Poly-$\mathbb{Z}$ group actions on Kirchberg algebras I

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Goal

Classify outer actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras up to KK-trivial cocycle conjugacy (as much as possible).

A C^*-algebra A is called a Kirchberg algebra if A is separable, nuclear, simple and purely infinite.
Kirchberg algebras are classified by KK-theory.

Theorem (Kirchberg-Phillips 2000)

Let A and B be Kirchberg algebras. T.F.A.E.

1. *A and B are stably isomorphic. i.e. $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.*
2. *A and B are KK-equivalent, i.e. $\exists$ invertible $x \in KK(A, B)$.*

Poly-$\mathbb{Z}$ groups

A (countable, discrete) group G is called a poly-$\mathbb{Z}$ group if G has a subnormal series

$$\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_l = G$$

such that $G_{i+1}/G_i \cong \mathbb{Z}$. The length l is independent from the choice of the series, and is called the Hirsch length of G.

In other words, a poly-$\mathbb{Z}$ group is a group of the form

$$(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \dots) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}.$$

Poly-$\mathbb{Z}$ groups with Hirsch length two are $\mathbb{Z}^2$ and the Klein bottle group $\langle a, b \mid bab = a^{-1} \rangle$.
The (discrete, three dimensional) Heisenberg group is a poly-$\mathbb{Z}$ group with Hirsch length three.

Group actions (1/2)

Goal

Classify outer actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras up to KK-trivial cocycle conjugacy (as much as possible).

- An action $\alpha : G \curvearrowright A$ is said to be outer if for any $g \in G \setminus \{1\}$, α_g is not an inner automorphism.
- Two actions $\alpha, \beta : G \curvearrowright A$ are said to be conjugate if there exists $\theta \in \mathrm{Aut}(A)$ such that $\alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ for all $g \in G$.
- A family of unitaries $(u_g)_g$ in A is called an α-cocycle if $u_g \alpha_g(u_h) = u_{gh}$ holds for all $g, h \in G$. $g \mapsto \mathrm{Ad}\, u_g \circ \alpha_g$ is also an action of G, and is called a cocycle perturbation of α.

Group actions (2/2)

Definition

Let $\alpha, \beta : G \curvearrowright A$ be two actions of G on A.

- α and β are said to be cocycle conjugate if β is conjugate to a cocycle perturbation of α,
  i.e. $\exists$ α-cocycle $(u_g)_g$, $\exists$ $\theta \in \mathrm{Aut}(A)$
  such that $\mathrm{Ad}\, u_g \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ holds for all $g \in G$.
- α and β are said to be KK-trivially cocycle conjugate
  if they are cocycle conjugate and one can make $KK(\theta) = 1$.

It is easy to see that if α and β are cocycle conjugate, then the crossed products $A \rtimes_\alpha G$ and $A \rtimes_\beta G$ are isomorphic.
Indeed,

$$a \mapsto \theta(a), \quad \lambda_g^\beta \mapsto u_g \lambda_g^\alpha$$

gives the isomorphism.

Overview

① Conjecture and partial answers

② Equivariant version of Nakamura's theorem

③ Uniqueness of outer G-actions on $\mathcal{O}_\infty$

④ Absorption of outer G-actions on $\mathcal{O}_\infty$

⑤ Stability

⑥ Classification

Conjecture (1/2)

In order to formulate our conjecture, we need to introduce principal $\mathrm{Aut}(A \otimes \mathbb{K})$-bundles associated with actions of G.

Let $\alpha : G \curvearrowright A$ be an action of a poly-$\mathbb{Z}$ group G on a unital Kirchberg algebra A. Let $\lambda : G \curvearrowright \ell^2 G$ be the left regular representation.
We denote by α^s the stabilization $\alpha \otimes \mathrm{Ad}\,\lambda : G \curvearrowright A \otimes \mathbb{K}(\ell^2 G)$.

Let BG be the classifying space and let EG be the total space. We can define the principal $\mathrm{Aut}(A \otimes \mathbb{K})$-bundle $\mathcal{P}_\alpha^s$ over BG as the quotient space of $EG \times \mathrm{Aut}(A \otimes \mathbb{K})$ by the equivalence relation

$$(g.x, \gamma) \sim (x, \alpha_g^s \circ \gamma)$$

for $x \in EG$, $g \in G$ and $\gamma \in \mathrm{Aut}(A \otimes \mathbb{K})$.

