Lecture 2

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Operator Algebras and Conformal Field Theory

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Jones index. $\mathcal{N} \subset \mathcal{M}$ inclusion of factors. $\mathcal{M}$ to be finite, namely there exists a (unique) tracial state $\omega = (\cdot\Omega, \Omega)$ on $\mathcal{M}$. With e the projection onto $\overline{\mathcal{N}\Omega}$, the von Neumann algebra generated by $\mathcal{M}$ and e

$$\mathcal{M}_1 = \langle \mathcal{M}, e \rangle = J_{\mathcal{M}} \mathcal{N}' J_{\mathcal{M}}$$

is a semifinite factor ($\exists$ unbounded trace).

$\mathcal{N} \subset \mathcal{M}$ has finite *index* $\stackrel{\text{def}}{=}$ $\mathcal{M}_1$ is finite. The index is defined as

$$[\mathcal{M} : \mathcal{N}] = \omega(e)^{-1}$$

with ω also denoting the trace of $\mathcal{M}_1$.

Jones thm.

$$\boxed{[\mathcal{M} : \mathcal{N}] \in \left\{ 4\cos^2 \frac{\pi}{n}, n \geq 3 \right\} \cup [4, \infty] \, .}$$

A *probabilistic definition* of the index was given by Pimsner and Popa through the inequality

$$\varepsilon(x) \geq \lambda x, \quad x \in \mathcal{M}^+,$$

$\lambda = [\mathcal{M} : \mathcal{N}]^{-1}$ where $\varepsilon : \mathcal{M} \to \mathcal{N}$ is the trace preserving conditional expectation.

$\mathcal{N} \subset \mathcal{M}$ any inclusion of factors, $\varepsilon : \mathcal{M} \to \mathcal{N}$ normal expectation:

$[\mathcal{M} : \mathcal{N}]_\varepsilon$ defined by Popa, Kosaki (e.g. by Pimsner-Popa inequality)

Minimal index (Hiai, L.)

$$[\mathcal{M} : \mathcal{N}] = \inf_\varepsilon [\mathcal{M} : \mathcal{N}]_\varepsilon = [\mathcal{M} : \mathcal{N}]_{\varepsilon_0}$$

where ε_0 is the unique *minimal conditional expectation*.

Joint modular structure. Sectors. $\mathcal{N} \subset \mathcal{M}$ type III factors. $J_\mathcal{N}$ and $J_\mathcal{M}$ modular conjugations of $\mathcal{N}$ and $\mathcal{M}$.

The unitary $\Gamma = J_\mathcal{N} J_\mathcal{M}$ implements a *canonical endomorphism* of $\mathcal{M}$ into $\mathcal{N}$

$$\boxed{\gamma(x) = \Gamma x \Gamma^*, \qquad x \in \mathcal{M}.}$$

Proof. $\Gamma\mathcal{M}\Gamma = J_{\mathcal{N}}J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}}J_{\mathcal{N}} = J_{\mathcal{N}}\mathcal{M}'J_{\mathcal{N}} \subset J_{\mathcal{N}}\mathcal{N}'J_{\mathcal{N}} = \mathcal{N}$.

γ depends on $J_{\mathcal{N}}$ and $J_{\mathcal{M}}$ only up to inners of $\mathcal{N}$; γ is canonical as a sector of $\mathcal{M}$:

The *sectors of* $\mathcal{M}$ are

$$\mathrm{Sect}(\mathcal{M}) = \mathrm{End}(\mathcal{M})/\mathrm{Inn}(\mathcal{M})$$

$\rho, \rho' \in \mathrm{End}(\mathcal{M})$, $\rho \sim \rho'$ iff there is a unitary $u \in \mathcal{M}$ such that $\rho'(x) = u\rho(x)u^*$ for all $x \in \mathcal{M}$.

$\mathrm{Sect}(\mathcal{M})$ is a *-*semiring*

Addition (direct sum): Let $\rho_1, \rho_2 \in \mathrm{End}(\mathcal{M})$; then $\rho \equiv \rho_1 \oplus \rho_2$

$$\rho : x \in \mathcal{M} \to \begin{bmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{M}) \simeq \mathcal{M}$$

naturally up to inners, thus in $\mathrm{Sect}(\mathcal{M})$.

