An operator algebra approach to the classification of certain fusion categories I

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Representations

For a finite group G, we denote by $\mathrm{Rep}(G)$ the category of finite dimensional representations of G over $\mathbb{C}$.

- A representation π of G is a group homomorphism $\pi : G \to GL(V_\pi)$, where $GL(V_\pi)$ is the set linear invertible transformations of a finite dimensional vector space V_π over $\mathbb{C}$.
- For two representations (π, V_π), (σ, V_σ), the morphism space is

$$\mathrm{Hom}_G(V_\pi, V_\sigma) = \{T \in \mathrm{Hom}(V_\pi, V_\sigma);\ T\pi(g) = \sigma(g)T,\ \forall g \in G\}.$$

- The tensor product representation is given by

$$\pi \otimes \sigma : G \ni g \mapsto \pi(g) \otimes \sigma(g) \in GL(V_\pi \otimes V_\sigma).$$

- The trivial representation $G \ni g \mapsto 1 \in \mathbb{C}$ satisfies

$$\pi \otimes 1 \cong 1 \otimes \pi \cong \pi.$$

Representations (continued)

• The contragradient representation π^* of π is given by $V_{\pi^*} = V_\pi^*$, the dual space of V_π, and $\pi^*(g)\xi^* = \xi^* \circ \pi(g)^{-1}, \quad \xi \in V_\pi^*$.

Let $\{e_i\}_{i=1}^n$ be a basis of V_π and let $\{e_i^*\}_{i=1}^n$ be the dual basis of V_π^*. Then $\sum_{i=1}^n e_i^* \otimes e_i$ is a G-invariant vector, and $1 \prec \pi^* \otimes \pi$.

• Every object is equivalent to a direct sum of simple objects. (Every representation is completely reducible.)

• There exist only finitely many equivalence classes of simple objects.

Tensor categories

A tensor category is a category $\mathcal{C}$ with
a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product,
an object I called the unit object,
a natural isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$,
natural isomorphisms $\lambda_X : I \otimes X \to X$ and $\rho_X : X \otimes I \to X$ satisfying
(1) The pentagon identity

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes 1_Z"'] \arrow[dr, "a_{(W \otimes X),Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,(X \otimes Y),Z}"'] & \circlearrowleft & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,(Y \otimes Z)}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "1_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

(2) The triangle identity

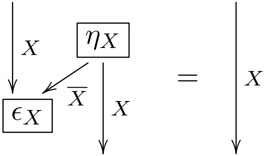
$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{X,I,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & \circlearrowleft & X \otimes (I \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Fusion categories

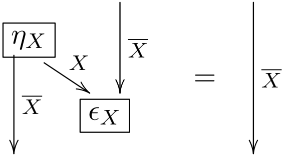
A fusion category $\mathcal{C}$ (over $\mathbb{C}$) is a rigid, linear, semisimple tensor category with only finitely many isomorphism classes of simple objects, such that $\mathrm{End}_{\mathcal{C}}(I) := \mathrm{Hom}_{\mathcal{C}}(I, I) = \mathbb{C}$.

Here $\mathcal{C}$ being rigid means that every object X has its dual (conjugate) $\overline{X}$, that is, there exist morphisms $\epsilon_X : X \otimes \overline{X} \to I$, and $\eta_X : I \to \overline{X} \otimes X$ such that the following compositions give the identity morphisms 1_X and $1_{\overline{X}}$ respectively:

$$\begin{tikzcd} X \arrow[r, "1_X \otimes \eta_X"] \arrow[rr, bend right, "1_X"'] & X \otimes \overline{X} \otimes X \arrow[r, "\epsilon_X \otimes 1_X"] & X, \end{tikzcd}$$



$$\begin{tikzcd} \overline{X} \arrow[r, "\eta_X \otimes 1_{\overline{X}}"] \arrow[rr, bend right, "1_{\overline{X}}"'] & \overline{X} \otimes X \otimes \overline{X} \arrow[r, "1 \otimes \epsilon_X"] & \overline{X}, \end{tikzcd}$$



Examples

Example

$\mathcal{C} = \mathrm{Rep}(G)$: the category of finite dimensional representations of a finite group G.

