Operator Algebras and Conformal Field Theory

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Boundary CFT on the half-plane $x > 0$ (H.K. Rehren, R.L.)

Stress-energy tensor left/right movers $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$: $T_L = T_L(t+x)$, $T_R = T_R(t-x)$.

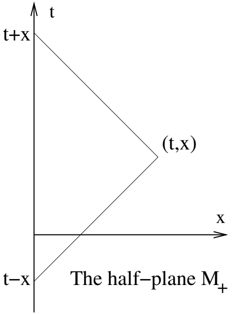
Boundary condition: no energy flow across the boundary:

$$T_{01}(t, x = 0) = 0 \qquad \Leftrightarrow \qquad T_L = T_R \equiv T.$$

so $T_{10} = T_{01}$, $T_{11} = T_{00}$ are of the form

$$T_{00}(t,x) = T(t+x) + T(t-x), \qquad T_{01}(t,x) = T(t+x) - T(t-x),$$

i.e., *bi-local* expressions in terms of T

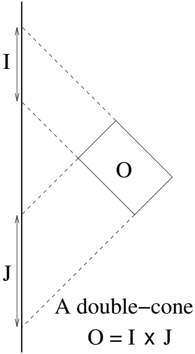


The chiral fields of a boundary CFT generate a net

$$O \mapsto A_+(O).$$

$A_+(O)$ is generated by chiral fields smeared in the variable $t+x$ over the interval I and in the variable $t-x$ over the interval J, where $O = I \times J$, $I > J$, is an open double-cone in M_+. So

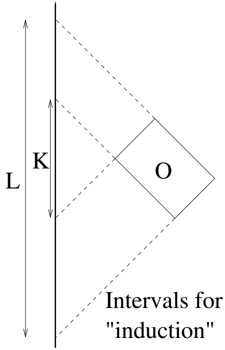
$$A_+(O) = A(I) \vee A(J) \qquad (O = I \times J, \ \ I > J).$$



chiral extension $\to$ boundary condition

If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to A), then the *induced net* is defined by

$$O \mapsto B_+^{ind}(O) := B(L) \cap B(K)'.$$

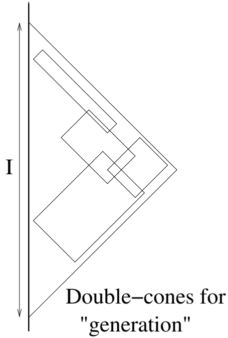


BCFT $\to$ non-local chiral net

A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{gen}(I)$ (the associated *boundary net*)

$$B^{gen}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L)$$

$W_L =$ left wedge spanned by I



If B is a chiral extension of A, then

$$(B_+^{ind})^{gen} = B$$

Conversely

$$(B_+^{gen})^{ind} = B_+^{dual}$$

where $B^{dual}(O) \equiv B(O')'$. Conclusion:

non-local chiral extensions of $A \leftrightarrow$ local extensions of A_+

Classification of non-local extensions

All irreducible (non-local) extensions of nets Vir_c, $c < 1$, are classified (Kawahigashi, Penning, Rehren, L.)

$$\Downarrow$$

All conformal (local) Boundary CFT with $c < 1$ are classified

Classification by modular invariants and α-induction (D. Evans, Y. Kawahigashi,...).

The semigroup $\mathcal{E}(\mathcal{A})$(E. Witten, R.L.)

Let $\mathcal{A}$ be a local Möbius covariant net of von Neumann algebras on $\mathbb{R}$

$$I \subset \mathbb{R} \text{ interval} \to \mathcal{A}(I)$$

T one-parameter unitary translation group. Then $T(t)\mathcal{A}(I)T(-t) = \mathcal{A}(I+t)$, T has positive generator P and $T(t)\Omega = \Omega$ where Ω is the vacuum vactor.

Let V be a unitary on $\mathcal{H}$ commuting with T. The following are equivalent:

(i) $V\mathcal{A}(I_2)V^*$ commutes with $\mathcal{A}(I_1)$ for all intervals I_1, I_2 of $\mathbb{R}$ such that $I_2 > I_1$ (I_2 is contained in the future of I_1).

(ii) $V\mathcal{A}(a,\infty)V^* \subset \mathcal{A}(a,\infty)$ for every $a \in \mathbb{R}$.

(iii) $V\mathcal{A}(0,\infty)V^* \subset \mathcal{A}(0,\infty)$.

