An operator algebra approach to the classification of certain fusion categories II

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August 2, 2016 at Sendai

$\mathrm{End}_0(M)$

Let M be a type III factor.
$\mathrm{End}_0(M)$ is a rigid tensor category with $\rho \otimes \sigma = \rho \circ \sigma$, and
$(\rho, \sigma) = \{T \in M;\ T\rho(x) = \sigma(x)T,\ \forall x \in M\}$.

Recall $\exists R_\rho \in (\mathrm{id}, \overline{\rho}\rho)$, $\exists \overline{R_\rho} \in (\mathrm{id}, \rho\overline{\rho})$ satisfying

$$\overline{R_\rho}^* \rho(R_\rho) = R_\rho^* \overline{\rho}(\overline{R_\rho}) = 1, \quad R_\rho^* R_\rho = \overline{R_\rho}^* \overline{R_\rho} = d(\rho).$$

If ρ is irreducible and $\rho = \overline{\rho}$, we have $\overline{R_\rho} = \epsilon R_\rho$, $\epsilon \in \{1, -1\}$.
We say that ρ is real (resp. pseudo-real) if $\epsilon = 1$ (resp. $\epsilon = -1$).
(In fact, ϵ=Frobenius-Schur indicator.)

If ρ is irreducible, $(\rho, \sigma\mu)$ is a Hilbert space with
$\langle T_1, T_2 \rangle := T_2^* T_1 \in (\rho, \rho) = \mathbb{C}$ for $T_1, T_2 \in (\rho, \sigma\mu)$.

Near-group category

Definition (Siehler 2003)

Let G be a finite group.
A near-group category with G is a fusion category $\mathcal{C}$ with $\mathrm{Irr}(\mathcal{C}) = G \sqcup \{\rho\}$.

The possible fusion rules are

$$[g][h] = [gh], \quad g, h \in G,$$

$$[g][\rho] = [\rho][g] = [\rho],$$

$$[\rho]^2 = \sum_{g \in G} [g] \oplus m[\rho], \quad m = 0, 1, 2, \ldots.$$

We denote by $R(G, m)$ the corresponding based ring.

Known classification results

Theorem (Tambara-Yamagami 1998)

$R(G,0)$ allows a categorification if and only if G is abelian. When G is abelian, the categorifications of $R(G,0)$ are in one-to-one correspondence with the data $\{(\epsilon, \langle\cdot,\cdot\rangle)\}$, where $\epsilon \in \{1,-1\}$ and $\langle\cdot,\cdot\rangle : G \times G \to \mathbb{T}$ is a non-degenerate symmetric bicharacter.

These categories are called Tambara-Yamagami categories.

Theorem (Ostrik, 2003)

$R(\{e\},m)$ allows a categorification if and only if $m = 1$. When $m = 1$, there exists a unique categorification up to Galois conjugate.

Known classification results (continued)

Theorem (Siehler 2003)

$R(G, |G|-1)$ allows a categorification if and only if G is a cyclic group and $q = |G|+1$ is a prime power.

Theorem (Etingof-Gelaki-Ostrik 2004)

$R(\mathbb{Z}_n, n-1)$ allows a categorification if and only if $q = n+1$ is a prime power.
Except for $n = 2, 3, 7$, there exists a unique categorification $\mathrm{Rep}(\mathbb{F}_q \rtimes \mathbb{F}_q^\times)$.
There are 3 categorifications for $n = 2$, and there are 2 for $n = 3, 7$.

The exceptions come from $H^3(\mathbb{F}_q, \mathbb{C}^\times)^{\mathbb{F}_q^\times}$.

General theorem

Let $\mathcal{C}$ be a near-group category with a finite group G.
Let $d = d(\rho) = \frac{m+\sqrt{m^2+4|G|}}{2}$.
We consider only C* fusion categories.

Theorem

Assume G is non-trivial and $m \neq 0$.
If d is rational, then either of the following holds:

(i) $m = |G| - 1$ *(already classified by Siehler and Etingof-Gelaki-Ostrik).*

(ii) G *is an extra-special 2-group of order* 2^{2a+1} *and* $m = 2^a$.
(a 2-group is extra-special if $[G,G] = Z(G) \cong \mathbb{Z}_2$*, e.g.* D_8 *and* Q_8*.)*
For each extra-special 2-group G *of order* 2^{2a+1}*, there exist exactly 3 categorifications of* $R(G, 2^a)$*.*

If d is irrational, then G is abelian and m is a multiple of $|G|$.
Moreover, the categorifications of $R(G,m)$ are completely classified by explicit polynomial equations.