Composition (monoidal product). Usual composition of maps

$$\rho_1 \cdot \rho_2(x) = \rho_1(\rho_2(x)), \qquad x \in \mathcal{M}$$

passes to the quotient Sect($\mathcal{M}$).

Conjugation. With $\rho \in$ End($\mathcal{M}$), choose a canonical endomorphism $\gamma_\rho : \mathcal{M} \to \rho(\mathcal{M})$. Then

$$\bar{\rho} = \rho^{-1} \cdot \gamma_\rho$$

well-defines a conjugation in Sect($\mathcal{M}$). Thus have

$$\boxed{\gamma_\rho = \rho \cdot \bar{\rho}}$$

Connes bimodules and sectors. $L^2(\mathcal{M})$ is a normal bimodule for $\mathcal{M}$

$$x, y \in \mathcal{M},\ \xi \in L^2(\mathcal{M}) \mapsto x\xi y \equiv xJy^*J\xi$$

If $\rho \in$ End($\mathcal{M}$) the bimodule $L^2_\rho(\mathcal{M})$ is $L^2(\mathcal{M})$ with left-rigth actions

$$x, y \in \mathcal{M},\ \xi \in L^2(\mathcal{M}) \mapsto \rho(x)\xi y \equiv xJy^*J\xi$$

All normal bimodules on $\mathcal{M}$ arise in this way up to unitary equivalence.

<u>Representation</u> <u>concepts make sense</u>.

$$\mathrm{Bimod}(\mathcal{M})_{/\sim} = \mathrm{Sect}(\mathcal{M})$$

$\mathrm{Ind}(\rho) \equiv [\mathcal{M} : \rho(\mathcal{M})]$.

Prop. $\rho \in \mathrm{End}(\mathcal{M})$ irreducible.

$$\mathrm{Ind}(\rho) < \infty \Leftrightarrow \rho\bar{\rho} \succ \iota \ \& \ \bar{\rho}\rho \succ \iota$$

Analytic def. of conjugate = algebraic def. of conjugate

One may represent objects with <u>non-integral</u> <u>dimension</u> $d(\rho) = \sqrt{\mathrm{Ind}(\rho)}$ as quantum groups, loop groups, infinite-dimensional Lie algebras, superselection sectors, . . .

The tensor category End(M).

Tensor category = category equipped with monoidal product (internal tensor product) on objects and arrows (+ natural compatibility conditions).

Tensor C^*-category = tensor category + arrows form a Banach space with an involution reversing directions. C^* property $||T^* \circ T|| = ||T||^2$ (Doplicher, Roberts).

$\mathcal{M}$ an infinite factor $\to$ End(M) is a *tensor C^*-category*:

Objects: = End(M)

Hom$(\rho, \rho') \equiv \{a \in M : a\rho(x) = \rho'(x)a \ \forall x \in M\}$

Composition of intertwiners (arrows): operator product

C^* *property*: obvious

Tensor product of objects: $\rho \otimes \rho' = \rho\rho'$

Tensor product of arrows: $\sigma, \sigma' \in \mathrm{End}(M)$, $t \in \mathrm{Hom}(\rho, \rho')$, $s \in \mathrm{Hom}(\sigma, \sigma')$,

$$t \otimes s \equiv t\rho(s) = \rho'(s)t \in \mathrm{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma') \ .$$

If ρ is irreducible (i.e. $\rho(M)' \cap M = \mathbb{C}$) and has finite index, then $\bar{\rho}$ is the unique sector such that $\rho\bar{\rho}$ contains the identity sector.

$\rho, \bar{\rho} \in \mathrm{End}(M)$ are conjugate as sectors iff $\exists$ isometries $v \in \mathrm{Hom}(\iota, \rho\bar{\rho})$ and $\bar{v} \in \mathrm{Hom}(\iota, \bar{\rho}\rho)$ such that

$$\boxed{\begin{aligned} (\bar{v}^* \otimes 1_{\bar{\rho}}) \cdot (1_{\bar{\rho}} \otimes v) \equiv \bar{v}^* \bar{\rho}(v) = \frac{1}{d}, \\ (v^* \otimes 1_{\rho}) \cdot (1_{\rho} \otimes \bar{v}) \equiv v^* \rho(\bar{v}) = \frac{1}{d}, \end{aligned}}$$

for some $d > 0$.