Boundary QFT models associated with semigroup elements

$\mathcal{A}$ local net on $\mathbb{R}$, $V \in \mathcal{E}(\mathcal{A})$ give a local, translation covariant QFT net $\mathcal{A}_V$ on $\mathbb{R}$

$$\mathcal{A}_V(O) = \mathcal{A}(I) \vee \mathcal{A}_V(J)$$

$O = I \times J$

Problem: find non trivial elements in the semigroup.

Inner functions

$\mathbb{S}_\pi \equiv \{z \in \mathbb{C} : 0 < \Im z < 1\}$ strip, $\mathbb{H}^\infty(\mathbb{S}_\pi)$ Hardy space.

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\pi) \Rightarrow$
$\exists\ \varphi(t) \equiv \lim_{s\to 1^+} \varphi(t+is)$ a.e. on $\mathbb{R}$
$\exists\ \varphi(t+i\pi) \equiv \lim_{s\to \pi^-} \varphi(t+is)$ a.e. on $\mathbb{R}$

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\pi)$ is an *inner function* if $|\varphi(z)| = 1$ for almost all $z \in \partial\mathbb{S}_\pi$.

Inner functions on the circle $\mathbb{D}$:

$$\varphi(z) = \alpha B(z) \exp\left(- \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \mathrm{d}\mu(e^{i\theta}) \right),$$

μ is a positive, Lebesgue singular measure on $\partial\mathbb{D}$, α is a constant with $|\alpha| = 1$, $B(z)$ is a Blaschke product: $B(z) \equiv \prod_{n=1}^\infty B_{a_n}(z)$, $a_n \in \mathbb{D}$, $\sum_{n=1}^\infty (1-|a_n|) < \infty$, $B_a(z) = \frac{|a|}{a}\frac{z-a}{1-\bar{a}z}$ (Blaschke factor).

Formula, notions go to $\mathbb{S}_\pi$,$\mathbb{S}_\infty$ by conformal identification:
$h(z) \equiv i\frac{1+z}{1-z}$,

$$\mathbb{D} \xrightarrow{h} \mathbb{S}_\infty \xrightarrow{\log} \mathbb{S}_\pi$$

Symmetric inner functions:

$\varphi \in \mathbb{H}^\infty(\mathbb{S}_\pi)$ inner is symmetric if $\varphi(q + i\pi) = \bar{\varphi}(q)$, $q \in \mathbb{R}$ a.e.

Scattering functions:

A scattering function is a symmetric inner function f on $\mathbb{S}_\pi$ s.t. $\varphi(-p) = \varphi(p)$.

Problem: construct QFT models from scattering function (cf. Lechner models)

Beurling-Lax theorem (1949-1959)

S shift operator on $H^2(\mathbb{D})$:

$$Sf(z) = zf(z)$$

A *closed S-invariant subspace* K of $H^2(\mathbb{D})$ has the form

$$K = \varphi H^2(\mathbb{D}), \quad \varphi \textit{ an inner function}$$

This implies: $f \in H^2$ (or $f \in H^p, p \geq 1$) has a factorization:

$$f(z) = \varphi(z)\psi(z)$$

φ is inner and ψ is outer $\psi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| \mathrm{d}t\right)$

Lax generalization to $H^2(\mathbb{S}_\infty)$, one-param. unitary translations in Fourier transform.

Real subspaces and inner functions (preliminaries)

$\mathcal{H}$ be a (complex) Hilbert space, H_1 a real Hilbert subspace of $\mathcal{H}$
K a selfadjoint operator on $\mathcal{H}$.
Suppose that

$$e^{itK}H_1 \subset H_1, \quad \forall t \geq 0 \ .$$

For f and its Fourier transform φ:

$$\varphi(K) = \int_{-\infty}^{\infty} f(t)e^{itK}dt \ .$$

Then, if $\mathrm{supp}(f) \subset \mathbb{R}^+$, we have $\varphi(K)H_1 \subset H_1$.
K has Lebesgue spectrum, taking limits $\to$ we have

$$\varphi(K)H_1 \subset H_1 \ , \quad \forall \varphi \in H^\infty(\mathbb{S}_\infty), \ \varphi \text{ symmetric,}$$

$\to$ every symmetric inner function φ on $\mathbb{S}_\infty$ gives a unitary
$V = \varphi(K)$ such that $VH_1 \subset H_1$.

Endomorphisms of standard subspaces

A *standard pair* of $\mathcal{H}$ is a pair (H, T) such that

- H is a standard subspace,
- T is a one-par. unitary group, with positive generator P, s.t. $T(t)H \subset H$, $t \geq 0$.

