Orbifold subfactors, central sequences and the relative Jones invariant κ

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Abstract

We introduce the relative version of the Jones invariant κ as a quadratic form over the relative Connes invariant $\chi(M, N)$ for a subfactor $N \subset M$ and study its basic properties. Among several properties, we prove that in the quantum $SU(n)_k$-orbifold constructions for subfactors, the flatness of the resulting connection is equivalent to the triviality of the relative κ of the original quantum $SU(n)_k$-subfactor.

1 Introduction

Our aim is to introduce the relative version of the Jones invariant κ for a subfactor $N \subset M$ as a quadratic form over the relative Connes invariant $\chi(M, N)$ by us in [21] and study its relation to the orbifold subfactors by us in [19], [7]. In particular, we prove that in the quantum $SU(n)_k$-orbifold construction for subfactors, the flatness of the resulting connection is equivalent to the triviality of the relative κ of the $SU(n)_k$-subfactor.

The subfactor theory initiated by V. F. R. Jones [17] has opened an exciting new series of interactions between the theory of operator algebras and other fields such as topology, quantum group theory, conformal field theory, and statistical mechanics. We here work on an interplay between analytic aspects of the subfactor theory and rational conformal field theory. A combinatorial machinery we use to study subfactors is Ocneanu's paragroups [27]. (See [7]–[10], [19]–[22], [28]–[34] for the paragroup theory.)

A systematic study of automorphisms of subfactors was initiated by Loi [26]. For a subfactor $N \subset M$, Loi introduced subfactor versions of two important classes of automorphisms; approximately inner automorphisms $\overline{\mathrm{Int}}(M, N)$ and centrally trivial automorphisms $\mathrm{Ct}(M, N)$. (His original classification problem of automorphisms of subfactors has been solved by S. Popa [40] based on Popa's deep classification result [39].) With these two classes, a relative version of the Connes invariant $\chi(M)$ in [3] was introduced by us in [21] as follows.

$$\chi(M,N) = \frac{\mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)}{\mathrm{Int}(M,N)},$$

where $\mathrm{Int}(M,N)$ is a class of inner automorphisms implemented by the unitaries of N.

One of the main results in [21] has revealed a relation between the relative Connes invariant χ and the orbifold construction for subfactors. Here we briefly recall the orbifold construction. This is a method to construct a new paragroup from a paragroup with a certain symmetry. That is, we construct a subfactor $N \times G \subset M \times G$ from a given subfactor $N \subset M$ and a certain action of a finite group G on the subfactor $N \subset M$. It was first used by us in [19] in order to realize principal graphs D_{2n} as well as to show impossibility of principal graphs D_{2n+1}. (This result on the Dynkin diagrams D_n was announced by Ocneanu [27] first. See the Appendix of [20] for his original proof, which is quite different.) The orbifold construction for the Dynkin diagrams A_{4n-3} gives D_{2n} and that for A_{4n-1} gives A_{4n-1} unchanged. This difference is called an obstruction for flatness in the orbifold construction. (The flatness is one of the axioms of paragroups.) That is, the obstruction prevent A_{4n-1} from changing into D_{2n+1} while vanishing of this obstruction makes A_{4n-3} change into D_{2n}. The orbifold construction has then been established as a general method in [7] and its relation to rational conformal field theory has been found in [47]. That is, the difference between the Dynkin diagrams D_{2n} and D_{2n+1} is clearly understood with the conformal dimensions of the Wess-Zumino-Witten model $SU(2)_k$. Also see [12], [13], [48] for more results on the orbifold construction.