The *minimal* d is the *dimension* $d(\rho)$; it is related to the minimal index by

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definion of the index)

$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$

$d(\rho_1\rho_2) = d(\rho_1)d(\rho_2)$

$d(\bar{\rho}) = d(\rho)$.

<u>Every</u> subset of End(M) having finite-index generate (by composition, subobjects, diret sum) a C^*-*tensor category with conjugates*.

Example 1. (Connes) G discrete (or locally compact) group,

π finite-dimensional unitary rep. of G on $\mathcal{H}$

$\lambda \otimes \pi$ acts on the left on $\ell^2(G) \otimes \mathcal{H}$

$\rho \otimes \iota$ acts on the right on $\ell^2(G) \otimes \mathcal{H}$

$\lambda \otimes \pi \sim \lambda$ (absorbing propery of λ) $\implies \ell^2(G) \otimes \mathcal{H}$ is a vN(G) bimodule with dimension dim$\mathcal{H}$.

Tensor product of reps. $\leftrightarrow$ tensor product of sectors.

Embedding an abstract tensor C^*-category $\mathcal{T}$. (Popa, Yamagami)

Every countable rigid tensor C^*-category is equivalent to a full sub-tensor C^*-category of End($\mathcal{M}$) for some factor $\mathcal{M}$.

$\boxed{\text{End}(\mathcal{M})\ \text{“universal” tensor } C^* \text{ tensor category}}$

Conformal Nets

Möbius covariants nets on S^1. A (local) *Möbius covariant net* $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

A. *Isotony.* $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

B. *Locality.* $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$

C. *Möbius covariance.* $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb},\ I \in \mathcal{I}.$$

D. *Positivity of the energy.* Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$ and unique U-invariant.

Sectors of $\mathcal{A}$. A *representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$, with π_I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

π is Möbius *covariant* if there is a projective unitary representation U_π of Möb on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in$ Möb.

Every rep. on a separable Hilbert space is equivalent to a DHR localised endomorphism, hence we may compose them.

<u>Localised end. naturally form a tensor C^*-category</u>

Equivalence classes of localised endoorphisms are the *sectors* of $\mathcal{A}$.

DHR statistics. ρ localized rep. in $I \in \mathcal{K}$, i.e. ρ acts identically on each $\mathcal{A}(I_1)$, $I_1 \subset I'$.

Choose $\rho_1 \sim \rho$ localized in $I_1 \subset I'$: $\rho_1 = u\rho(\cdot)u^*$ with $u \in \mathcal{A}(\tilde{I})$. <u>Two choices $\pm$ of $\tilde{I} \supset I \cup I_1$</u> are possible up to deformation.

$\rho\rho_1 = \rho_1\rho$ gives $\epsilon = u^*\rho(u) \in \rho^2(\mathcal{A})'$ $({\epsilon^+}^* = \epsilon^-)$

$\epsilon_i \equiv \rho^{i-1}(\epsilon)$, $i \in \mathbb{N}$,

$$\begin{cases} \epsilon_i\epsilon_j = \epsilon_j\epsilon_i & \text{if } |i-j| \geq 2, \\ \epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1} \end{cases}$$

unitary rep. of $\mathbb{B}_\infty$, the *statistics* of ρ.

There is an expectation $\varepsilon : \mathcal{A}(I) \to \rho(\mathcal{A}(I))$.

ρ irreducible: *statistics parameter* $\lambda_\rho = \varepsilon(\epsilon)$

$\lambda_\rho = \kappa_\rho / d_{\mathrm{DHR}}(\rho)$ with $d_{\mathrm{DHR}}(\rho) > 0$ and $\kappa_\rho \in \mathbb{T}$.

