Actions of amenable groups on the Cantor set and their crossed products

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Basic objects

Our basic objects of study will be free minimal actions

$$G \curvearrowright X$$

of countably infinite amenable groups on the Cantor set, in particular those which are uniquely ergodic.

Question

When is $C(X) \rtimes G$ classifiable?

Classifiability

By results of Elliott-Gong-Lin-Niu and Tikuisis-White-Winter, and incorporating the Kirchberg-Phillips classification:

Theorem

The class of simple separable infinite-dimensional unital C^*-algebras which satisfy the UCT and have finite nuclear dimension is classified by the Elliott invariant.

Since crossed products by actions of amenable groups satisfy the UCT by a theorem of Tu, the question of whether $C(X) \rtimes G$ is classifiable thus boils down to the problem of whether it has **finite nuclear dimension**.

Nuclear dimension

The **nuclear dimension** $\dim_{\mathrm{nuc}}(A)$ of a C*-algebra A is the least integer $d \geq 0$ such that for all $\Omega \Subset A$ and $\delta > 0$ there exists an (Ω, δ)-commuting diagram

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\varphi"'] & & A \\ & F_0 \oplus \cdots \oplus F_d \arrow[ur, "\psi"'] & \end{tikzcd}$$

such that the F_i are finite-dimensional C*-algebras, φ is a c.p.c. map, and $\psi|_{F_i}$ is an order-zero c.p.c. map for each i. If no such d exists, we define it to be ∞.

Nuclear dimension

Two possible methods for showing $\dim_{\mathrm{nuc}}(C(X) \rtimes G) < \infty$:

1. Develop an analogous notion of dimension for dynamics and establish an inequality relating it to nuclear dimension.
2. Verify $\mathcal{Z}$-stability, which is known to imply finite nuclear dimension when the extreme tracial states form a nonempty compact set.

$\mathcal{Z}$-stability

Theorem (Hirshberg-Orovitz)

Let A be a simple separable unital nuclear C^*-algebra. Suppose that for every $n \in \mathbb{N}$, $\Omega \Subset A$, and $\varepsilon > 0$ there exist an order-zero c.p.c. map $\varphi : M_n \to A$ and a $v \in A$ such that

1. $vv^* = 1_A - \varphi(1_{M_n})$,
2. $v^*v \leq \varphi(e_{11})$,
3. $\|[a, \varphi(b)]\| < \varepsilon$ *for all* $a \in \Omega$ *and norm-one* $b \in M_n$.

Then A is $\mathcal{Z}$-stable.

Strict comparison

Definition

Let $G \curvearrowright X$ be an action on the Cantor set. Let A and B be clopen subsets of X. We say that A is **subequivalent** to B if there are a clopen partition $\{A_1, \dots, A_n\}$ of A and $s_1, \dots, s_n \in G$ such that the sets $s_1 A_1, \dots, s_n A_n$ are pairwise disjoint and contained in B.

Definition

An action $G \curvearrowright X$ on the Cantor set is said to have **strict comparison** if, for all clopen sets $A, B \subseteq X$, A is subequivalent to B whenever $\mu(A) < \mu(B)$ for all G-invariant Borel probability measures μ on X.

Strict comparison

Proposition (Glasner-Weiss)

A minimal $\mathbb{Z}$-action on the Cantor set has strict comparison.

Strict comparison

An action $G \curvearrowright X$ on a compact space is **strictly ergodic** if its minimal and uniquely ergodic.

Theorem

Let $G \curvearrowright X$ be a strictly ergodic free action of a countably infinite amenable group on the Cantor set. Suppose that the action has strict comparison. Then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

Castles

Let $G \curvearrowright X$ be an action on a set.

A **tower** is a pair (S, B) where $B \subseteq X$ and $S \Subset G$ are such that the sets sB for $s \in S$ are pairwise disjoint.

The set B is the **base** of the tower, the set S its **shape**, and the sets sB for $s \in S$ its **levels**.

A **castle** is a finite collection $\{(S_i, B_i)\}_{i=1}^{n}$ of towers such that the sets $S_i B_i$ are pairwise disjoint.