Conjecture (2/2)

Conjecture (Izumi 2010)

Let A be a unital Kirchberg algebra and let G be a poly-$\mathbb{Z}$ group.
Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright A$ be outer actions.
The following are equivalent.

1. α and β are KK-trivially cocycle conjugate.
2. There exists a base point preserving isomorphism between $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$.

The second condition is equivalent to the following:
there exists a continuous map $\Phi : EG \to \mathrm{Aut}(A \otimes \mathbb{K})_0$
such that $\Phi(x_0) = \mathrm{id}$ and $\Phi(g.x) \circ \alpha^s_g = \beta^s_g \circ \Phi(x)$.

Partial answers (1/2)

When $G = \mathbb{Z}$, its classifying space $B\mathbb{Z}$ is $\mathbb{T}$ and its total space $E\mathbb{Z}$ is $\mathbb{R}$. So, the conjecture says that $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(A)$ are KK-trivially cocycle conjugate if and only if

$$\exists \Phi : [0,1] \to \mathrm{Aut}(A \otimes \mathbb{K})_0, \quad \Phi(0) = \mathrm{id},\ \Phi(1) \circ \alpha^s = \beta^s \circ \Phi(0),$$

i.e. α^s is homotopic to β^s.
This is known to be equivalent to $KK(\alpha) = KK(\beta)$.

Theorem (Nakamura 2000)

The conjecture is true for $G = \mathbb{Z}$.

Partial answers (2/2)

Theorem (Izumi-M)

Let G be a poly-$\mathbb{Z}$ group. If A is either $\mathcal{O}_2$, $\mathcal{O}_\infty$ or $\mathcal{O}_\infty \otimes B$ with B being a UHF algebra of infinite type, there exists a unique cocycle conjugacy class of outer G-actions on A.

Theorem (Izumi-M)

Let A be a unital Kirchberg algebra and let G be a poly-$\mathbb{Z}$ group. All asymptotically representable outer actions of G on A are mutually KK-trivially cocycle conjugate.

Note that $\mathcal{P}^s_\alpha$ is trivial if α is asymptotically representable.

Theorem (Izumi-M)

When G is a poly-$\mathbb{Z}$ group with Hirsch length ≤ 3, the conjecture is true.

Obstruction theory

Let $\alpha, \beta : G \curvearrowright A$ be outer actions of a poly-$\mathbb{Z}$ group G.
How can we know whether $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$ are isomorphic or not?

The classical obstruction theory says (at least in principle) that we can determine it by computing relevant cohomology classes in

$$H^n(G, \pi_{n-1}(\mathrm{Aut}(A \otimes \mathbb{K})_0)), \quad 1 \leq n \leq \dim BG.$$

For Kirchberg algebras A, the homotopy groups $\pi_n(\mathrm{Aut}(A \otimes \mathbb{K}))$ are computed by Dadarlat 2007:

$$\pi_0(\mathrm{Aut}(A \otimes \mathbb{K})) \cong KK(A, A)^{-1},$$

$$\pi_n(\mathrm{Aut}(A \otimes \mathbb{K})) \cong \begin{cases} KK(A, A) & n \text{ is even} \\ KK(A, SA) & n \text{ is odd,} \end{cases} \qquad n \geq 1.$$

Central sequence algebra

Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

$$I_\omega = \{(a_n)_n \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|a_n\| = 0\}$$

is an ideal of $\ell^\infty(\mathbb{N}, A)$. We define the limit algebra A^ω and the central sequence algebra A_ω by

$$A^\omega = \ell^\infty(\mathbb{N}, A)/I_\omega \quad \text{and} \quad A_\omega = A^\omega \cap A'.$$

We also need the continuous version, namely

$$A^\flat = C_b([0,\infty), A)/C_0([0,\infty), A),$$

$$A_\flat = A^\flat \cap A'.$$

A group action $\alpha : G \curvearrowright A$ naturally extends to A^ω, A_ω, $A^\flat$ and $A_\flat$, for which we use the same symbol α.