Thm. Assume (H, T) to be irreducible and let V be a unitary on $\mathcal{H}$. The following are equivalent:

(*i*) $VH \subset H$ and V commutes with T,

(*ii*) $V = \psi(Q)$ with $Q \equiv \log P$ and ψ is the boundary value of a symmetric inner function in $H^\infty(\mathbb{S}_\pi)$.

The semigroup $\mathcal{E}(H)$ of endomorphisms of (H, T) is isomorphic to the semigroup of symmetric inner functions on the strip $0 < \Im z < \pi$.

Note: Compare with the Beurling-Lax theorem.

Constructing models (E. Witten, R.L.)

$\mathcal{A}$ free field on $\mathbb{R}$ acting on the Fock space $F(\mathcal{H})$.

H standard subspace of $\mathcal{H} \to$ von Neumann algebra on $F(\mathcal{H})$

$$\mathcal{A}(H) = \{W(h) : h \in H\}''$$

Take $H = H(0, \infty)$.

$$V \in \mathcal{E}(H) \to \Gamma(V) \in \mathcal{E}(\mathcal{A})$$

therefore

symmetric inner function $\to V \in \mathcal{E}(\mathcal{A}) \to$ Boundary QFT net $\mathcal{A}_V$ on M_+

In particular

$$\varphi \text{ scattering function} \to \text{Boundary QFT}$$

More general BQFT's

$\mathcal{A} = \mathcal{A}_N$ Buchholz-Mach-Tododorv extension of $U(1)$-current net:

symmetric inner function Hölder continuous at 0 & $V \in \mathcal{E}(\mathcal{A})$

$$\downarrow$$

Boundary QFT net $\mathcal{A}_V$ on M_+

More models: Bischoff, Lechner, Tanimoto

Problem: Non-trivial elements of $\mathcal{E}(\mathcal{A})$ for loop group, Virasoro models, etc.

Phase boundaries, (Bischoff, Kawahigashi, Rehren, L.

$M_L \equiv \{(t,x) : x < 0\}$, $M_R \equiv \{(t,x) : x > 0\}$ left and right half Minkowski plane.

A (transmissive) *phase boundary* is given by specifying two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on M, covariantly represented on the same Hilbert space $\mathcal{H}$; $\mathcal{B}^L$ and $\mathcal{B}^R$ both contain a common chiral subnet $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. Initially $\mathcal{B}^{L/R}$ is defined on $M_{L/R}$

$$M_L \supset O \mapsto \mathcal{B}^L(O) \; ; \qquad M_R \supset O \mapsto \mathcal{B}^R(O) \; ,$$

yet $\mathcal{B}^{L/R}$ extends on the entire M by covariance. Indeed, the chiral nets $\mathcal{A}_\pm$ on $\mathbb{R}$ contain the unitaries implementing the local diffeomorphisms, and hence both nets $\mathcal{B}^L$ and $\mathcal{B}^R$ share the same unitary representation of the symmetry group $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$.

Causality requires that the algebras $\mathcal{B}^L(O_1)$ and $\mathcal{B}^R(O_2)$ commute whenever $O_1 \subset M_L$ and $O_2 \subset M_R$ are spacelike separated. By diffeomorphism covariance, $\mathcal{B}^R$ is thus right local with respect to $\mathcal{B}^L$, i.e. if O_1 is spacelike to O_2 *and* O_2 is to the left of O_R, then we have $[\mathcal{B}^L(O_2), \mathcal{B}^R(O_1)] = 0$.

Given a phase boundary, we consider the von Neumann algebras generated by $\mathcal{B}^L(O)$ and $\mathcal{B}^R(O)$:

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O) \; , \quad O \in \mathcal{K} \; .$$

$\mathcal{D}$ is another extension of $\mathcal{A}$, but $\mathcal{D}$ is in general non-local, but relatively local w.r.t. $\mathcal{A}$. $\mathcal{D}(O)$ may have non-trivial center. In the completely rational case, $\mathcal{A}(O) \subset \mathcal{D}(O)$ has finite Jones index, so the center of $\mathcal{D}(O)$ is finite dimensional; by standard arguments, we may cut down the center to $\mathbb{C}$ by a minimal projection of the center, and we may then assume $\mathcal{D}(O)$ to be a factor, as we will do for simplicity in the following.