Approximate invariance

Let G be a discrete group. Let $F \Subset G$ and $\delta > 0$. We say that a set $A \Subset G$ is (F, δ)-*invariant* if

$$\frac{|FA \Delta A|}{|A|} < \delta.$$

When $e \in F$ this is the same as $|FA| < (1 + \delta)|A|$. It implies that

$$|\{s \in A : Fs \subseteq A\}| < (1 + |F|\delta)|A|.$$

Castles

Theorem (Ornstein-Weiss)

Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of a countably infinite amenable group. Let $F \Subset G$ and $\delta, \varepsilon > 0$. Then there exists a measurable castle whose shapes are (F, δ)-invariant and whose levels have union of measure at least $1 - \varepsilon$.

Castles

Proposition

Let $G \curvearrowright X$ be a free minimal action of a countably infinite amenable group on the Cantor set and let μ be a G-invariant Borel probability measure on X. Let $F \Subset G$ and $\delta, \varepsilon > 0$. Then there exists a clopen castle whose shapes are (F, δ)-invariant and whose levels have union of μ-measure at least $1 - \varepsilon$.

Jewett-Krieger theorem

A **topological model** for a p.m.p. action $G \curvearrowright (X, \mu)$ is an action $G \curvearrowright Y$ on a compact space and a G-invariant regular Borel probability measure ν on Y such that the actions $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are measure conjugate.

Theorem (Jewett, Krieger)

Every ergodic p.m.p. transformation has a strictly ergodic topological model.

Jewett-Krieger theorem

Theorem

Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of a countable amenable group and let H be a subgroup of G isomorphic to $\mathbb{Z}$ such that the restriction $H \curvearrowright (X, \mu)$ is ergodic. Then there is a strictly ergodic free topological model $G \curvearrowright Y$ for $G \curvearrowright (X, \mu)$ such that the restriction $H \curvearrowright Y$ is strictly ergodic.

The proof uses tiling technology, to which we will shortly turn.

Classifiability

Theorem
Let $G \curvearrowright X$ be a strictly ergodic free action of a countable amenable group on the Cantor set. Suppose that there is a subgroup $H \subseteq G$ isomorphic to $\mathbb{Z}$ such that the restriction $H \curvearrowright X$ is strictly ergodic. Then the action $G \curvearrowright X$ has strict comparison, and hence $C(X) \rtimes G$ is $\mathcal{Z}$-stable by a previous theorem.

Combining the previous two theorems:

Theorem
Let G be a nontorsion countably infinite amenable group. Then there is a strictly ergodic free action $G \curvearrowright X$ on the Cantor set such that $C(X) \rtimes G$ is classifiable.

Tilings of amenable groups

Theorem (Ornstein-Weiss)

*Let $\varepsilon > 0$. Let $F \Subset G$ and $\delta > 0$. Then there exist (F, δ)-invariant shapes $S_1, \ldots, S_n \Subset G$ which ε-**quasitile** every sufficiently left invariant set $A \Subset G$.*

This means there exist sets $C_i \Subset G$ (tile centres) such that the tiles $S_i c$ for $i = 1, \ldots, n$ and $c \in C_i$

(i) *are ε-disjoint and*

(ii) *proportionally cover all but ε of A.*

Tilings of amenable groups

Theorem (Weiss)

Suppose that G is residually finite and amenable. Let $F \Subset G$ and $\delta > 0$. Then there is an (F, δ)-invariant set $S \Subset G$ and a finite-index normal subgroup $N \subseteq G$ such that

$$G = \bigsqcup_{t \in N} St.$$

Tilings of amenable groups

Theorem (Downarowicz-Huczek-Zhang)

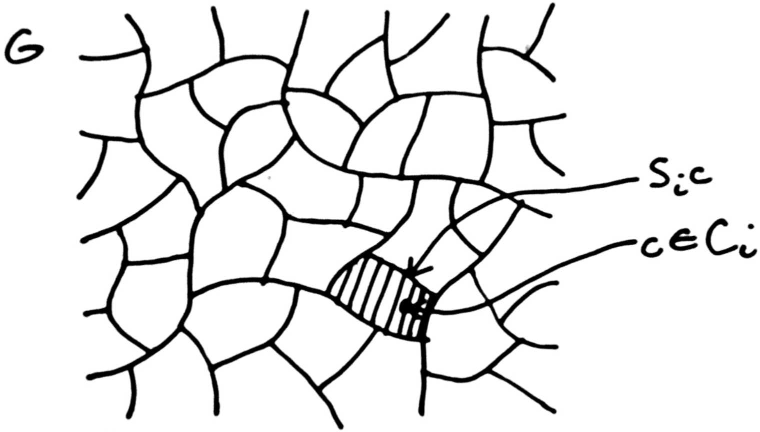
Suppose that G is amenable. Let $F \Subset G$ and $\delta > 0$. Then there is a tiling of G by translates of finitely many (F, δ)-invariant shapes.