Asymptotical representability

A group action $\alpha : G \curvearrowright A$ is said to be asymptotically representable if there exists a family of continuous maps $u_g : [0, \infty) \to U(A)$ ($g \in G$) such that

$$u_g(t)u_h(t) - u_{gh}(t) \to 0, \quad u_g(t)\ a\ u_g(t)^* \to \alpha_g(a)$$

$$\alpha_g(u_h(t)) - u_{ghg^{-1}}(t) \to 0.$$

In other words, $(u_g)_g \subset U(A^\flat)$ forms a unitary representation satisfying $u_g a u_g^* = \alpha_g(a)$ and $\alpha_g(u_h) = u_{ghg^{-1}}$ in $A^\flat$.
It is easy to see that $a \mapsto a$, $\lambda_g^\alpha \mapsto u_g$ give rise to a unital homomorphism $A \rtimes_\alpha G \to A^\flat$.
When G is a poly-$\mathbb{Z}$ group and A is a unital Kirchberg algebra, we will prove that asymptotically representable, outer actions of G on A are unique up to KK-trivial cocycle conjugacy.

Approximate representability is also defined in a similar way.

Equivariant Nakamura's theorem (1/3)

Our first goal is to show the following.

- Outer actions $\mu : G \curvearrowright \mathcal{O}_\infty$ are unique up to cocycle conjugacy.
- Any outer action $\alpha : G \curvearrowright A$ is cocycle conjugate to $\alpha \otimes \mu : G \curvearrowright A \otimes \mathcal{O}_\infty$.

To this end, we discuss an equivariant version of Nakamura's theorem.

Let G be a group and let $N \lhd G$ be a normal subgroup with $G/N \cong \mathbb{Z}$. Suppose that G is generated by N and $\xi \in G$.
Let $\alpha : G \curvearrowright A$ be an action of G on a unital C^*-algebra.
The automorphism $\alpha_\xi \in \mathrm{Aut}(A)$ extends to $\tilde{\alpha}_\xi \in \mathrm{Aut}(A \rtimes_\alpha N)$ by

$$\tilde{\alpha}_\xi(x) = \alpha_\xi(x) \quad \forall x \in A \quad \text{and} \quad \tilde{\alpha}_\xi(\lambda^\alpha_g) = \lambda^\alpha_{\xi g \xi^{-1}} \quad \forall g \in N.$$

Equivariant Nakamura's theorem (2/3)

Our setting is as follows:

- $G \cong N \rtimes \mathbb{Z}$ is a group generated by $N \lhd G$ and $\xi \in G$.
- Two outer actions $\alpha, \beta : G \curvearrowright A$ on a unital Kirchberg algebra A are given, and assume that $\alpha|N$ and $\beta|N$ are asymptotically representable.
- $\beta|N$ is a cocycle perturbation of $\alpha|N$. Thus, there exists an α-cocycle $(u_g)_{g\in N}$ such that $\beta_g = \mathrm{Ad}\, u_g \circ \alpha_g$ for $g \in N$.

We like to compare $\tilde{\alpha}_\xi \in \mathrm{Aut}(A \rtimes_\alpha N)$ and $\tilde{\beta}_\xi \in \mathrm{Aut}(A \rtimes_\beta N)$. Define an isomorphism $\theta : A \rtimes_\beta N \to A \rtimes_\alpha N$ by $\theta(a) = a$ for $a \in A$ and $\theta(\lambda_g^\beta) = u_g \lambda_g^\alpha$ for $g \in N$.

Theorem (Equivariant Nakamura's theorem)

If $KK(\tilde{\alpha}_\xi) = KK(\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})$, then $\alpha : G \curvearrowright A$ is KK-trivially cocycle conjugate to $\beta : G \curvearrowright A$.

Equivariant Nakamura's theorem (3/3)

Theorem (Equivariant Nakamura's theorem)

If $KK(\tilde{\alpha}_\xi) = KK(\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})$, then $\alpha : G \curvearrowright A$ is KK-trivially cocycle conjugate to $\beta : G \curvearrowright A$.

In the case that N is trivial (i.e. $G = \mathbb{Z}$), the theorem above becomes

$$KK(\alpha_\xi) = KK(\beta_\xi) \quad \Longrightarrow \quad \alpha_\xi \sim \beta_\xi \quad (KK\text{-trivial cc}),$$

which is the original version of Nakamura's theorem.