Polynomial equations for the categorifications of $R(G,|G|)$

$\langle\cdot,\cdot\rangle : G\times G\to\mathbb{T}$: non-degenerate symmetric bicharacter,
$a: G\to\mathbb{T}$, $b: G\to\mathbb{C}$, $c\in\mathbb{T}$,

$$\langle g,h\rangle = a(g)a(h)\overline{a(g+h)}, \quad a(-g)=a(g),$$

$$c^3 = \frac{1}{\sqrt{|G|}}\sum_{g\in G}\overline{a(g)},$$

$$b(0)=\frac{-1}{d}, \quad b(g)=\frac{\overline{ca(g)}}{\sqrt{|G|}}\sum_{h\in G}\langle g,h\rangle b(h),$$

$$\overline{b(g)} = a(g)b(-g),$$

$$|b(g)| = \frac{1}{\sqrt{|G|}}, \quad g\in G\setminus\{0\},$$

$$\sum_{g\in G} b(g+h)b(g+k)\overline{b(g)} = \overline{\langle h,k\rangle}b(h)b(k) - \frac{c}{d\sqrt{|G|}}.$$

Polynomial equations for the categorification of $R(G, |G|)$ (continued)

Evans-Gannon determined the solutions for $\#G \leq 13$, and they always exist except for $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

G	# (solutions/$\mathrm{Aut}(G)$)
$\mathbb{Z}_2$	2
$\mathbb{Z}_3$	2
$\mathbb{Z}_4$	2
$\mathbb{Z}_5$	3
$\mathbb{Z}_6$	4
$\mathbb{Z}_7$	2
$\mathbb{Z}_8$	8
$\mathbb{Z}_9$	2
$\mathbb{Z}_{10}$	4
$\mathbb{Z}_{11}$	4
$\mathbb{Z}_{12}$	4
$\mathbb{Z}_{13}$	4

Higher multiplicity case

$G = \mathbb{Z}_2 \Rightarrow m \leq 2$ (Ostrik).

$G = \mathbb{Z}_3 \Rightarrow m \leq 6$ (Larson),
For $m = 6$, there exist exactly two near-group categories (Liu-Snyder, Evans-Pugh, M.-I.).

$G = \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$: There is no near-group categories for $m = 8$ (M.-I.).

Strategy for the proof

Step I: Show the group part has trivial $H^3(G, \mathbb{T})$ obstruction, and a privileged lifting.

Step II: Construct a unitary representation of G and show a character formula.

Step III: Rational abelian case:
Use group actions on factors and intermediate subfactors to reduce the problem to cohomology computation.

Step IV: Rational non-commutative case:
Necessity of $G =$ extra-special 2-groups: Use group action on factors and intermediate subfactors.
Existence: Cuntz algebra endomorphisms.

Step III: Irrational case.
Quadratic form: Construct 3 unitary representations.
Existence: Cuntz algebra endomorphisms.

Lifting

Assume $\mathcal{C} \subset \mathrm{End}_0(M)$ is a near-group category with $G \neq \{e\}$ and $m \neq 0$. Then $\mathrm{Irr}(\mathcal{C}) = \{[\alpha_g]\}_{g \in G} \sqcup \{[\rho]\}$.

Since $[\alpha_g][\rho] = [\rho]$, we may assume $\alpha_g \circ \rho = \rho$.
($\mathrm{Ad}^{\exists} U_g \circ \alpha_g = \rho$, replace α_g with $\mathrm{Ad}\, U_g \circ \alpha_g$).

We get $\alpha_g \circ \alpha_h = \alpha_{gh}$.
($\rho = \alpha_g \circ \alpha_h \circ \rho$ and $\alpha_g \circ \alpha_h = \mathrm{Ad}^{\exists} U_{g,h} \circ \alpha_{gh}$
$\Rightarrow \mathrm{Ad}\, U_{g,h} \circ \rho = \rho \Rightarrow U_{g,h} \in \mathbb{T} \Rightarrow \alpha_g \circ \alpha_h = \alpha_{gh}$.)