In [21] and [9], we have seen that for the quantum $SU(n)_k$ subfactors $N \subset M$ arising from the WZW models $SU(n)_k$, the relative Connes invariant $\chi(M,N)$ is equal to $\mathbf{Z}_d$ with $d = (n,k)$ and that this $\mathbf{Z}_d$ action is exactly the one we use in the orbifold construction. In particular, the result for $SU(2)_k$ means the following. For an approximately finite dimensional (AFD) subfactor $N \subset M$ with principal graph A_n ($n > 3$), the relative Connes invariant $\chi(M,N)$ is $\mathbf{Z}_2$ for odd n and 0 for even n. That is, the relative Connes invariant χ does not see the obstruction for flatness in the orbifold construction. This is rather unsatisfactory because it is a general belief that algebraic property (such as flatness) should be equivalent to analytic property of ultraproducts/central sequences as long as we have a certain amenability condition such as strong amenability of subfactors. Our aim in this paper is to remove this insufficiency by introducing the relative Jones invariant κ as a finer invariant than the relative Connes invariant χ.

We also study a relation between the orbifold construction and the central subfactor construction of Ocneanu [27].

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2 Definition and basic properties

Let $N \subset M$ be a subfactor of type II_1 with finite index. (Later we work on AFD II_1 factors, but we do not need this assumption for a general definition.) We first define the relative Jones invariant κ as a quadratic form over the relative Connes invariant χ, which is always an abelian group.

The Jones invariant κ was introduced in [15], [16] in a single factor case. We make the following definition based on the single factor theory in [2], [15], [16], [42].

Choose $\alpha, \beta \in \mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)$. By $\beta \in \overline{\mathrm{Int}}(M,N)$, we have unitaries $\{u_n\}_n$ in N with $\beta = \lim_{n\to\infty} \mathrm{Ad}(u_n)$. We look at $\alpha \cdot \beta \cdot \alpha^{-1}$. On one hand, this is equal to $\mathrm{Ad}(u) \cdot \beta = \lim_{n\to\infty} \mathrm{Ad}(uu_n)$ for some unitary $u \in N$ because $\chi(M,N)$ is abelian, and on the other hand this is clearly equal to $\lim_{n\to\infty} \mathrm{Ad}(\alpha(u_n))$. These imply that the sequence $\{u_n^* u^* \alpha(u_n)\}_n$ is central in M. We also know that this sequence is a Cauchy sequence by the following estimate.

$$\begin{aligned} \|u_n^* u^* \alpha(u_n) - u_m^* u^* \alpha(u_m)\|_2 &= \|u_m u_n^* u^* \alpha(u_n u_m^*) - u^*\|_2 \\ &\to 0, \qquad \text{as } n, m \to \infty, \end{aligned}$$

because of the central triviality of α. Thus there exists a scalar $\kappa(a,\beta)$ with modulus one such that $\lim_{n\to\infty} u_n^* u^* \alpha(u_n) = \kappa(\alpha,\beta)$. A standard argument shows that this number $\kappa(\alpha,\beta)$ does not depend on the choice of u and $\{u_n\}_n$. A direct computation using $\alpha \cdot \beta \cdot \alpha^{-1} = \lim_{n\to\infty} \mathrm{Ad}(uu_n)$ shows that $\kappa(a,\beta) = \kappa(\mathrm{Ad}(v)\cdot\alpha, \mathrm{Ad}(w)\cdot\beta)$ for unitaries $v, w \in N$, thus κ is a well-defined map from $\chi(M,N) \times \chi(M,N)$ to $\mathbf{T}$, the set of complex numbers with modulus one.

We also define $\kappa(\alpha) = \kappa(\alpha,\alpha)$ for $\alpha \in \chi(M,N)$. Then we have the following lemma in the exactly same way as in [2, Proposition 1.7], which means that this κ is a quadratic form over $\chi(M,N)$.

Lemma 2.1 *1. For $\alpha \in \chi(M,N)$, we have $\kappa(\alpha) = \kappa(\alpha^{-1})$.*

2. The form $b_\kappa(\alpha,\beta) = \kappa(\alpha\beta)\overline{\kappa(\alpha)\kappa(\beta)}$ is symmetric and bilinear.

All the general results in [2] hold in our subfactor settings.