$d_{\mathrm{DHR}}(\rho)$ is the *statistical dimension* of ρ;

Index-statistics thm.

$$\boxed{\text{DHR dim. } d(\rho) = \sqrt{\text{Jones index Ind}(\rho)}}$$

$$\begin{array}{c} \text{tensor category} \\ \text{End. local. in } I \end{array} \xrightarrow[\text{restriction}]{\text{full functor}} \begin{array}{c} \text{tensor category} \\ \text{End. of } \mathcal{A}(I) \end{array}$$

$$\text{Hom}(\rho, \sigma) = \text{Hom}(\rho_I, \sigma_I)$$

Local intertwiners = global intertwiners (Guido,L.)
In particular

$$\boxed{\text{Superselection sectors} \longrightarrow \text{Sect}(\mathcal{M}).}$$

<u>Subfactor theory contains all local information</u>.

Index-statistics thm. gives by Jones' thm:

$$\boxed{d(\rho) \in \{2\cos\frac{\pi}{n}, n \geq 3\} \cup [2, \infty].}$$

$\rho^2 = \rho_1 \oplus \cdots \oplus \rho_n$ irred. decomposition.

$n \leq 3$, in particular for "small" index, statistics is <u>classified</u> by the braid group rep., that is by Jones and Kauffman knot/link invariants.

We have

$4 < d(\rho)^2 < 6$

$\Rightarrow d(\rho)^2 = 5,\ 5.049\ldots, 5.236\ldots, 5.828\ldots$

(Rehren, L.) while Jones index values $\supset [4, \infty)$!.

(Locally normal) universal algebra.

$$\begin{array}{ccc} \mathcal{A}(I) & \xrightarrow{\iota_I} & C^*(\mathcal{A}) \\ {\scriptstyle \pi_I}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ B(\mathcal{H}) & = & B(\mathcal{H}) \end{array}$$

Locally norm. reps of $\mathcal{A} \leftrightarrow$ Endom. of $C^*(\mathcal{A})$

$\Downarrow$

Fusion of representations

$\downarrow$

$\boxed{\text{End}(C^*(\mathcal{A})) \text{ is braided tensor category}}$

$||$

canonical intertwiners $\varepsilon(\rho, \sigma) : \rho\sigma \to \sigma\rho$

(Fredenhagen, Rehren, Schroer)

Thm. (Carpi, Conti, Weiner) If $\mathcal{A}$ is rational (finitely many irr. reps, all with finite index) then

$$C^*(\mathcal{A}) \simeq F_1 \oplus F_2 \oplus \cdots \oplus F_n$$

F_k type I_∞ factors. So $C^*(\mathcal{A})$ is a von Neumann algebra, with finite dimensional center!

Conformal spin-statistics thm. (Guido, L.) π rep. of $\mathcal{A}$, λ_ρ DHR statistics parameter

$$\boxed{\kappa_\rho \equiv \mathrm{ph}(\lambda_\rho) = e^{2\pi i h_\rho}}$$

$h_\rho = =$ *spin*, i.e. lowest eigenvalue of L_ρ.

Proof. (some argument) $I_1 =$ upper half-circle, $I_2 =$ right half-circle ρ automorphism localized in $I_1 \cap I_2$.

$\rho|_{\mathcal{A}(I_i)} \to$ Araki-Connes-Haagerup unitary standard implementation V_i

V_1 and V_2 commute up to a phase

$$V_1 V_2 = \mu V_2 V_1.$$

μ algebraic invariant & geometric invariant: compare the two aspects...

Diff(S^1) and the Virasoro algebra. Diff(S^1) = smooth oriented diffeomorphisms of S^1. The (complexification of) Lie algebra of Diff(S^1) is Vect(S^1) (Witt algebra)

$$[L_m, L_n] = (m-n)L_{m+n}, \quad L_n = ie^{int}\frac{\mathrm{d}}{\mathrm{d}t}$$

The *Virasoro algebra* is the unique, non-trivial one-dim. central extension of the Witt alg.

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}}$$

and $[L_n, c] = 0$. c is called *central charge*.

Unitary irreducible representation of Virasoro alg. on Hilbert space $\mathcal{H}$ $\updownarrow$ Irr. family of operators L_n on $\mathcal{H}$ and $c \in \mathbb{R}$ with Virasoro relations and $L_n^* = L_{-n}$.

L_1, L_{-1}, L_0 = generators of complex span of $s\ell(2,\mathbb{R})$ (Lie algebra of Möbius group):

$[L_1, L_0] = L_1,\ [L_{-1}, L_0] = -L_{-1},\ [L_1, L_{-1}] = 2L_0.$

$L_0 \stackrel{\text{def}}{=}$ *conformal Hamiltonian* (= generator of rotations).

