish a 'poly-$\mathbb{Z}$ version' of stability.

Let $\alpha : G \curvearrowright A$ be an action of a poly-$\mathbb{Z}$ group G on a unital (not necessarily separable) C^*-algebra, and let $\beta : G \curvearrowright \mathcal{O}_\infty$ be an outer action (which is unique up to cocycle conjugacy).

We say that α accepts $(\mathcal{O}_\infty, \beta)$ if for any separable subset $S \subset A^\omega$ there exists a unital homomorphism $\varphi : \mathcal{O}_\infty \to A^\omega \cap S'$ such that $\varphi \circ \beta_g = \alpha_g \circ \varphi$ for all $g \in G$.

- If A is separable, then by the McDuff type theorem, $\alpha : G \curvearrowright A$ is cocycle conjugate to $\alpha \otimes \beta : G \curvearrowright A \otimes \mathcal{O}_\infty$.
- If A is a unital Kirchberg algebra and α is outer, then α accepts $(\mathcal{O}_\infty, \beta)$ (by the absorption theorem).

Stability (2/2)

Theorem (poly-$\mathbb{Z}$ stability)

Let $\alpha : G \curvearrowright A$ be an action of a poly-$\mathbb{Z}$ group G on a unital separable C^*-algebra A, which accepts $(\mathcal{O}_\infty, \beta)$.
Let $I \subset A$ be an ideal. Suppose that a family $(x_g)_{g \in G}$ of continuous maps from $[0,1] \times [0, \infty)$ to $U(I)$ satisfies

$$x_g(0,t) = 1, \quad \lim_{t \to \infty} \max_{s \in [0,1]} \|[x_g(s,t), a]\| = 0 \quad \forall g \in G, \ a \in A,$$

$$\lim_{t \to \infty} \max_{s \in [0,1]} \|x_g(s,t)\alpha_g(x_h(s,t)) - x_{gh}(s,t)\| = 0 \quad \forall g, h \in G.$$

Then there exists a continuous map $y : [0, \infty) \to U(I)$ such that

$$\lim_{t \to \infty} \|[y(t), a]\| = 0 \quad \forall a \in A,$$

$$\lim_{t \to \infty} \|x_g(1,t) - y(t)\alpha_g(y(t)^*)\| = 0 \quad \forall g \in G.$$