We note that as in single factor cases, we can define κ with ultraproducts as follows. We fix a free ultrafilter ω over $\mathbf{N}$. Let U be a unitary in N^ω implementing an approximately inner automorphism β. Then we get $a(U) = \kappa(\alpha,\beta)uU$ as above.

It is also possible the relative version of the invariant Ω as the obstruction of the kernel $\chi(M,N) \to \mathrm{Aut}(M,N)/\mathrm{Int}(M,N)$, but we omit details because all the concrete examples below arising from the orbifold construction have trivial Ω.

3 Orbifold subfactors and the relative κ

Let $SU(n)_k$ be the Wess-Zumino-Witten model for $SU(n)$ with level k. Then we can construct a paragroup from it as in [1] and thus we have a corresponding subfactor

$N \subset M$ of the AFD II_1 factor. These subfactors are isomorphic to the Wenzl subfactors arising from Hecke algebras of type A [45]. Based on [47], it has been shown in [9, Section 6] that $\chi(M,N)$ for this subfactor is $\mathbf{Z}_d$, where $d=(n,k)$. In these cases, the $\mathbf{Z}_d$ action is realized in a concrete way as in [7] and thus we know that the relative obstruction Ω is trivial. In particular, in the Definition of κ, we can take the unitary u to be 1. We will compute κ for these subfactors and identify κ with the obstruction for flatness in the orbifold construction. We fix n,k with $d>1$. Thus we have a subfactor $N\subset M$. We denote the global index of $N\subset M$ by $\tilde\tau$, which is the summation of the normalized Perron-Frobenius weights $\mu(x)^2$ for all the even vertices x of the principal graph. (See [27], [28], [21], [10] for a general theory of the global index.)

We denote the $\mathbf{Z}_d$ action on $N\subset M$ by α. Xu's results in [47], [48] are summarized as follows.

Proposition 3.1 *In the $\mathbf{Z}_d$-orbifold construction for the quantum $SU(n)_k$ subfactor, the resulting connection is flat if and only if d is odd or $2d$ divides k.*

We have the following lemma.

Lemma 3.2 *The global index of the orbifold subfactor $N\times_\alpha \mathbf{Z}_d \subset M\times_\alpha \mathbf{Z}_d$ is given by $\tilde\tau/d$ if d is odd or $2d$ divides k and by $2\tilde\tau/d$ if d is even and $2d$ does not divide k.*

Proof: The principal graph of the orbifold subfactor is computed from d,k as in [47], [48]. Then the description of the principal graph in [7, Figure 1.8] gives the global indices as desired. Q.E.D.

Our next aim is to compute $(N\times_\alpha\mathbf{Z}_d)^\omega\cap(M\times_\alpha\mathbf{Z}_d)'$ and $(M\times_\alpha\mathbf{Z}_d)^\omega\cap(M\times_\alpha\mathbf{Z}_d)'$ for a fixed free ultrafilter ω over $\mathbf{N}$. We first compute $(N\times_\alpha\mathbf{Z}_d)^\omega\cap M'$ and $(M\times_\alpha\mathbf{Z}_d)^\omega\cap M'$. Choose a sequence of unitaries $\{u_n\}_n$ in N so that $\alpha_1=\lim_{n\to\infty}\mathrm{Ad}(u_n)$. We denote by U the element in N^ω corresponding to the sequence $\{u_n\}_n$. We denote by u the implementing unitary for α_1 in the crossed products. A general element in $N\times_\alpha\mathbf{Z}_d$ is written as $\sum_{j=0}^{d-1}X_j u^j$ where $X_j\in N^\omega$. Suppose this element commutes with M. Then we have $X_j u^j x = x X_j u^j$ for all $x\in M$ and $j=0,1,\dots d-1$. Because $uxu^*=\alpha_1(x)=UxU^*$ for $x\in M$, the condition we have is $X_jU^j\in N^\omega\cap M'$. So a general element in $(N\times_\alpha\mathbf{Z}_d)^\omega\cap M'$ is expressed as $\sum_{j=0}^{d-1}Y_j(U^*u)^j$, where each Y_j is in $N^\omega\cap M'$. Similarly, we know that a general element in $(M\times_\alpha\mathbf{Z}_d)^\omega\cap M'$ is expressed as $\sum_{j=0}^{d-1}Z_j(U^*u)^j$, where each Z_j is in $M^\omega\cap M'$.