α has trivial H^3-obstruction and a privileged lifting to an action.

Cuntz algebra endomorphisms

Choose an isometry $S_e \in (\mathrm{id}, \rho^2)$.
Then $S_e^* \rho(S_e) = \frac{\epsilon}{d}$, where $d = d(\rho) = (m + \sqrt{m^2 + 4|G|})/2$.

Set $S_g = \alpha_g(S_e) \in (\alpha_g, \rho^2)$.
Choose an ONB $\{T_i\}_{i=1}^m$ of (ρ, ρ^2).

$\{S_g\}_{g \in G} \cup \{T_i\}_{i=1}^m$ satisfies the Cuntz algebra $\mathcal{O}_{|G|+m}$-relation, that is, having mutually orthogonal ranges with summation 1.

Moreover α_g and ρ preserve the $*$-algebra generated by $\{S_g\}_{g \in G} \cup \{T_i\}_{i=1}^m$.

Proof on board.

Character formula

Since $[\rho\alpha_g] = [\rho]$, $\exists U(g) \in (\rho, \rho\alpha_g)$.
Since $U(g)S_e \in (\mathrm{id}, \rho\alpha_g\rho) = (\mathrm{id}, \rho^2) = \mathbb{C}S_e$, normalize $U(g)$ by $U(g)S_e = S_e$.

$\{U(g)\}_{g\in G}$ is a unitary representation of G in $(\rho, \rho\alpha_g) \subset (\rho^2, \rho\alpha_g\rho) = (\rho^2, \rho^2)$.

Since $[\rho^2] = \sum_{g\in G}[\alpha_g] + m[\rho]$,

$$(\rho^2, \rho^2) = \bigoplus_{g\in G} \mathbb{C}S_gS_g^* \oplus B(\mathcal{K}),$$

where $\mathcal{K} = (\rho, \rho^2)$, and we have decomposition

$$U(g) = \sum_{h\in G} \chi_h(g)S_hS_h^* + U_{\mathcal{K}}(g).$$

Compute the categorical trace of the both sides on board.

Character formula (continued)

$$(1+\frac{m}{|G|}d(\rho))\,\mathrm{Tr}(\lambda_g)=\sum_{h\in G}\chi_h(g)+d(\rho)\,\mathrm{Tr}(U_{\mathcal{K}}(g)).$$

Lemma

$d=\frac{m+\sqrt{m^2+4|G|}}{2}\in\mathbb{R}\setminus\mathbb{Q}\Rightarrow G$ is abelian, m is a multiple of $|G|$, and

$$\bigoplus_{h\in G}\chi_h\cong\lambda,$$

$$U_{\mathcal{K}}\cong\frac{m}{|G|}\lambda.$$

When d is rational (integer), $s=1+\frac{m}{|G|}d(\rho)\in\mathbb{N}$.
$d(\rho)^2=|G|+md(\rho)\Rightarrow(s-1)^2|G|^2=sm^2\Rightarrow t=\frac{m}{s-1}\in\mathbb{N}$.

Character formula (continued)

Lemma

$d \in \mathbb{Q}$ *(in fact $\mathbb{N}$)* $\Rightarrow \exists s, t \in \mathbb{N}$ *such that* $|G| = st^2$, $m = (s-1)t$, $d = st$,

$$\mathrm{Tr}(\lambda_g) = \frac{1}{s} \sum_{h \in G} \chi_h(g) + t\,\mathrm{Tr}(U_{\mathcal{K}}(g)).$$

(i) $t = 1 \Rightarrow \chi_h = 1$ *and* $1 \oplus U_{\mathcal{K}} \cong \lambda$.

(ii) $t > 1 \Rightarrow G$ *is non-abelian,* $\#\,\mathrm{Hom}(G, \mathbb{T}) = t^2$ *and*

$$\bigoplus_{h \in G} \chi_h \equiv s \bigoplus_{\chi \in \mathrm{Hom}(G,\mathbb{T})} \chi.$$

Let $\hat{G}^\dagger = \hat{G} \setminus \mathrm{Hom}(G, \mathbb{T})$. *Then* $t | \dim \pi$ *for all* $\pi \in \hat{G}^\dagger$, *and*

$$U_{\mathcal{K}} \cong \bigoplus_{\pi \in \hat{G}^\dagger} \frac{\dim \pi}{t} \pi.$$

Rational abelian case

Assume $m = |G| - 1 \Rightarrow d(\rho) = |G|$.