The proof of this theorem consists of the following three parts.

- $\exists u : [0, \infty) \to U(A)$ s.t. $\lim_{t\to\infty} \operatorname{Ad} u(t) \circ \tilde{\alpha}_\xi = \theta \circ \tilde{\beta}_\xi \circ \theta^{-1}$
- α_ξ has 'Rohlin projections' in $(A_\omega)^{\alpha|N}$ (and the same for β_ξ)
- Evans-Kishimoto intertwining argument

Proof of Equiv. Nakamura's thm (1/3)

First, we want to find $u : [0, \infty) \to U(A)$ such that $\lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \tilde{\alpha}_\xi = \theta \circ \tilde{\beta}_\xi \circ \theta^{-1}$.

By $KK(\tilde{\alpha}_\xi) = KK(\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})$, there exists a continuous path of unitaries $v : [0, \infty) \to U(A \rtimes_\alpha N)$ such that

$$\lim_{t\to\infty} \mathrm{Ad}\, v(t) \circ \tilde{\alpha}_\xi = \theta \circ \tilde{\beta}_\xi \circ \theta^{-1}.$$

Now, $\alpha|N : N \curvearrowright A$ is asymptotically representable,
and so there exists a unital homomorphism $\pi : A \rtimes_\alpha N \to A^\flat$
such that $\pi(a) = a$ for all $a \in A$.
By 'sending' v by π to A,
we obtain the desired $u : [0, \infty) \to U(A)$.

Proof of Equiv. Nakamura's thm (2/3)

Next, we want to find 'Rohlin projections' for α_ξ in $(A_\omega)^{\alpha|N}$.

Since $\alpha : G \curvearrowright A$ is outer, we can show that the automorphisms $\alpha_\xi^n \in \mathrm{Aut}((A_\omega)^{\alpha|N})$ are outer for every $n \in \mathbb{N}$.

Furthermore, by using that $\alpha|N : N \curvearrowright A$ is approximately representable, we can show that $(A_\omega)^{\alpha|N}$ is purely infinite simple.

Then, we can modify Nakamura's argument and obtain the following: for any $m \in \mathbb{N}$, there exist projections $e_0, e_1, \ldots, e_{m-1}$, $f_0, f_1, \ldots, f_m$ in $(A_\omega)^{\alpha|N}$ such that

$$\sum_{i=0}^{m-1} e_i + \sum_{j=0}^{m} f_j = 1,$$

$$\alpha_\xi(e_i) = e_{i+1}, \quad \alpha_\xi(f_j) = f_{j+1}.$$

Proof of Equiv. Nakamura's thm (3/3)

We have obtained

- $u : [0, \infty) \to U(A)$ s.t. $\lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \tilde{\alpha}_\xi = \theta \circ \tilde{\beta}_\xi \circ \theta^{-1}$,
- Rohlin projections for α_ξ in $(A_\omega)^{\alpha|N}$. (the same for β_ξ)

Then, we can apply Evans-Kishimoto intertwining argument for the automorphisms $\tilde{\alpha}_\xi$ and $\theta \circ \tilde{\beta}_\xi \circ \theta^{-1}$ of $A \rtimes_\alpha N$.

The essence of this argument is described as follows:
We have $u \in U(A^\flat)$ such that

$$(\mathrm{Ad}\, u \circ \tilde{\alpha}_\xi)(a) = (\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})(a) \quad \forall a \in A.$$

Using the Rohlin property of $\tilde{\alpha}_\xi$ (and $\tilde{\beta}_\xi$), we can construct $v \in U(A^\flat)$ such that $\|u - v\tilde{\alpha}_\xi(v^*)\| \approx 0$. Then

$$(\mathrm{Ad}\, v \circ \tilde{\alpha}_\xi \circ \mathrm{Ad}\, v^*)(a) \approx (\theta \circ \tilde{\beta}_\xi \circ \theta^{-1})(a) \quad \forall a \in A,$$

which shows $\tilde{\alpha}_\xi$ and $\tilde{\beta}_\xi$ are 'conjugate'.

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