In other words, we can write

$$G = \bigsqcup_{i=1}^{n} \bigsqcup_{c \in C_i} S_i c$$

where each S_i is a finite (F, δ)-invariant set.

Tilings of amenable groups



Tilings of amenable groups

Definition

Let $L \subseteq G$ and $F \Subset G$. Let $\{F_n\}$ be a Følner sequence for G. Define the *lower Banach density* of L by

$$\underline{D}(L) = \limsup_{n\to\infty} \inf_{s\in G} \frac{|L \cap F_n s|}{|F_n|}$$

The *upper Banach density* is defined similarly.

Proposition

The above limit supremum is in fact a limit and it doesn't depend on the Følner sequence. Similarly for upper Banach density.

Tilings of amenable groups

Theorem (Rado)

Let B and A be sets, and let $b \mapsto F_b$ be an assignment to each element of B a finite subset of A such that

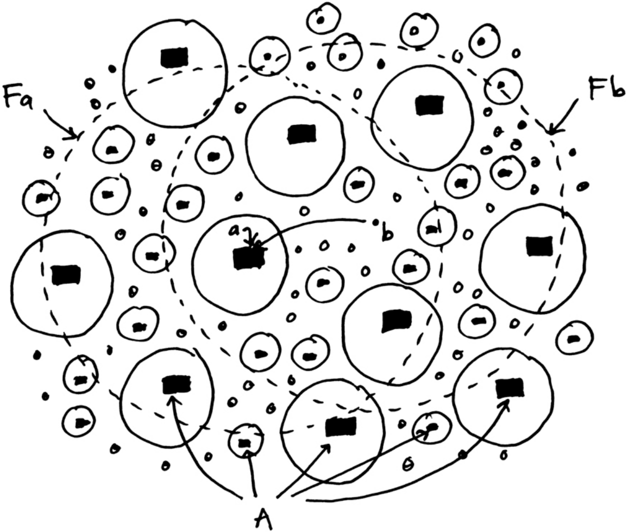
$$|E| \leq \left| \bigcup_{b \in E} F_b \right|$$

for every finite set $E \subseteq B$. Then there is an injection

$$\varphi : B \to A$$

such that $\varphi(b) \subseteq F_b$ for all $b \in B$.

Tilings of amenable groups



Castles

Theorem (Ornstein-Weiss)

Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of a countably infinite amenable group. Let $F \Subset G$ and $\delta > 0$. Let $\varepsilon > 0$. Then there exists a measurable castle whose shapes are (F, δ)-invariant and whose levels have union of measure at least $1 - \varepsilon$.

Castles

Theorem

Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of a countably infinite amenable group. Let $F \Subset G$ and $\delta > 0$. Let $\varepsilon > 0$. Then there exists a measurable castle whose shapes are (F, δ)-invariant and whose levels have conull union.

One can use this to establish the more general versions of the Jewett-Krieger theorem.

Generic free minimal actions

Theorem

Let G be a countably infinite amenable group. Then a generic free minimal action $G \curvearrowright X$ admits a sequence of clopen castles which partition X and have shapes which become more and more invariant.

Generic free minimal actions

Proposition

Suppose that the action $G \curvearrowright X$ admits a sequence of clopen castles which partition X and have shapes which become more and more invariant. Then $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

Theorem

Let G be a countably infinite amenable group. Then for a generic free minimal action $G \curvearrowright X$ the crossed product $C(X) \rtimes G$ is $\mathcal{Z}$-stable.

Actions as in the proposition have strict comparison. Does every free minimal action on the Cantor set have strict comparison?