Stable actions and central extensions

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A central sequence $(x_n)_n$: $\forall y \quad [x_n, y] \to 0$ as $n \to \infty$.

★ (Murray-von Neumann 1943): $\mathsf{L}\mathfrak{S}_\infty \not\simeq \mathsf{L}F_2$.

★ (McDuff 1970): $M \otimes \mathsf{L}\mathfrak{S}_\infty \simeq M \iff$

$\exists$ two central seq in M, (x_n), (y_n), s.t. $[x_n, y_n] \not\approx 0$.

Let $\mathcal{R}$ be an ergodic II_1 equivalence relation on (X, μ).

E.g. The hyperfinite equivalence relation:

$$\mathcal{R}_0 = \mathcal{R}(\textstyle\bigoplus_{\mathbb{N}} \mathbb{Z}/2 \curvearrowright \prod_{\mathbb{N}} \mathbb{Z}/2).$$

The sequence $V_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots) \in \bigoplus_{\mathbb{N}} \mathbb{Z}/2$ is asymptotically central (a.c.) in $[\mathcal{R}_0]$.

★ (Jones-Schmidt 1987): $\mathcal{R} \times \mathcal{R}_0 \simeq \mathcal{R} \iff$

$\exists$ a.c. seq $(V_n)_n$ in $[\mathcal{R}]$ which move points of X sufficiently.

Such an $\mathcal{R}$ is called **stable**. A countable group G is called **stable** if G has a free pmp action $G \curvearrowright X$ with $\mathcal{R}_{G \curvearrowright X}$ stable.

★ (J-S): If $\mathcal{R}_{G \curvearrowright X}$ is stable, then G is **inner amenable**, i.e.,

$\exists$ G-conjugacy invariant mean m on G s.t. $\forall g \in G$ $m(\{g\}) = 0$.

Construction: Let $\mathcal{R} = \mathcal{R}_0 \times \mathcal{R}_1$ and $V_n \in [\mathcal{R}_0] \subset [\mathcal{R}]$.

Let m be a limit of the ℓ^1-functions $f_n \colon G \ni g \mapsto \mu(\{V_n = g\})$.

Baumslag-Solitar group: $G = \mathsf{BS}(2,3) = \langle\, a,\, t \mid ta^2t^{-1} = a^3 \,\rangle$.

★ (K 2012): A stable action of G is constructed explicitly.

An ingredient: $G \twoheadrightarrow \Gamma := \mathbb{Z}[3/2] \underset{\times 3/2}{\rtimes} \mathbb{Z}$, $a \mapsto (1,0)$, $t \mapsto (0,1)$.

$$\Gamma \curvearrowright X := \prod_{\mathbb{Z}_-}\{0,1,2\} \times \prod_{\mathbb{N}}\{0,1,\dots,5\} \times \prod_{\mathbb{Z}_+}\{0,1\}.$$

$((x_{-n}),(z_m),(x_n)) \in X \overset{1:1}{\longleftrightarrow}$ a formal sum

$$\cdots + x_{-2}(3/2)^{-2} + x_{-1}(3/2)^{-1} + z + x_1(3/2) + x_2(3/2)^2 + \cdots,$$

where $z := z_0 + 6z_1 + 6^2z_2 + \cdots \in \mathbb{Z}_6 = \prod_{\mathbb{N}}\{0,1,\dots,5\}$.

Define $a \curvearrowright X$ by "$+1$" and $t \curvearrowright X$ by "$\times 3/2$".

Obs. The following seq is a.c. in $\Gamma \ltimes L^\infty(X)$:

$$V_n = (a^{2^{2n}3^n} + a^{2^{2n-1}3^{n+1}} + a^{2^{2n-2}3^{n+2}} + \cdots + a^{2^n3^{2n}})/n.$$

★ (Vaes 2009):

$\exists$ an ICC inner-amenable group V s.t. LV is non-Gamma.

$\rightsquigarrow$ solves Effros's problem in 1975.

★ (K 2012): The group V is stable.

★ (Deprez-Vaes 2016): completed the picture of all implications between two of inner-amenability, stability, Gamma, and McDuff of a group G.

Obs. If $\mathcal{R} = \mathcal{R}_0 \times \mathcal{R}_1$, the pair $(\mathcal{R}, \mathcal{R}_0 \times \mathcal{I}_1)$ does not have (T).

Question. A group G is stable $\underset{?}{\iff}$

(1) G is inner amenable with a conj-inv mean m and

(2) the pair $(G, \underset{??}{\operatorname{supp}}\, m)$ does not have (T).