Positive energy unitary rep. U of Diff(S^1): $L_0 \geq 0$. Thus sp$U \subset \{h, h+1, h+2, \cdots\}$, $h \geq 0$. h is called *lowest weight*.

For every possible value of c and h $\exists!$ irr. pos. energy rep. $V_{c,h}$ of Diff(S^1). Possible values

(Friedan, Qui, Shenker ‘86):

$$\boxed{c = 1 - \frac{6}{n(n+1)} \quad \text{or} \quad \mathrm{c} \geq 1}$$

$$h_{p,q} = \frac{((n+1)p - nq)^2 - 1}{4n(n+1)},$$

$1 \leq p \leq n-1$, $1 \leq q \leq n$, $p, q \in \mathbb{N}$, $(p.q) \sim (n-p, n+1-q)$. All values are taken (Goddard, Kent, Olive ‘86).

Reps. with the same c have *fusion* (internal tensor product).

Long standing problem: <u>is there a relation between Jones index discrete series and Virasoro central charge discrete series</u>? We shall see an interplay below.

Conformal nets. A local conformal net $\mathcal{A}$ is a local Möbius covariant net s.t. $\exists$ proj. unitary rep. U of Diff(S^1), extending the Möbius rep., s.t.

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1),$$
$$U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'),$$

$\mathrm{Diff}(I) \stackrel{\mathrm{def}}{=} \{g \in \mathrm{Diff}(S^1) : g(t) = t\ \forall t \in I'\}$.

U is <u>unique</u> (Carpi,Weiner), hence canonical.

Virasoro nets Vir$_c$.

$$\mathrm{Vir}_c(I) \equiv V_c(\mathrm{Diff}(I))''$$

$V_c \equiv V_{c,h=0}$ (*vacuum representation*).

Buchholz Shultz-Mirbach, Carpi, recently completed by Weiner:

<u>Reps of Vir$_c$ net $\leftrightarrow$ Unitary reps of Virasoro$_c$ algebra</u>

in particular $V_{c,h}$ and $V_{c,h'}$ are locally equivalent

$\mathcal{A}$ (local) conformal net, Haag duality implies

$$U(\mathrm{Diff}(I)) \subset \mathcal{A}(I),$$

U is direct sum of reps $V_{c,h}$ with the same central charge c: the <u>central charge of $\mathcal{A}$</u>

$$\boxed{\begin{array}{c} \mathcal{A} \supset \mathrm{Vir}_c \\ \text{every local conformal net} \\ \text{is an extension of a Virasoro net} \end{array}}$$

On the other hand Vir_c is <u>minimal</u>, no nontrivial subnet (Carpi):

<u>universal role of Vir_c</u>

A (irred.) *representation* π of $\mathcal{A}$ on $\mathcal{H}$ is diffeomorphism *covariant* if $\exists$ projective unitary rep. U_π of $\mathrm{Diff}(S^1)$ extending the rep. U_π of Möb s.t.

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

Automatic diff. covariance: D'Antoni, Fredenhagen, Koester,

Complete rationality. Problem: characterize <u>intrinsically</u> a "rational" net (= finitely many irr. sectors, all with $d(\rho) < \infty$)

Def. $\mathcal{A}$ is *completely rational* if

- The *μ-index* $\mu_{\mathcal{A}}$ is finite, i.e.

$$\mu_{\mathcal{A}} \equiv [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty$$

$E = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, $\hat{\mathcal{A}}(E) = \mathcal{A}(E')'$ (failure of Haag duality for disconneted regions).

$\mu_{\mathcal{A}} < \infty$ for $SU(N)$ loop group models (F. Xu). General theory (Kawahigashi, Müger, L.)

The μ-index is equal to global index:

$$\boxed{\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2}$$

sum of the indeces of all irreducible sectors

For a completely rational net:

• $\mathcal{A}$ is *rational* and the representation tensor category is *modular* has <u>non-degenerate braiding</u>

• $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is the quantum double inclusion of Rehren, L.

• All irreducible extensions of $\mathcal{A}$ have <u>finite Jones index</u> (by Izumi, Popa, L.)