The universal construction

A phase boundary is a transmissive boundary with chiral observables $\mathcal{A}_{2\mathrm{D}} = \mathcal{A}_+ \otimes \mathcal{A}_-$. The phases on both sides of the boundary are given by a pair of Q-systems $A^L = (\Theta^L, W^L, X^L)$ and $A^R = (\Theta^R, W^R, X^R)$ in the sectors of $\mathcal{A}_{2\mathrm{D}}$, describing local 2D extensions $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^L_{2\mathrm{D}}$ and $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^R_{2\mathrm{D}}$.
Now consider the braided product Q-systems (Evans, Pinto)

$$(\Theta = \Theta^L \circ \Theta^R, W = W^L \times W^R, X = (1 \times \epsilon^{\pm}_{\Theta^L,\Theta^R} \times 1) \circ (X^L \times X^R))$$

and the corresponding extensions $\mathcal{A}_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}}$. The original extensions $\mathcal{B}^L_{2\mathrm{D}}$, $\mathcal{B}^R_{2\mathrm{D}}$ are intermediate

$$\mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^L_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}} \qquad \mathcal{A}_{2\mathrm{D}} \subset \mathcal{B}^R_{2\mathrm{D}} \subset \mathcal{D}^{\pm}_{2\mathrm{D}},$$

and the nets $\mathcal{D}^{\pm}_{2\mathrm{D}}$ are generated by $\mathcal{A}_{2\mathrm{D}}$ and two sets of charged fields $\Psi^L_{\sigma\otimes\tau}$ ($\sigma \otimes \tau \prec \Theta^L$) and $\Psi^R_{\sigma\otimes\tau}$ ($\sigma \otimes \tau \prec \Theta^R$), suppressing possible multiplicity indices.

The braided product Q-system determines their commutation relations among each other:

$$\Psi^R_{\sigma\otimes\tau}\Psi^L_{\sigma'\otimes\tau'} = \epsilon^{\pm}_{\sigma'\otimes\tau',\sigma\otimes\tau} \cdot \Psi^L_{\sigma'\otimes\tau'}\Psi^R_{\sigma\otimes\tau}.$$

$\epsilon^{-}_{\sigma'\otimes\tau',\sigma\otimes\tau} = \mathbf{1}$ whenever $\sigma' \otimes \tau'$ is localized to the spacelike left of $\sigma \otimes \tau$. Thus, the choice of $\pm$-braiding ensures that $\mathcal{B}^L$ is left-local w.r.t. $\mathcal{B}^R$, as required by causality. Thus

$$\Theta = (\Theta^L, W^L, X^L) \times^- (\Theta^R, W^R, X^R),$$

Universal construction:

The extension $\mathcal{D}$ of $\mathcal{A}$ defined by the above Q-system implements a transmissive boundary condition in the sense. It is universal in the sense that every irreducible boundary condition appears as a representation of $\mathcal{D}$.

Cf. the work of Fröhlich, Fuchs, Runkel, Schweigert (Euclidean setting)

Kac-Wakimoto formula

Kac-Wakimoto formula (conjecture)

Let $\mathcal{A}$ be a conformal net, ρ representations of $\mathcal{A}$, then

$$\lim_{t\to 0^+} \frac{\mathrm{Tr}(e^{-tL_{0,\rho}})}{\mathrm{Tr}(e^{-tL_0})} = d(\rho)$$

Analog of the Kac-Wakimoto formula (theorem)

ρ a representation of $\mathcal{A}$:

$$(\xi, e^{-2\pi K_\rho}\xi) = d(\rho)$$

where $K\rho$ is the generator of the dilations δ_I and ξ is any vector cyclic for $\rho(\mathcal{A}(I'))$ such that $(\xi, \rho(\cdot)\xi)$ is the vacuum state on $\mathcal{A}(I')$.

A classification of KMS states (Camassa, Tanimoto, Weiner, L.)

How many KMS states do there exist?

Completely rational case

Thm $\mathcal{A}$ completely rational: only one KMS state (geometrically constructed) $\beta = 2\pi$
exp: net on $\mathbb{R}$ $\mathcal{A} \to$ restriction of $\mathcal{A}$ to $\mathbb{R}^+$

$$\text{exp} \restriction \mathcal{A}(I) = \mathrm{Ad}U(\eta)$$

η diffeomorphism, $\eta \restriction I =$ exponential

$$\text{geometric KMS state on } \mathcal{A}(\mathbb{R}) = \text{vacuum state on } \mathcal{A}(\mathbb{R}^+) \circ \text{exp}$$

$$\varphi_{\mathrm{geo}} = \omega \circ \text{exp}$$

Scaling with dilation, we get the geometric KMS state at any give $\beta > 0$.

Comments

About the proof:

Essential use of the *thermal completion* and *Jones index*.