Next we compute $(N\times_\alpha\mathbf{Z}_d)^\omega\cap(M\times_\alpha\mathbf{Z}_d)'$. Let $\kappa=\kappa(\alpha_1,\alpha_1)$. Then we have $u(U^*u)u^*=\bar\kappa$ and $\kappa^d=1$. Let l be the minimal positive integer with $\kappa^l=1$. Note that the quadratic form $\kappa(\cdot,\cdot)$ is trivial if and only if $l=1$. Then a general element in $(N\times_\alpha\mathbf{Z}_d)^\omega\cap(M\times_\alpha\mathbf{Z}_d)'$ is expressed as $\sum_{j=0}^{d/l-1}Y_j(U^*u)^{jl}$, where each Y_j is in $N^\omega\cap M'$.

We need the following lemma.

Lemma 3.3 *The action $\mathrm{Ad}(U^*u)$ gives an automorphism σ of $N^\omega\cap M'\subset M_\omega$ and σ^j is outer for $j=1,2,\dots,d-1$ on both factors $N^\omega\cap M'$ and M_ω.*

Proof: The action $\mathrm{Ad}(u)$ acts trivially on $N^\omega \cap M'$ and freely on M_ω. Because $UNU^* = N$ and $UMU^* = M$, it is clear that σ gives an automorphism of $N^\omega \cap M' \subset M_\omega$.

Suppose that σ^j is inner on $N^\omega \cap M'$ for some j with $0 < j < d$. Then we have a unitary $V \in N^\omega \cap M'$ with $VXV^* = U^{-j}XU^j$ for all $X \in N^\omega \cap M'$. Then we have $U^jV \in (N^\omega \cap M')' \cap N^\omega$. By Lemma 3.3 in [18] and the proof of the Central Freedom Lemma in [21], we get $(N^\omega \cap M')' \cap N^\omega$. It means we have a unitary $v \in N$ such that

$$\mathrm{Ad}(v)(x) = \mathrm{Ad}(U^jV)(x) = \mathrm{Ad}(U^j)(x) = \alpha_j(x),$$

for all $x \in M$, which contradicts the freeness of α.

The freeness of σ on M_ω is proved similarly. Q.E.D.

By Lemma 3.3, we get

$$[(N \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' : N^\omega \cap M'] = \frac{d}{l}.$$

By [4], the action α acts M_ω freely and we get $[M_\omega : M_\omega^\alpha] = d$ and

$$[(M \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' : M_\omega^\alpha] = d.$$

By the central triviality of α, we get

$$\begin{aligned} &N^\omega \cap M' \subset M_\omega^\alpha \subset (M \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)', \\ &N^\omega \cap M' \subset (N \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' \subset (M \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)'. \end{aligned}$$

Because $[M_\omega^\alpha : N^\omega \cap M'] = \tilde{\tau}/d$, the identity

$$\begin{aligned} & [(M \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' : M_\omega^\alpha][M_\omega^\alpha : N^\omega \cap M'] \\ = & [(M \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' : (N \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)'] \\ & \times [(N \times_\alpha \mathbf{Z}_d)^\omega \cap (M \times_\alpha \mathbf{Z}_d)' : N^\omega \cap M'], \end{aligned}$$

together with Lemma 3.2 and Proposition 3.1 implies that if the resulting connection in the orbifold construction is flat, then $l = 1$ and that if the resulting connection in the orbifold construction is not flat, then $l = 2$. Thus we have the following theorem.