This equiv is supported by "Connes-Weiss + Jones-Schmidt" and the following:

★ (K 2013): $C < G$ central, (G, C) not (T) $\Rightarrow$ G stable.

★ (Tucker-Drob 2014): Let G be a linear inner-amenable group. Then there exists a canonical $A \lhd G$ such that

- any G-conj-inv mean on G is supported on A, and

- G is stable iff the pair (G, A) does not have (T).

Stability of stability.

★ For a group G and a finite index $H < G$: G i.a. $\Rightarrow$ H i.a.

Pf (T-D). Let m be a conj-inv mean on G. $\exists g_0 \in G \; m(g_0 H) > 0$.

$$m * m(H) = \int_G m(gH)\, dm(g) \geq \int_{g_0 H} m(gH)\, dm(g) = m(g_0 H)^2 > 0.$$

$\square$

★ (T-D 2016): If $G \curvearrowright X$ is stable, then any ergodic component of $H \curvearrowright X$ is stable.

The first step is to show: For a.c. seq V_n for $G \curvearrowright X$, after passing to a subsequence, for any large n, if $m > n$ is large enough, then the measure of the set $\{ V_n \circ V_m \in H \}$ is uniformly positive.

★ (K 2016): $C < G$ central, G/C stable $\Rightarrow$ G stable.

Note: The case when $\#C < \infty$ is essential.

Cor (T-D + K). For a finite $N \triangleleft G$, G/N stable $\Rightarrow$ G stable.

Cor. Stability is invariant under virtual isomorphism.

2-cocycles.

Let Γ be a group and C an abelian group. Classically well-known:

$$\sigma \in Z^2(\Gamma, C) \quad \text{(i.e., } \sigma(gh,k)\sigma(g,h) = \sigma(g,hk)\sigma(h,k))$$

$$\overset{1:1}{\longleftrightarrow} \qquad 1 \to C \to G_\sigma \underset{\underset{s}{\leftarrow}}{\to} \Gamma \to 1 \quad \text{central ext.}$$

($\exists$ a section s with $\sigma(g,h)s(gh) = s(g)s(h)$.)

Generalized into the groupoid setting (Series 1981):

$$\sigma \in Z^2(\mathcal{R}, C) \quad \overset{1:1}{\longleftrightarrow} \quad 1 \to C \times X \to \mathcal{G}_\sigma \to \mathcal{R} \to 1 \quad \text{central ext.}$$

Künneth?

Suppose $\mathcal{R} = \mathcal{E} \times \mathcal{F}$ on $X \times Y$. Is $H^2(\mathcal{R})$ isomorphic to

$$(H^2(\mathcal{E}) \otimes H^0(\mathcal{F})) \oplus (H^1(\mathcal{E}) \otimes H^1(\mathcal{F})) \oplus (H^0(\mathcal{E}) \otimes H^2(\mathcal{F}))?$$

Not correct: $\mathcal{R}_0 = \mathcal{R}_0 \times \mathcal{R}_0$.

Q. For $\sigma \in Z^2(\mathcal{R}, C)$, what are $(H^0 \otimes H^2)$- and $(H^1 \otimes H^1)$-parts?

Given $\mathcal{R} = \mathcal{E} \times \mathcal{F}$ on $X \times Y$ and $\sigma \in Z^2(\mathcal{R}, C)$, take a section s:

$$1 \to C \to \mathcal{G}_\sigma \underset{\overleftarrow{s}}{\to} \mathcal{R} \to 1, \quad \sigma(g,h) = s(gh)^{-1}s(g)s(h).$$

(1) For $x \in X$, define $\sigma_x \in Z^2(\mathcal{F})$ by $\sigma_x = \sigma \restriction \{e_x\} \times \mathcal{F}$.

(2) For $g = (x', x) \in \mathcal{E}$, define $\sigma_g \colon \mathcal{F} \to C$ by

$$\sigma_g(h) = s(e_x, h)^{-1}s(g, e_y)^{-1}s(e_{x'}, h)s(g, e_y) \text{ for } h = (y', y) \in \mathcal{F}.$$

★ $\sigma_x(h,k)\sigma_g(hk)^{-1}\sigma_g(h)\sigma_g(k) = \sigma_{x'}(h,k)$, where $g = (x', x)$.

$\rightsquigarrow$ $\sigma_x \sim \sigma_{x'}$ if $(x', x) \in \mathcal{E}$.

$\rightsquigarrow$ σ_g is a 1-cocycle iff $\sigma_x = \sigma_{x'}$.