$\mathcal{A}$ net on $\mathbb{R}$, φ KMS state:

In the GNS representation we apply Wiesbrock theorem

$$\mathcal{A}(\mathbb{R}^+) \subset \mathcal{A}(\mathbb{R}) \text{ hsm modular inclusion} \to \text{new net } \mathcal{A}_\varphi$$

Want to prove duality for $\mathcal{A}_\varphi$ in the KMS state, but $\mathcal{A}_\varphi$ satisfies duality up to finite Jones index.

Iteration of the procedure...

Conjecture: $\mathcal{A} \subset \mathcal{B}$ finite-index inclusion of conformal nets, $\varepsilon : \mathcal{B} \to \mathcal{A}$ conditional expectation. If φ is a translation KMS on $\mathcal{A}$ then $\varphi \circ \varepsilon$ is a translation KMS on $\mathcal{B}$.

Non-rational case: $U(1)$-current model

The primary (locally normal) KMS states of the $U(1)$-current net are in one-to-one correspondence with real numbers $q \in \mathbb{R}$; each state φ^q is uniquely determined by

$$\varphi^q(W(f)) = e^{iq\int f\,dx} \cdot e^{-\frac{1}{4}\|f\|^2_{S_\beta}}$$

where $\|f\|^2_{S_\beta} = (f, S_\beta f)$ and $\widehat{S_\beta f}(p) := \coth\frac{\beta p}{2}\widehat{f}(p)$.

Geometric KMS state: $\varphi_{\mathrm{geo}} = \varphi^0$

Any other primary KMS state

$$\varphi^q = \varphi_{\mathrm{geo}} \circ \gamma_q.$$

where

$$\gamma_q(W(f)) = e^{iq\int_{\mathbb{R}} f dx} W(f).$$

Virasoro net: $c = 1$

(With $c < 1$ there is only one KMS state: the net is completely rational)

Primary KMS states of the Vir_1 net are in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ is uniquely determined by its value on the stress-energy tensor T:

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right)\int f\,dx.$$

The geometric KMS state corresponds to $q = 0$, and the corresponding value of the 'energy density' $\frac{\pi}{12\beta^2} + \frac{q^2}{2}$ is the lowest in the set of the KMS states.

(We construct these KMS states by composing the geometric state with automorphisms on the larger $U(1)$-current net.)

Virasoro net: $c > 1$

There is a set of primary (locally normal) KMS states of the Vir_c net with $c > 1$ w.r.t. translations in one-to-one correspondence with positive real numbers $|q| \in \mathbb{R}^+$; each state $\varphi^{|q|}$ can be evaluated on the stress-energy tensor

$$\varphi^{|q|}\left(T\left(f\right)\right) = \left(\frac{\pi}{12\beta^2} + \frac{q^2}{2}\right) \int f \, dx$$

and the geometric KMS state corresponds to $q = \frac{1}{\beta}\sqrt{\frac{\pi(c-1)}{6}}$ and energy density $\frac{\pi c}{12\beta^2}$.

Are they all? Probably yes...

Rotation KMS states: Tanimoto's talk

Non-equilibrium thermodynamics

The purpose of non-equilibrium thermodynamics is to study physical systems that are not in thermodynamic equilibrium but can be basically described by thermal equilibrium variables. It thus deals with systems that are in some sense near equilibrium.
Although the research on non-equilibrium thermodynamics has been effectively pursued for decades with important achievements, the general theory still missing. The framework is even more incomplete in the quantum case, non-equilibrium quantum statistical mechanics.
Non-equilibrium thermodynamics deals with inhomogeneous systems. A typical model system is given by two infinite reservoirs, initially in equilibrium at different temperatures and different chemical potentials, set in contact at the boundary with an energy flux from one reservoir to the other; possibly the global system may incorporate a probe between the two reservoirs.

Non-equilibrium steady states

KMS states:

$\mathfrak{A}$ a C^*-algebra, τ a one-parameter group of automorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ a dense *-subalgebra of $\mathfrak{A}$. A state ω of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if

(a) $F_{XY}(t) = \omega\big(X\tau_t(Y)\big)$,
(b) $F_{XY}(t + i\beta) = \omega\big(\tau_t(Y)X\big)$,

where $A(S_\beta)$ is the algebra of functions analytic in the strip $S_\beta = \{0 < \Im z < \beta\}$, bounded and continuous on the closure $\bar{S}_\beta$.