Theorem 3.4 *For the quantum $SU(n)_k$ subfactor $N \subset M$, which has $\chi(M, N) = \mathbf{Z}_d$ with $d = (k, n)$, the relative Jones invariant κ is trivial if and only if the resulting connection in the $\mathbf{Z}_d$-orbifold construction is flat.*

For other orbifold subfactors arising from a connected, simply connected, compact and simple Lie group G as in [47], we have a similar result.

4 Relation to the quantum double construction

Here we discuss the result in the previous section from the viewpoint of topological quantum field theory (TQFT) and rational conformal field theory (RCFT).

A. Ocneanu has found that a paragroup gives a Turaev-Viro type TQFT in three dimensions. (See [30], [32], [8].) He further discovered that the system of M_∞-M_∞ bimodules of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ realizes combinatorial data of RCFT related to this TQFT. (See [31], [32], [33], [34], [10].) He had noticed that the asymptotic inclusion and the central sequence subfactor as above give essentially same paragroups from a given paragroup. (See [29], [21], and the Appendix below.) These constructions give the quantum double in the sense of Drinfel′d [6] if the original paragroup is really a finite group, as noticed by Ocneanu. (See [34], [10], [5], [24] and the Appendix below.) These altogether mean that the central sequence subfactor construction is an analytic subfactor analogue of the quantum double construction. Roughly speaking, this is a machinery to produce a higher symmetry from an original algebraic data. For example, the Yang-Baxter equation is one expression of such a higher symmetry. (See [31], [34], [10].)

Let $N \subset M$ be a subfactor of the AFD II_1 factor corresponding to the WZW-model $SU(n)_k$ as above. Let α be the action of $\mathbf{Z}_d$ used in the orbifold construction, where $d = (k, n)$. We set $P = N^\omega \cap M'$ and $Q = M_\omega$ for a fixed free ultrafilter ω over $\mathbf{N}$. Then the central sequence subfactor $P \subset Q$ has an intermediate subfactor Q^α because of the central triviality of α. (This was a key observation in [21].) From the above viewpoint related to the quantum double, the subfactor $P \subset Q$ should be a very nice subfactor, and then it is rather unsatisfactory that it has a "classical" intermediate subfactor $Q^\alpha \subset Q$. So it is tempting to look at the subfactor $P \subset Q^\alpha$ by removing the "classical symmetry". This step does not require triviality of the relative κ. Also note that this new subfactor $P \subset Q^\alpha$ has the same index value as the global index of the orbifold subfactor $N \times_\alpha \mathbf{Z}_d \subset M \times_\alpha \mathbf{Z}_d$ if and only if the relative κ is trivial by Lemma 3.2. This suggests that the central sequence subfactor of the orbifold subfactor might have the same paragroup as $P \subset Q^\alpha$, but this is not the case. To get the correct description of the central sequence subfactor of the orbifold construction, we need the following lemma.

Lemma 4.1 *The action* $\mathrm{Ad}(U^* u)$ *gives an automorphism* σ *of a factor* Q^α *and* σ^j *is outer for* $j = 1, 2, \ldots, d-1$.

Proof: For $X \in Q^\alpha$, we have $\sigma(X) = U^* X U$. Because $\kappa = \pm 1$ now, we get $U^* X U \in Q^\alpha$.

Suppose we have $\sigma^j = \mathrm{Ad}(V)$ for some unitary $V \in Q^\alpha$ on Q^α for some $j = 1, 2, \ldots, d-1$. Then we get

$$U^j V \in (Q^\alpha)' \cap M^\omega = (M^\omega \cap (M \times_\alpha \mathbf{Z}_d)')' \cap M^\omega = M,$$

as in the proof of Lemma 3.3. This again gives a contradiction. Q.E.D.

We now assume that d is odd or $2d$ divides k, which is the condition for the flatness in the orbifold construction and the triviality of the relative κ as in Theorem 3.4. Then Lemmas 3.3 and 4.1 imply the following theorem. The assumption on d, k is necessary to get $Q^\alpha \times_\sigma \mathbf{Z}_d$.