• $\mathcal{A}$ is strongly additive (Xu, L.)

$$\mathcal{A}(I \smallsetminus \{\text{point}\}) = \mathcal{A}(I)$$

Loop group and coset models. G compact Lie group,

LG loop group, i.e. $LG = \{g : t \in S^1 \to G\}$ (smooth maps with pointwise multiplication),

$U : LG \to B(\mathcal{H})$ pos. energy unitary rep. of LG, i.e. the action of Diff(S^1) on Aut(LG) is implements by a pos. energy rep.

Vacuum irr. reps. (pos. energy) U of LG (0 eigenvalue of L_0) are labeled by a parameter, the *level* of U. Fix a level ℓ rep. U:

$$\mathcal{A}(I) \equiv \{U(g), g \in LG : \ g(t) = t, \ t \in I'\}''$$

is a conformal net.

$H \subset G$ closed subgroup

$$\mathcal{B}(I) \equiv \{U(g), g \in LH : \ g(t) = 1, \ t \in I'\}''$$

conformal subnet.

$C(I) = \mathcal{B}(I)' \cap \mathcal{A}(I)$ coset model of $H \subset G$.

$\boxed{\text{Vir}_c = \text{coset } SU(2)_{m-1} \subset SU(2)_{m-1} \times SU(2)_1}$

$c = 1 - \frac{6}{m(m+1)}$ (GKO, Xu, Carpi, Kawahigashi, L.).

$\Rightarrow$ Vir$_c$ is completely rational $c < 1$

$\Rightarrow$ All extensions of Vir$_c$ have finite Jones index

$\Rightarrow$ Sectors of Vir$_c$ have finite index (Loke)

The classification problem for the discrete series (Kawahigashi, L.)

$$\boxed{\begin{array}{c}\text{Classify conformal nets with } c<1\\ \Updownarrow \\ \text{Classify all irreducible extensions of } \mathrm{Vir}_c\end{array}}$$

Verlinde-Rehren matrices. $\mathcal{A}$ rational, i.e. finitely many irr. sectors $\rho_o = \mathrm{id}, \rho_1, \ldots \rho_n$

$$Y_{ij} \equiv d_i d_j \Phi_j(\varepsilon(\rho_j, \rho_i)^* \varepsilon(\rho_i, \rho_j)^*)$$

ε non degenerate $\Leftrightarrow$ $|\sigma|^2 = \sum d_i^2$ with $\sigma \equiv \sum \kappa_i^{-1} d_i^2$

$$S \equiv |\sigma|^{-1} Y, \qquad T \equiv \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \mathrm{Diag}(\kappa_i)$$

$$\begin{aligned} SS^\dagger &= TT^\dagger = \mathrm{id}, \\ STS &= T^{-1} S T^{-1}, \\ S^2 &= C, \\ TC &= CT, \end{aligned}$$

where $C_{ij} = \delta_{i\bar{j}}$. In our case ($\mathrm{Vir}_c$) $C = \mathrm{id}$. $\Rightarrow$ $\underline{T \text{ and } S \text{ generate unitary rep. of } SL(2,\mathbb{Z})}$.

Modular invariants. Given a unitary, finite-dim. rep. of $SL(2,\mathbb{Z})$, a *modular invariant* is a matrix $Z \in \text{Mat}(\mathbb{Z}_+)$, $Z_{00} = 1$, s.t.

$$ZU = UZ$$

• Rational net with non-degenerate braiding $\to$ unitary rep. of $SL(2,\mathbb{Z})$ $\to$ modular invariants

• Thus (KLM): complete rational nets $\to$ modular invariants

• Capelli, Itzykson, Zuber '87: ADE classification of modular invariants for Vir_c, $c < 1$

• Böckenhaur, Evans, Kawahigashi 2000: $\mathcal{A} \subset \mathcal{B}$ conformal nets, $[\mathcal{B} : \mathcal{A}] < \infty$, then

$$\alpha - \text{induction} \longrightarrow \text{modular invariants}$$

$$Z_{\mu\nu} = \text{dimHom}(\alpha^+_\mu, \alpha^-_\nu)$$

$\alpha^\pm_\mu$ = extension of DHR sector μ of $\mathcal{A}$ to right/left solitonic sector of $\mathcal{B}$ (Roberts, Rehren-L., Xu)

Q-systems. Recall: $\mathcal{M}$ factor, $\rho \in \mathrm{End}(\mathcal{M})$ then

$$\gamma_\rho = \rho\bar{\rho}$$

<u>Converse problem</u>: given $\gamma \in \mathrm{End}(M)$, when is γ canonical?

