Lecture 1

Operator Algebras and Conformal Field Theory

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The “$ax + b$” group

The (proper) “$ax + b$” group: transformations on $\mathbb{R}$

$$g = x \mapsto ax + b, \qquad a > 0, b \in \mathbb{R}.$$

One parameter subgroups: *translations* $x \mapsto x + b$ and *dilations* $x \mapsto ax$.

Unitary representations of the group on a Hilbert space $\mathcal{H}$: two one-parameter unitary U and V groups on $\mathcal{H}$

$$V(s)U(t)V(-s) = U(e^s t)$$

$\exists$ unique unitary, irreducible representation of G with "positive energy" (where translations have no non-zero fixed points),

because generator of V and log of generator of U satisfies CCR.

Modular Tomita-Takesaki theory.

$\mathcal{M}$ von Neumann alg. on $\mathcal{H}$. $\Omega \in \mathcal{H}$ cyclic and separating for $\mathcal{M}$,

$$L^\infty(\mathcal{M}) \equiv \mathcal{M}, \quad L^2(\mathcal{M}) = \mathcal{H} \quad L^1(\mathcal{M}) = \mathcal{M}_*,$$

where $\mathcal{M}_*$ is the predual of $\mathcal{M}$ (normal linear functionals),

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow[\text{isometric}]{x \mapsto x^*} & \mathcal{M} \\
{\scriptstyle x \to x\Omega}\big\downarrow & & \big\downarrow{\scriptstyle x \to x\Omega} \\
L^2(\mathcal{M}) & \xrightarrow[\text{non isometric}]{x\Omega \overset{S_0}{\longmapsto} x^*\Omega} & L^2(\mathcal{M})
\end{array}$$

S the closure of the anti-linear operator S_0, $S = J\Delta^{1/2}$ polar decomposition, thus $\Delta = S^*S > 0$ positive selfadjoint, J anti-unitary involution:

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

$$J\mathcal{M}J = \mathcal{M}'$$

$t \to \sigma_t^\omega = \mathrm{Ad}\Delta^{it}$ canonical "evolution" associated with ω (*modular automorphisms*).

Exponential Hilbert space

$\mathcal{H}$ Hilbert space. The Fock space

$$e^{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathrm{sym}}^{\otimes^n}$$

is generated by the exp vectors $e^h = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes^n}$;

The Weyl unitaries:

$$W(h)e^k \equiv e^{-\frac{1}{2}(h,h)} e^{-(h,k)} e^{h+k}$$

satisfy $W(h+k) = e^{i\Im(h,k)} W(h)W(k)$.

H real linear subspace of $\mathcal{H} \to$ von Neumann algebra on $e^{\mathcal{H}}$

$$\mathcal{A}(H) = \{W(h) : h \in H\}''$$

First and second quantisation

First quantisation: map

$$\mathcal{O} \subset \mathbb{R}^d \mapsto H(\mathcal{O}) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

Second quantisation: map

$$\mathcal{O} \subset \mathbb{R}^d \mapsto \mathcal{A}(\mathcal{O}) \text{ v.N. algebra on } e^{\mathcal{H}}$$

The <u>free</u> QFT is determined by the QM structure

$$\begin{tikzcd} \mathcal{O} \arrow[rr] \arrow[dr] & & \mathcal{A}(\mathcal{O}) \\ & H(\mathcal{O}) \arrow[ur] & \end{tikzcd}$$

$\mathcal{A}(\mathcal{O}) = \mathcal{A}(H(\mathcal{O}))$

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace. *Symplectic complement*:

$$H' = \{\xi \in H : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

$H' = (iH)^{\perp}$ (real orthogonal complement), so $H'' = H$ and

$$H_1 \subset H_2 \Leftrightarrow H_2' \subset H_1'$$

H is *cyclic* if $(\overline{H + iH} = \mathcal{H})$ and *separating* if $(H \cap iH = \{0\})$.

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. H is standard iff H' is standard.