Theorem 4.2 *Under the above assumptions, the central sequence subfactor of the orbifold subfactor $N \times_\alpha \mathbf{Z}_d \subset M \times_\alpha \mathbf{Z}_d$ is given as $P \times_\sigma \mathbf{Z}_d \subset Q^\alpha \times_\sigma \mathbf{Z}_d$.*

The above theorem means that the central sequence subfactor of the simultaneous crossed product subfactor is given as the simultaneous crossed product subfactor of the central sequence subfactor with a "classical" intermediate subfactor removed. In short, the removal of the classical intermediate subfactor corresponding to the cyclic group $\mathbf{Z}_d$ appears as the "commutator" of two operations; the simultaneous crossed product (orbifold construction) and the central sequence subfactor. The flatness condition that d is odd or $2d$ divides k is necessary to keep a "higher symmetry" in this procedure.

In the above viewpoint related to the quantum double, we can say that the orbifold construction removes a redundant classical symmetry of $\mathbf{Z}_d$-type from a quantum subfactor and that the flatness condition is required to keep a high symmetry of the quantum double type in this procedure. This "removal" is also related to the elimination of degeneracy of the finite systems of N-N bimodules of the quantum $SU(n)_k$ subfactors in the sense of [34]. For example, the A_{odd} subfactors have degenerate systems of bimodules, and the orbifold construction tries to eliminate this degeneracy. This is successful if and only if the obstruction for flatness vanishes.

5 Appendix

The aim of this Appendix is to include proofs of two theorems obtained by A. Ocneanu in [29]. These two statements are logically independent of the main body of this paper, but give clear motivation of this work by showing that a genuine analytic construction of the central sequence subfactor can be regarded as an analogue of the quantum double construction of Drinfel′d [6] as in Section 4.

First, we recall the setting of [21]. Let $N \subset M$ be an irreducible AFD subfactor of type II_1 with finite index and finite depth. Let ω be a free ultrafilter over $\mathbf{N}$. We compare the higher relative commutants of the central sequence subfactor $N^\omega \cap M' \subset M_\omega$ and the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. (These two constructions were introduced in [27]. See [29], [21] for details.)

Our first aim is to give a proof of the following Theorem.

Theorem 5.1 *The dual canonical commuting square of the central sequence subfactor $N^\omega \cap M' \subset M_\omega$ is anti-isomorphic to the canonical commuting square of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$.*

This was claimed in Ocneanu's Tokyo lectures in 1990 [29] with a brief sketch of an outline of his proof. This is also one of the two statements left without a complete proof in [21, Remark 2.16]. (A complete proof of the other statement left in [21, Remark 2.16] was supplied in [10, Section 3] based on [33].)

In the above statement, the word "dual" means that we make a basic construction once. This word was missing in [29, page 42] by a fault of the recorder (the author of this paper) as pointed out in [21, Remark 2.16]. The last statement of this Remark 2.16 in [21] was also slightly incorrect, because the correct form should have an "anti"-isomorphism, not an isomorphism. In short in the paragroup terminology, we can say that the central sequence subfactor and the asymptotic inclusion give mutually dual and opposite paragroups.

It was already proved in [21, Lemma 2.14] that the dual canonical commuting square of the central sequence subfactor is contained in the canonical commuting square of the asymptotic inclusion with a trace-preserving injective anti-homomorphism, so we only have to prove the converse inclusion.

We recall some notations in [21]. We use $\tilde{\tau}$ for the index $[M_\omega : N^\omega \cap M']$, which is the global index of the original subfactor $N \subset M$. We choose a generating tunnel

$$\cdots \subset M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots$$

and set $A_{k,l} = M_k' \cap M_l$, $A_{k,\infty} = \bigvee_l A_{k,l} = M_k' \cap M_\infty$, $A_{-\infty,l} = \bigvee_k A_{k,l} = M_l$, and $A_{-\infty,\infty} = \bigvee_{k,l} A_{k,l} = M_\infty$. We also set $P_0 = N^\omega \cap M'$, $P_1 = M_\omega$. We construct the Jones tower $P_0 \subset P_1 \subset P_2 \subset P_3 \cdots$ within M_∞^ω as in [21, Lemma 2.13]. We denote the Jones projection for the subfactor $P_0 \subset P_1$ by $\tilde{e}$. For a general subalgebra R of M_∞^ω, we write R^c for $R' \cap M_\infty^\omega$. By [21, Lemma 2.13], the sequence $\cdots P_3^c \subset P_2^c \subset P_1^c \subset P_0^c$ is a tunnel.