Non-equilibrium statistical mechanics:

A *non-equilibrium steady state* **NESS** ω of $\mathfrak{A}$ satisfies property (a) in the KMS condition, for all X, Y in a dense *-subalgebra of $\mathfrak{B}$, but not necessarily property (b). (Ruelle)

E.g. tensor product of KMS states at different temperatures.

NESS in CFT. (S. Hollands, R.L.)

Let us consider two local conformal nets $\mathcal{B}^L$ and $\mathcal{B}^R$ on the Minkowski plane M and both containing the same chiral net $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. For the moment $\mathcal{B}^{L/R}$ is completely rational, and we use the uniqueness of the geometric KMS state later we get on the case where chemical potentials are present.

Before contact. The two systems $\mathcal{B}^L$ and $\mathcal{B}^R$ are, separately, each in a thermal equilibrium state. KMS states $\varphi^{L/R}_{\beta_{L/R}}$ on $\mathfrak{B}^{L/R}$ at inverse temperature $\beta_{L/R}$ w.r.t. τ, possibly with $\beta_L \neq \beta_R$.
The two systems $\mathcal{B}^L$ and $\mathcal{B}^R$ live independently in their own half plane M_L and M_R and their own Hilbert space. The composite system is described by the net on $M_L \cup M_R$ given by

$$M_L \supset O \mapsto \mathcal{B}^L(O)\,, \qquad M_R \supset O \mapsto \mathcal{B}^R(O)\,.$$

The C^*-algebra of the composite system is $\mathfrak{B}^L(M_L) \otimes \mathfrak{B}^R(M_R)$ and the state of the system is

$$\varphi = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(M_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(M_R)}\ ;$$

φ is a stationary state, NESS but not KMS.

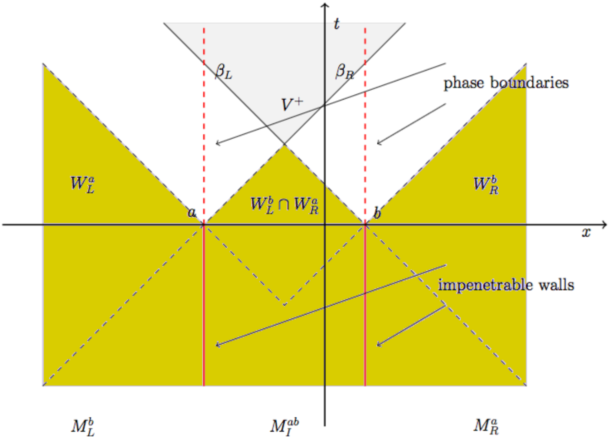


Figure 3: Spacetime diagram of our setup in the special case $t_1 = t_2 = 0$. The initial state ψ is set up in the shaded region before the system is in causal contact with the phase boundaries. In W_L^a resp. W_R^b, we have a thermal equilibrium state at inverse temperatures β_L resp. β_R. In the diamond shaped shaded region $W_L^b \cap W_R^a$, we have an essentially arbitrary probe state. Again, every time-translated causal diamond will eventually enter V^+.

After contact.

At time $t = 0$ we put the two systems $\mathcal{B}^L$ on M_L and $\mathcal{B}^R$ on M_R in contact through a totally transmissible phase boundary and the time-axis the defect line. We are in the phase boundary case, with $\mathcal{B}^L$ and $\mathcal{B}^R$ now nets on M acting on a common Hilbert space $\mathcal{H}$. With $O_1 \subset M_L$, $O_2 \subset M_R$ double cones, the von Neumann algebras $\mathcal{B}^L(O_1)$ and $\mathcal{B}^R(O_2)$ commute if O_1 and O_2 are spacelike separated, so $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ commute.
We want to describe the state ψ of the global system after time $t = 0$. As above, we set

$$\mathcal{D}(O) \equiv \mathcal{B}^L(O) \vee \mathcal{B}^R(O)$$

The origin $\mathbf{0}$ is the only $t = 0$ point of the defect line; the observables localized in the causal complement $W_L \cup W_R$ of the $\mathbf{0}$ thus do not feel the effect of the contact, so ψ should be a natural state on $\mathfrak{D}$ that satisfies

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)}, \quad \psi|_{\mathfrak{B}^R(W_R)} = \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)} \ .$$

In particular, ψ is to be a *local thermal equilibrium state* on $W_{L/R}$ in the sense of Buchholz.