The problem is finding a "square root" ρ.

The conjugate equations give conditions:

γ canonical with finite index

$$\Downarrow$$

$\exists$ isometry $T \in \mathrm{Hom}(\iota, \gamma)$, and a co-isometry $S \in \mathrm{Hom}(\gamma^2, \gamma)$

$$\boxed{\begin{array}{c} SS = S\gamma(S) \\ S\gamma(T) \in \mathbb{C}\backslash\{0\} \ , \quad ST \in \mathbb{C}\backslash\{0\} \end{array}}$$

Def. A *Q-system* is a triple (γ, T, S) where $\gamma \in \mathrm{End}(M)$, $T \in \mathrm{Hom}(\iota, \gamma)$ is an isometry,

$S \in \mathrm{Hom}(\gamma^2, \gamma)$ is a co-isometry satisfying the above relations.

Thm. Q-system $(\gamma, T, S) \to$ finite-index subfactor $\mathcal{N} \subset \mathcal{M}$ with $\gamma : \mathcal{M} \to \mathcal{N}$ canonical endomorphism.

$\exists$ bijection

$$\boxed{\text{subfactors} \leftrightarrow \text{Q-systems}}$$

Application 1: Quantum double (Rehren, L.), see below.

Application 2: Duality for finite-dimensional complex semisimple Hopf algebras (L.).

Two Q-systems (ρ, T_1, S_1) and (ρ, T_2, S_2) are *equivalent* if $\exists u \in \mathrm{Hom}(\rho, \rho)$ satisfying

$$T_2 = uT_1, \qquad uS_1 = S_2 u\rho(u).$$

Equivalence of Q-systems $\Leftrightarrow$ inner conjugacy of subfactors.

$$N \subset M \qquad \overset{\text{Jones construction}}{\underset{\text{can. endomorphism}}{\longleftrightarrow}} \qquad \tilde{M} \supset M$$

<u>Problem</u>: classify Q-systems up to equivalence when a system of endomorphisms is given and ρ is a direct sum of endomorphisms in the system.

Izumi-Kosaki cohomology for Q-systems: <u>finite</u> groups.

Classification of local extensions of the Virasoro nets (Kawahigashi, L.)

• Consider the Cappelli-Itzykson-Zuber classification of the modular invariants for the Virasoro nets with central charge $c = 1 - 6/m(m+1) < 1$, $m = 2, 3, 4, \ldots$.

• Show that each "type I" modular invariant is realized with α-induction for an extension $\mathrm{Vir}_c \subset \mathcal{M}$ as in Bockenhauer-Evans-Kawahigashi

• Use Q-system to detect the local extension of Vir_c, $c < 1$

$$\Downarrow$$

Classification of local conformal nets, $c = 1 - \frac{6}{m(m+1)}$

m	Labels for Z
n	(A_{n-1}, A_n)
$4n+1$	(A_{4n}, D_{2n+2})
$4n+2$	(D_{2n+2}, A_{4n+2})
11	(A_{10}, E_6)
12	(E_6, A_{12})
29	(A_{28}, E_8)
30	(E_8, A_{30})

Thm. (Kawahigashi,L.) Local conformal nets with $c < 1$ are classified by pair of Dynkin diagrams $A - D_{2n} - E_{6,8}$ s.t. the difference of Coxeter numbers is 1.

Simple current extensions. The simple current extensions of index 2

The four exceptional cases.

$(E_6, A_{12}), (E_8, A_{30})$ coset constructions (conjectuered by Böckenhauer-Evans

(A_{10}, E_6) coset construction (Köster)

One *new example* (A_{28}, E_8), most probably not constructable as coset.

Case $c = 1$ classified by Xu, Carpi (with a spectral condition, probably always true)

Many new models by mirror symmetry F. Xu.