We have $P_0^c = \bigvee_k A_{-k,\infty}^\omega$ by [21, Lemma 2.9] and $P_1^c = M \vee (M' \cap M_\infty)^\omega = \bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega$ by the Central Freedom Lemma, [21, Lemma 2.2]. With the double commutant theorem in M_∞ [27, page 137] (also see [39]) and the Central Freedom Lemma, we get $P_1^{cc} = P_1$ and $P_0^{cc} = \bigcap_{k \geq 0} M_{-k}^\omega = P_0$. Choose a Jones projection $f \in P_1$, i.e., $E_{P_0}(f) = \tilde{\tau}^{-1}$. We then need the following lemma.

Lemma 5.2 *In the above context, the inclusion $P_1^c \subset P_0^c \subset \langle P_0^c, f \rangle$ is standard.*

Proof: First note that $E_{P_1^c}(\tilde{e}) = \tilde{\tau}^{-1}$. Because $P_2^c \subset P_1^c \subset P_0^c = \langle P_1^c, \tilde{e} \rangle$ is standard, a general element of P_0^c is a linear combination of elements of the form $x\tilde{e}y$ with $x, y \in P_1^c$. For such x, y, we get

$$f(x\tilde{e}y)f = xf\tilde{e}fy = \tilde{\tau}^{-1}xfy = fE_{P_1^c}(x\tilde{e}y).$$

Thus it is now enough to prove that the central support of f in $\langle P_0^c, f \rangle$ is 1. This is proved as in [21, Lemma 2.7]. Q.E.D.

We set $Q_1 = \langle P_0^c, f \rangle$. Then $P_1^c \subset P_0^c \subset Q_1$ and $Q_1^c \subset P_0^{cc} \subset P_1^{cc}$ are both standard. By [21, Lemma 2.12], we can construct a Jones tower

$$P_1^c \subset P_0^c \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset M_\infty^\omega$$

so that

$$\cdots Q_3^c \subset Q_2^c \subset Q_1^c \subset P_0^{cc} \subset P_1^{cc}$$

is a tunnel. We first have

$$P_0' \cap P_k \subset P_k^{cc} \cap P_0^c = (P_k^c)' \cap P_0^c.$$

With the trace preserving anti-isomorphism, we can identify $(P_k^c)' \cap P_0^c$ with $P_0^{cc} \cap Q_k$, which is contained in $P_0 \cap Q_k^{cc} = (Q_k^c)' \cap P_0$, which is again contained in $P_0' \cap P_k$ with the trace-preserving anti-isomorphism. This shows that $P_0' \cap P_k = (P_k^c)' \cap P_0^c$. With this and the results in [21], we get the Theorem as desired.

Our second object is to give a proof of the following theorem, which was also obtained by Ocneanu in [29].

Theorem 5.3 *Let N be an AFD factor of type II_1 and M be the crossed product $N \times G$ of N with an outer action of a finite group G. Then the central sequence subfactor $N^\omega \cap M' \subset M_\omega$ is of the form $Q \times G \in Q \times (G \times G)$, where Q is some factor of type II_1 with an outer action of $G \times G$ and G is embedded into $G \times G$ with a map $g \mapsto (g, g)$.*

Because a clear outline of a proof of this Theorem is already in [29], we will just fill its details for the sake of completeness.