Since $\mathfrak{B}^L(M_L)$ and $\mathfrak{B}^R(M_R)$ are not independent, the existence of such state ψ is not obvious. Clearly the C^*-algebra on $\mathcal{H}$ generated by $\mathfrak{B}^L(W_L)$ and $\mathfrak{B}^R(W_R)$ is naturally isomorphic to $\mathfrak{B}^L(W_L) \otimes \mathfrak{B}^R(W_R)$ ($\mathfrak{B}^L(W_L)''$ and $\mathfrak{B}^R(W_R)''$ are commuting factors) and the restriction of ψ to it is the product state $\varphi^L_{\beta_L}|_{\mathfrak{B}^L(W_L)} \otimes \varphi^R_{\beta_R}|_{\mathfrak{B}^R(W_R)}$.

Construction of the doubly scaling automorphism:
Let $\mathcal{C}$ be a conformal net on $\mathbb{R}$. Given $\lambda_-, \lambda_+ > 0$, there exists an automorphism α of the C^*-algebra $\mathfrak{C}(\mathbb{R} \smallsetminus \{0\})$ or $\mathfrak{D}(\check{M})$ such that

$$\alpha|_{\mathfrak{C}(-\infty,0)} = \delta_{\lambda_-} \ , \quad \alpha|_{\mathfrak{C}(0,\infty)} = \delta_{\lambda_+} \ ,$$

Then we construct an automorphism on the C^*-algebra $\mathfrak{D}(x \pm t \neq 0)$

$$\alpha|_{\mathfrak{D}(W_L)} = \delta_{\lambda_L} \ , \quad \alpha|_{\mathfrak{D}(W_R)} = \delta_{\lambda_R} \ .$$

where δ_λ is the λ-dilation automorphism of $\mathfrak{A}_\pm(\mathbb{R})$.

There exists a natural state $\psi \equiv \psi_{\beta_L,\beta_R}$ on $\mathfrak{D}(x \pm t \neq 0)$ such that $\psi|_{\mathfrak{B}(W_{L/R})}$ is $\varphi^{L/R}_{\beta_L/\beta_R}$.

The state ψ is given by $\psi \equiv \varphi \cdot \alpha_{\lambda_L,\lambda_R}$, where φ is the geometric state on $\mathfrak{D}$ (at inverse temperature 1) and $\alpha = \alpha_{\lambda_L,\lambda_R}$ is the above automorphism with $\lambda_L = \beta_L^{-1}$, $\lambda_R = \beta_R^{-1}$.

It is convenient to extend the state ψ to a state on $\mathfrak{D}$ by the Hahn-Banach theorem. *By inserting a probe ψ the state will be normal.*

The large time limit. Waiting a large time we expect the global system to reach a stationary state, a non equilibrium steady state. The two nets $\mathcal{B}^L$ and $\mathcal{B}^R$ both contain the same net $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$. And the chiral net $\mathcal{A}_\pm$ on $\mathbb{R}$ contains the Virasoro net with central charge $c_\pm$. In particular $\mathcal{B}^L$ and $\mathcal{B}^R$ share the same stress energy tensor.

Let $\varphi^+_{\beta_L}$, $\varphi^-_{\beta_R}$ be the geometric KMS states respectively on $\mathfrak{A}_+$ and $\mathfrak{A}_-$ with inverse temperature β_L and β_R; we define

$$\omega \equiv \varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R} \cdot \varepsilon \ ,$$

so ω is the state on $\mathfrak{D}$ obtained by extending $\varphi^+_{\beta_L} \otimes \varphi^-_{\beta_R}$ from $\mathfrak{A}$ to $\mathfrak{D}$ by the conditional natural expectation $\varepsilon : \mathfrak{D} \to \mathfrak{A}$. Clearly ω is a stationary state, indeed:

ω is a NESS on $\mathfrak{D}$ with $\beta = \min\{\beta_L, \beta_R\}$.

We now want to show that the evolution $\psi \cdot \tau_t$ of the initial state ψ of the composite system approaches the non-equilibrium steady state ω as $t \to +\infty$.