H standard subspace $\to$ anti-linear operator $S : D(S) \subset \mathcal{H} \to \mathcal{H}$,

$$S : \xi + i\eta \to \xi - i\eta, \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H} \to H_S = \{\xi \in D(S) : S\xi = \xi\}$ is a standard subspace:

$$H \leftrightarrow S \text{ is a bijection}$$

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then J is an anti-unitary involution, $\Delta > 0$ is non-singular and $J\Delta J = \Delta^{-1}$.

$$H \leftrightarrow (J, \Delta) \text{ is a bijection.}$$

Modular theory for standard subspaces

$$\Delta^{it} H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

(real subspace analog of) Borchers theorem

H standard subspace, T a one-parameter group with positive generator s.t. $T(s)H \subset H, \;\; s > 0$.

Then:

$$\begin{cases} \Delta^{it} T(s) \Delta^{-it} = T(e^{-2\pi t} s) \\ JT(s)J = T(-s), \;\; t, s \in \mathbb{R} \end{cases}$$

(positive energy) representation of the proper "$ax + b$" group!

Proof Based on the analytic extension of $T(s)$ on the upper half-plane by positivity of the energy and the analytic extension of $(\xi, \Delta^{-is}\eta)$ on the trip $0 < \Im z < 1$ by the KMS condition.

Consequence: If T has no non-zero fixed vector, the pair (H, T) is unique up to multiplicity

Note: Setting $K = T(1)H$ we have

$$\Delta_H^{-it} K = \Delta_H^{-it} T(1)H = T(e^{2\pi t})\Delta_H^{it} H = T(e^{2\pi t})H \subset K, \ t > 0$$

$K \subset H$ is a *half-sided modular inclusion*, i.e. $\Delta_H^{-it} K \subset K$, $t > 0$.

(real subspace analog of) Wiesbrock-Araki-Zsido theorem

Let $K \subset H$ be a half-sided modular inclusion of standard subspaces. Then $K = T(1)H$ as above

$$\text{translation generator} = \frac{1}{2\pi}(\log \Delta_H - \log \Delta_K)$$

Therefore Δ_H^{-it} and Δ_K^{-is} generate a representation of the "$ax + b$" group.

Symmetries $\leftrightarrow$ Standard subspaces in certain relative positions

The Möbius group

$\mathrm{SL}(2,\mathbb{R}) = 2 \times 2$ real matrices with determinant one acts on $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$: $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as

$$g : x \mapsto gx \equiv \frac{ax+b}{cx+d}$$

Kernel $= \{\pm 1\}$. Möb $\equiv \mathrm{SL}(2,\mathbb{R})/\{1,-1\}$. We identify $\bar{\mathbb{R}} \sim S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$

$$C : x \in \mathbb{R} \mapsto -\frac{x-i}{x+i} \in S^1 \ ,$$

$\mathrm{SL}(2,\mathbb{R}) \sim SU(1,1)$ by C.

Three one-parameter subgroups of **G**: *rotation* R, *dilation* δ, *translation* τ

$$R(\theta)z = e^{i\theta}z \text{ on } S^1; \quad \delta(s)x = e^s x \text{ on } \mathbb{R}; \quad \tau(t)x = x + t \text{ on } \mathbb{R} \ .$$

The set of all intervals of S^1 will be denoted by $\mathcal{I}$. Note that Möb acts transitively on $\mathcal{I}$. If $I \in \mathcal{I}$, we denote by I' the interior of the complement of I in S^1, which is an interval.
Given any interval I, we now define two one-parameter subgroups of $\mathbf{G}$, the *dilation* δ_I and the *translation* group τ_I associated with I. Let I_1 be the upper semi-circle, i.e. the interval $\{e^{i\theta}, \theta \in (0, \pi)\}$, that corresponds to the positive real line $\mathbb{R}_+$ in the real line picture. We set $\delta_{I_1} \equiv \delta$, and $\tau_{I_1} \equiv \tau$. Then, if I is any interval, we chose $g \in \mathbf{G}$ such that $I = gI_1$ and set

$$\delta_I \equiv g\delta_{I_1}g^{-1}, \qquad \tau_I \equiv g\tau_{I_1}g^{-1}.$$

δ_I is well defined; while τ_I is defined only up to a rescaling.
If I is an open interval or half-line of $\mathbb{R}$ we write τ_I or δ_I to denote the translation or dilation group associated with $C(I)$ thus, for example, $\tau_{(0,\infty)} = \tau_{I_1} = \tau$.