Let $C < G$ be a central subgroup, $G/C \curvearrowright X$ a stable action, and $\mathcal{R} := \mathcal{R}_{G/C \curvearrowright X} = \mathcal{R}_0 \times \mathcal{R}_1$. (Our aim is to find a stable action of G.)

We have the central extension of groupoids:

$$1 \to C \times X \to G \ltimes X \underset{\overleftarrow{s}}{\to} \mathcal{R} \to 1,$$

where G acts on X through the map $G \twoheadrightarrow G/C$.

$\exists\, \mathcal{G}_0 < G \ltimes X$ a section of $\mathcal{R}_0 \times \mathcal{I}_1$ (because $H^2(\mathcal{R}_0, C) = 0$).

Obs. If $\exists$ a.c. seq V_n in $[\mathcal{G}_0]$ asymp. commuting with $s(\mathcal{I}_0 \times \mathcal{R}_1)$, then it will be a.c. in $[G \ltimes X]$. (This is enough for G to be stable.)

★ Let V_n be an a.c. seq in $[\mathcal{R}_0]$. We have the 1-cocycle-like map $\sigma_{(V_n x, x)} \colon \mathcal{R}_1 \to C$ assigned to each $x \in X_0$. Then

$$\sigma_{(V_n x, x)} \approx 0 \iff V_n \text{ asymp. commutes with } s(\mathcal{I}_0 \times \mathcal{R}_1).$$

The problem is reduced to:

Can one find an a.c. seq V_n in $[\mathcal{R}_0]$ with $\sigma_{(V_n x, x)} \approx 0$?

In general:

Obs (very simple). Let $\Gamma \curvearrowright X$ be an action and $\alpha\colon \Gamma \times X \to C$ a 1-cocycle. We define $\Gamma \curvearrowright \widetilde{X} := X \times C$ by $\gamma(x,c) = (\gamma x, \alpha(\gamma,x)c)$. Then $\alpha \sim 0$ on $\Gamma \ltimes \widetilde{X}$.

Pf. Define $\varphi\colon \widetilde{X} \to C$ by $\varphi(x,c) = c$. Then $\alpha(\gamma,x) = \varphi(\gamma\tilde{x})\varphi(\tilde{x})^{-1}$ for $\tilde{x} = (x,c) \in \widetilde{X}$. $\square$

Thm. Let $C < G$ be central and $\Gamma := G/C \curvearrowright X$ stable with an a.c. seq V_n. Then there exists an extension $\Gamma \curvearrowright \widetilde{X}$ of the action $\Gamma \curvearrowright X$ such that after passing to a subsequence, some lift of V_n to $G \ltimes \widetilde{X}$ is a.c. in $G \ltimes \widetilde{X}$.

We define $\sigma_n\colon \mathcal{I}_0 \times \mathcal{R}_1 \to C$ by $\sigma_n = \sigma_{(V_n x, x)}$ on $\{e_x\} \times \mathcal{R}_1$.

★★ $\exists\, \mathcal{S}_n < \mathcal{I}_0 \times \mathcal{R}_1$ s.t. $\sigma_n \restriction \mathcal{S}_n$ is a 1-cocycle and $\mathcal{S}_n \to \mathcal{I}_0 \times \mathcal{R}_1$ as $n \to \infty$.

Let $\tau\colon \mathcal{R} \to \Gamma$ be the projection. The subrelation $\mathcal{S}_n$ is given by:

$$\{\, (e_x, (y', y)) \mid \gamma V_n(x, y) = V_n \gamma(x, y), \text{ where } \gamma := \tau(e_x, (y', y)) \,\}.$$

The proof that $\sigma_n \restriction \mathcal{S}_n$ is a 1-cocycle follows from the equation

$$\sigma_x(h, k)\sigma_n(hk)^{-1}\sigma_n(h)\sigma_n(k) = \sigma_{x'}(h, k), \text{ where } x' := V_n x,$$

and that our 2-cocycle comes from the central *group*-extension.

More involved: co-induced actions of equivalence relations, approximation of the cocycle $\sigma_n \restriction \mathcal{S}_n$ by cocycles independent of X_0, etc.

Question (a natural generalization of Thm). Let $\mathcal{R}$ be a stable equivalence relation and C an abelian group. Given $\sigma \in Z^2(\mathcal{R}, C)$, can one find an action $\mathcal{R} \curvearrowright \widetilde{X}$ such that if $\widetilde{\mathcal{R}}$ denotes the equivalence relation for that action, then the central extension of $\widetilde{\mathcal{R}}$ associated with σ is stable?