Note that:

$$\psi|_{\mathcal{D}(O)} = \omega|_{\mathcal{D}(O)} \text{ if } O \in \mathcal{K}(V_+)$$

Proposition

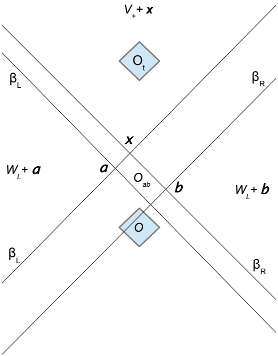
For every $Z \in \mathfrak{D}$ we have:

$$\lim_{t\to+\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \ .$$

Proof. Let $Z \in \mathcal{D}(O)$ with $O \in \mathcal{K}(M)$. If $t > t_O$, we have $\tau_t(Z) \in \mathfrak{D}(V_+)$ as said, so

$$\psi\big(\tau_t(Z)\big) = \omega\big(\tau_t(Z)\big) = \omega(Z) \ , \quad t > t_O \ ,$$

because of the stationarity property of ω. Therefore the limit holds true if Z belongs to the norm dense subalgebra of $\mathfrak{D}$ generated by the $\mathcal{D}(O)$'s, $O \in \mathcal{K}$, hence for all $Z \in \mathfrak{D}$ by the density approximation argument. $\square$



Spacetime diagram of our setup.

We now get back in the phase boundary framework, but we suppose here that $\mathcal{A}_\pm$ is the above net $\mathcal{C}$ generated by the $U(1)$-current $J^\pm$ (thus $\mathcal{B}^{L/R}$ is non rational with central charge $c = 1$).
Given $q \in \mathbb{R}$, the β-KMS state $\varphi_{\beta,q}$ on $\mathfrak{D}$ with charge q is defined by

$$\varphi_{\beta,q} = \varphi^+_{\beta,q} \otimes \varphi^-_{\beta,q} \cdot \varepsilon \ ,$$

where $\varphi^\pm_{\beta,q}$ denote the state characterized by the previous lemma and theorem on $\mathcal{A}_\pm$. $\varphi_{\beta,q}$ satisfies the β-KMS condition on $\mathfrak{D}$ w.r.t. the one-parameter automorphism group $t \mapsto \tau_t \cdot \alpha_t$, where τ is the time-translation one-parameter automorphism group of $\mathcal{D}$ and α a one-parameter subgroup of the gauge group of $\mathfrak{D}$.

Similarly as above we have:

Given $\beta_{L/R} > 0$, $q_{L/R} \in \mathbb{R}$, there exists a state ψ on $\mathfrak{D}$ such that

$$\psi|_{\mathfrak{B}^L(W_L)} = \varphi_{\beta_L, q_L}|_{\mathfrak{B}^L(W_L)} \ , \qquad \psi|_{\mathfrak{B}^R(W_R)} = \varphi_{\beta_R, q_R}|_{\mathfrak{B}^R(W_R)} \ .$$

and for every $Z \in \mathfrak{D}$ we have:

$$\lim_{t \to +\infty} \psi\big(\tau_t(Z)\big) = \omega(Z) \ .$$

We can explicitly compute the expected value of the asymptotic NESS state ω on the stress energy tensor and on the current (chemical potential enters):

Now $\omega = \varphi^+_{\beta_L,q_L} \otimes \varphi^-_{\beta_R,q_R} \cdot \varepsilon$ is a steady state is a NESS and ω is determined uniquely by $\beta_{L/R}$ and the charges $q_{L/R}$

$$\varphi^+_{\beta_L,q_L}\big(J^+(0)\big) = q_L \ , \qquad \varphi^-_{\beta_R,q_R}\big(J^-(0)\big) = q_R \ .$$

We also have

$$\varphi^+_{\beta_L,q_L}\big(T^+(0)\big) = \frac{\pi}{12\beta_L^2} + \frac{q_L^2}{2} \ , \qquad \varphi^-_{\beta_R,q_R}\big(T^-(0)\big) = \frac{\pi}{12\beta_R^2} + \frac{q_R^2}{2} \ .$$

In presence of chemical potentials $\mu_{L/R} = \frac{1}{\pi}q_{L/R}$, the large time limit of the two dimensional current density expectation value (x-component of the current operator J^μ) in the state ψ is, with $J^x(t,x) = J^-(t+x) - J^+(t-x)$

$$\lim_{t\to+\infty}\psi\big(J^x(t,x)\big) = \varphi^-_{\beta_L,q_L}\big(J^-(0)\big) - \varphi^+_{\beta_R,q_R}\big(J^+(0)\big) = -\pi(\mu_L - \mu_R) \ ,$$

whereas on the stress energy tensor

$$\lim_{t\to+\infty}\psi\big(T_{tx}(t,x)\big) = \varphi^+_{\beta_L,q_L}\big(T^+(0)\big) - \varphi^-_{\beta_R,q_R}\big(T^-(0)\big)$$

$$= \frac{\pi}{12}\big(\beta_L^{-2} - \beta_R^{-2}\big) + \frac{\pi^2}{2}\big(\mu_L^2 - \mu_R^2\big) \ ,$$

(cf. Bernard-Doyon)