Nets of standard subspaces

A local Möbius covariant net H of standard subspaces on S^1 is a map

$$I \in \mathcal{I} \to H(I) \subset \mathcal{H}$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies H(I_1) \subset H(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb $= PSL(2, \mathbb{R})$ on $\mathcal{H}$ s.t.

$$U(g)H(I) = H(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U is positive.
- **E.** *Irreducibility*. $\overline{lin.span}\{H(I), I \in \mathcal{I}\} = \mathcal{H}$

Some consequences

- *Reeh-Schlieder theorem*: Each $H(I)$ is a standard subspace.

  proof: $\xi \perp H(I) \Rightarrow \xi \perp T(s)H(I_0) = H(I_0 + s)$, if $\bar{I}_0 \subset I$, by analytic ext on $\Im z > 0$
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(H(I), \Omega)$, are

$$U(\delta_I(2\pi t)) = \Delta_I^{it}, \; t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  proof: Use Borchers theorem
- *Haag duality*: $H(I)' = H(I')$

  proof: Use the geomtric meaning of J_I

Converse construction (Brunetti, Guido, L.)

Given a positive energy unitary representation U of (proper) Möb on $\mathcal{H}$ we set

$$H(I) \equiv \{\xi \in \mathcal{H} : S_I \xi = \xi\}, \qquad S_I \equiv J_I \Delta_I^{1/2}$$

where Δ_I is *by definition* given by $\Delta_I^{-it} = U(\delta_I(2\pi t))$ with Λ_I one-parameter group of "dilations" associated with I.

Then H *is a local* Möb-*covariant net of standard subspaces*
Therefore:

Local net os standard subspaces

$\updownarrow$

Unitary, positive energy representation of $PSL(2, \mathbb{R})$.

Möbius covariant nets

A local Möbius covariant net $\mathcal{A}$ on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

$\mathcal{I} \equiv$ family of proper intervals of S^1, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \; I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U (conformal Hamiltonian) is positive.
- **E.** *Existence of the vacuum*. $\exists!$ U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

Consequences

$H(I) \equiv \overline{\mathcal{A}(I)_{sa}\Omega}$ is a standard subspace, therefore:

- *Irreducibility*: $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(H)$.
- *Reeh-Schlieder theorem*: Ω is cyclic and separating for each $\mathcal{A}(I)$.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I), \Omega)$, are

$$U(\delta_I(2\pi t)) = \Delta_I^{it}, \ \ t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

  (Frölich-Gabbiani, Guido-L.)
- *Haag duality*: $\mathcal{A}(I)' = \mathcal{A}(I')$
- *Factoriality*: $\mathcal{A}(I)$ is III_1-factor (in Connes classification)
- *Additivity*: $I \subset \cup_i I_i \implies \mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$ (Fredenhagen, Jorss).

Split property

$\mathcal{A}$ satisfies the *split* property if the von Neumann algebra

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

(natural isomorphism) if $\bar{I}_1 \cap \bar{I}_2 = \varnothing$.

$$\mathrm{Tr}(e^{-tL_0}) < \infty, \ \forall t > 0 \implies \text{split} .$$

A recent result by Morinelli, Tanimoto, Weiner shows that the spit property follows by conformal symmetries!

The spit property is general and will be assumed.