We denote the implementing unitaries in M by λ_g, the projections in M_1 corresponding to the group elements by f_g, and the implementing unitaries in M_2 by ρ_g, where g is an element of G. Note that we have relations $\lambda_g f_h \lambda_g^* = f_{gh}$, $\rho_g f_h \rho_g^* = f_{hg^{-1}}$ for $g, h \in G$.

First we claim that $N^\omega \cap M' \subset M_\omega \subset \langle M_\omega, f_1 \rangle$ is standard. Note that for $x = (x_n) \in M_\omega$, we get $E_{N^\omega \cap M'}(x) = (E_N(x_n))$ because M is the crossed product by a group action. Thus we have $f_1 x f_1 = E_{N^\omega \cap M'}(x) f_1$ for $x \in M_\omega$. So it is enough (as in [21, Lemma 2.7]) to prove that the central support q of f_1 in $\langle M_\omega, f_1 \rangle$ is 1. The Central Freedom Lemma implies $(M_\omega)' \cap M_1^\omega = M \vee (M' \cap M_1)^\omega = M$, so we get $E_{(M_\omega)' \cap M_1}(f_1) = E_m(f_1) = 1/n$, where n is the order of the group G. Then we have $1/n = E_{(M_\omega)' \cap M_1}(f_1) = E_{(M_\omega)' \cap M_1}(q f_1) = q/n$, and the claim is proved.

Next, let $P = N' \cap M_1^\omega$. By the Central Freedom Lemma, we get $P' \cap P = (N \vee (M_1' \cap M_1')^\omega) \cap N' = \mathbf{C}$, so P is a factor. We define an action of $G \times G$ on P by $\mathrm{Ad}(\lambda_g \cdot \rho_h)$ for $(g, h) \in G \times G$ and claim that this action is outer. Suppose that there exists a unitary $U \in N' \cap M_1^\omega$ with $\mathrm{Ad}U = \mathrm{Ad}(\lambda_g \cdot \rho_h)$ for some $(g, h) \in G \times G$. Then $\lambda_g \rho_h U^* \in (N' \cap M_1^\omega)' \cap M_2^\omega = N$, where we used the Central Freedom Lemma again. This implies that U is in M_2 and hence in $N' \cap M_1^\omega \cap M_2 = N' \cap M_1$. By $\lambda_g \rho_h U^* \in N$ and $U \in M_1$, we get $\rho_h \in M_1$ and hence $h = 1$. By $\lambda_g U^* \in N$, we get $U \in M$ and then $U \in N' \cap M_1$ implies $U \in \mathbf{C}$ and $g = 1$, and thus the outerness claim is proved. We regard G as a subgroup of $G \times G$ with the diagonal embedding as in the Theorem.

It is easy to see that M_ω is equal to the fixed point algebra $P^{G \times G}$ with this action. We next claim that $P^G = \langle M_\omega, f_1 \rangle$. The inclusion $\langle M_\omega, f_1 \rangle \subset P^G$ is trivial. Because

$[P : P^G] = n$, it is enough to show $[P : \langle M_\omega, f_1\rangle] \leq n$. By the commuting square condition, we get $[N' \cap M_1^\omega : N' \cap M^\omega] \leq n$, and it is easy to see $[N' \cap M^\omega : M' \cap M^\omega] \leq n$. Thus we have $[N' \cap M_1^\omega : M' \cap M^\omega] \leq n^2$. Because $M' \cap M^\omega \subset \langle M_\omega, f_1\rangle \subset P$ and $[\langle M_\omega, f_1\rangle : M_\omega] = n$, we get $[P : \langle M_\omega, f_1\rangle] \leq n$, as desired.

With these, we have proved that the basic construction of the central sequence subfactor is of the form $P^{G\times G} \subset P^G$, which proves the Theorem.

With these two theorems, we can conclude that the paragroup of the asymptotic inclusion of $N \subset M = N \times G$ is given by $N^{G\times G} \subset N^G$.

(After this work, the author has learned an argument of M. Izumi with which one can compute the paragroup of the asymptotic inclusion of $N \subset M = N \times G$ directly.)

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