Since $\alpha_g \circ \rho = \rho$, we have $N = \rho(M) \subset M^G \subset M$ with $[M : M^G] = [M^G : N] = |G|$.

Let $\kappa : M^G \hookrightarrow M$.
Then $\exists \mu : M \to M^G$ with $\rho = \kappa\mu$, $d(\kappa) = d(\mu) = \sqrt{|G|}$.

Lemma

$\exists \theta \in \mathrm{Aut}(M^G)$ *such that* $\rho = \kappa\theta\overline{\kappa}$*.*

Proof.

$\rho = \overline{\rho} \Rightarrow \rho = \overline{\mu}\,\overline{\kappa} \Rightarrow \overline{\mu}\mu \prec \rho^2 \Rightarrow [\overline{\mu}\mu] = \sum_{g\in G}[\alpha_g] \Rightarrow \mathrm{Ad}^{\exists} U_g \circ \alpha_g \circ \overline{\mu} = \overline{\mu}$.
$\alpha_g \circ \rho = \rho \Rightarrow U_g \in \mathbb{T} \Rightarrow \overline{\mu}(M^G) = M^G$. □

Rational abelian case (continued)

Assume G is abelian for simplicity.
Then $[\overline{\kappa}\kappa] = \sum_{\chi \in \hat{G}} [\beta_\chi]$.

Let $H = [\beta_{\hat{G}}]$ and $\Gamma =< H \cup [\theta] >\subset \mathrm{Out}(M^G)$.

Fusion rules of $\rho \Rightarrow$
$\Gamma = H \sqcup H[\theta]H$, and $\Gamma \curvearrowright \Gamma/H$ is sharply 2-transitive $\Rightarrow$
$\Gamma = \mathbb{F}_q \rtimes \mathbb{F}_q^\times$ and $H = \mathbb{F}_q^\times$.

Our categories are classified by $H^3(\mathbb{F}_q, \mathbb{T})^{\mathbb{F}_q^\times} \subset H^3(\Gamma, \mathbb{T})$.

Rational non-abelian case

In the previous case, we had

$$\rho(M) \subset \rho(M) \rtimes G = M^G \subset M.$$

In the rational non-abelian case, we have

$$\rho(M) \subset \rho(M) \rtimes [G, G] = M^G \subset \rho(M) \rtimes G = M^{[G,G]} = M.$$

More complicated argument using two intermediate subfators (and induction reduction argument between $[G, G]$ and G) are necessary.

Irrational case

When $d(\rho)$ is irrational, $\langle g, h\rangle = \chi_h(g)$ is a non-degenerate symmetric bicharacter.

Recall $U_{\mathcal{K}}(g) \in B(\mathcal{K})$, where $\mathcal{K} = (\rho, \rho^2)$, is given by $\mathcal{K} \ni T \mapsto U(g)T$.

Three representations on board.

Definition

Let $\mathcal{H}(G)$ be the universal C*-algebra generated by three unitary representations v_0, v_1, v_2 of G, and a unitary w of period 3 satisfying

$$v_{i+1}(g)v_i(h) = \langle h, g\rangle v_i(h)v_{i+1}(g),$$

$$w^*v_i(g)w = v_{i+1}(g),$$

where $i \in \mathbb{Z}/3\mathbb{Z}$.

Irrational case (continued)

Lemma

$\exists 3|G|$ irreducible representations of $\mathcal{H}(G)$, realized in $B(\ell^2(G))$ as

$$\pi_{a,c}(v_0(g))f(h) = \langle g, h\rangle f(h),$$

$$\pi_{a,c}(v_1(g))f(h) = f(h+g),$$

$$\pi_{a,c}(v_2(g))f(h) = a(h)\overline{a(h-g)}f(h-g),$$

$$\pi_{a,c}(w)f(h) = \frac{c}{\sqrt{n}}\sum_k a(h)\overline{\langle h, k\rangle}f(k),$$

where $a : G \to \mathbb{T}$ and $c \in \mathbb{T}$ satisfy

$$a(g+h)\langle g, h\rangle = a(g)a(h),$$

$$c^3 \sum_{g\in G} a(g) = \sqrt{n}.$$