$U(1)$ current net

The unitary, irreducible, positive energy representations U of Möb are classified by the lowest eigenvalue ℓ of the conformal Hamiltonian L_0, the generator of the rotation group, $\ell = 1, 2, \ldots$. Let $I \mapsto H_\ell(I)$ the net associated with the ℓ representation

$$\mathcal{A}_\ell(I) \equiv \mathcal{A}(H_\ell(I))$$

the net of von Neumann algebras on the Fock space. $\mathcal{A}_1$ is the $U(1)$-current net, $\mathcal{A}_{1+\ell}$ the net associated with the ℓ derivative of the $U(1)$ current.

H_1 can be realized as the completion of $C^\infty(S^1)$ modulo constants, with scalar product

$$(f, g) = \sum_{n=0}^{\infty} n\hat{f}(n)\hat{g}(-n)$$

f, g real. Real functions with support in I generate $H_1(I)$.

Representations

A *representation* π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$, with π_I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}} \restriction \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I} \ .$$

π is Möbius *covariant* if there is a projective unitary representation U_π of Möb on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in$ Möb.
Version of DHR argument: given I and π rep. of $\mathcal{A}$, $\exists$ an endomorphism $\rho \simeq \pi$ of $\mathcal{A}$ localized in I; i.e. $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$.
Proof. $\mathcal{A}(I)$ is a type III factor, thus only one normal rep.
– Fix I: choose $\rho \simeq \pi$, $\pi_{I'} = \mathrm{id}$.
– By Haag duality $\rho_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$.

Example. Let $\mathcal{A}$ be the local conformal net on S^1 associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_{\mathbb{R}}(\mathbb{R}),\ \mathrm{supp} f \subset I\}''$$

where W is the representation of the Weyl commutation relations

$$W(f)W(g) = e^{-i\int fg'} W(f+g)$$

associated with the vacuum state ω

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty p|\tilde{f}(p)|^2 \mathrm{d}p$$

where $\tilde{f}$ is the Fourier transform of f.

Buchholz-Mack-Todorov sectors and exensions

There is a one parameter family $\{\alpha_q, q \in \mathbb{R}\}$ of irreducible sectors and all have index 1.

$$\alpha_q(W(f)) \equiv e^{2i\int Ff} W(f), \quad F \in C^\infty, \quad \int F = q \ .$$

The "crossed product" of $\mathcal{A}_N$ by a single α_q, $N = \frac{1}{2}q^2$ is a net on S^1, local iff N is an integer. $\mathcal{A}_N$ is maximal iff N is a product of distinct primes.

Examples: $\mathcal{A}_1$ associated with level 1 $\widehat{su(2)}$-Kac-Moody algebra with $c = 1$, $\mathcal{A}_2$ Bose subnet of free complex Fermi field net, $\mathcal{A}_3$ appears in the $\mathbb{Z}_4$-parafermion current algebra analyzed by Zamolodchikov and Fateev, and in general $\mathcal{A}_N$ is a coset model $SO(4N)_1/SO(2N)_2$.

2-dimensional CFT

$M = \mathbb{R}^2$ Minkowski plane.

$\begin{pmatrix} T_{00} & T_{10} \\ T_{01} & T_{11} \end{pmatrix}$ conserved and traceless stress-energy tensor.

As is well known, $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields,

$$T_L = T_L(t + x), \quad T_R = T_R(t - x).$$

Left and right movers.

Ψ_k family of conformal fields on M: T_{ij} + *relatively local fields*
$\mathcal{O} = I \times J$ double cone, I, J intervals of the chiral lines $t \pm x = 0$

$$\mathcal{A}(\mathcal{O}) = \{e^{i\Psi_k(f)}, \mathrm{supp} f \subset \mathcal{O}\}''$$

then by relative locality

$$\mathcal{A}(\mathcal{O}) \supset \mathcal{A}_L(I) \otimes \mathcal{A}_R(J)$$

$\mathcal{A}_L, \mathcal{A}_R$ chiral fields on $t \pm x = 0$ generated by T_L, T_R and other chiral fields

(completely) rational case: $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{A}(\mathcal{O})$ finite Jones index