Example

$\mathcal{C} = \mathrm{Vec}_G$: the category of G-graded finite dimensional vector spaces.
For $V = \bigoplus_{g \in G} V_g$, $W = \bigoplus_{g \in G} W_g$ in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(V, W) = \{T \in \mathrm{Hom}(V, W);\ TV_g \subset W_g\}.$$

$$(V \otimes W)_g = \bigoplus_{kl = g} V_k \otimes W_l.$$

Goal

Fusion categories appear in many fields of mathematics and mathematical physics, e.g.

- Representation theory of (quantum) groups,
- Conformal field theory,
- Operator Algebras (Jones' theory of subfactors).

Goal: To classify near-group categories and more general quadratic categories by using operator algebras, more specifically, Cuntz algebra endomorphisms.

Advantage: Working on concrete categories.
Detailed information about the fusion categories can be available, e.g. Drinfeld centers, 6j-symbols, outer automorphism groups...

Disadvantage: Brute force method.
A more natural (geometric?) method is desired in order to obtain an infinite series.

Grothendieck ring and PF-dimension

Let $\mathcal{C}$ be a fusion category.
$\mathrm{Irr}(\mathcal{C})$=the set of equivalence classes of simple objects in $\mathcal{C}$.
The Grothendieck ring $K(\mathcal{C})$ of a fusion category $\mathcal{C}$ is $K(\mathcal{C}) = \mathbb{Z}\,\mathrm{Irr}(\mathcal{C})$ with multiplication

$$[X] \cdot [Y] = \sum_Z N_{X,Y}^Z [Z],$$

where X, Y, Z are simple and $N_{X,Y}^Z = \dim \mathrm{Hom}_{\mathcal{C}}(Z, X \otimes Y)$.

By the Perron-Frobenius theorem, there exists a unique ring homomorphism $d_{\mathrm{PF}} : K(\mathcal{C}) \to \mathbb{R}$ with $d_{\mathrm{PF}}(X) \geq 1$ for simple X.
$d_{PF}(X)$ is called the Perron-Frobenius dimension of X.

Example

For $\mathcal{C} = \mathrm{Rep}(G)$, $d_{PF}(\pi) = \dim V_\pi$.

Categorification

Definition

Given a based ring R, a fusion category $\mathcal{C}$ with $R \cong K(\mathcal{C})$ is called a categorification of R.

Example

Vec_G is a categorification of the group ring $\mathbb{Z}G$.

Problem

Given a based ring R, classify the categorifications $\mathcal{C}$ of R.

This is a non-trivial problem even for $\mathbb{Z}G$.

Categorification (continued)

Let G be a finite group, and assume $K(\mathcal{C}) \cong \mathbb{Z}G$.
For each pair $g, h \in \mathrm{Irr}(\mathcal{C}) = G$, choose an isomorphism $f_{g,h} : g \otimes h \to gh$.

The diagram

$$\begin{tikzcd}
(g \otimes h) \otimes k \arrow[rr, "a_{g,h,k}"] \arrow[d, "f_{g,h} \otimes 1_k"'] & & g \otimes (h \otimes k) \arrow[d, "1_g \otimes f_{h,k}"] \\
gh \otimes k \arrow[dr, "f_{gh,k}"'] & & g \otimes (hk) \arrow[dl, "f_{g,hk}"] \\
& ghk &
\end{tikzcd}$$

is not necessarily commutative, and it gives a number $\omega(g,h,k) \in \mathbb{C}^\times$.
The pentagon identity implies $\omega \in Z^3(G, \mathbb{C}^\times)$.

Theorem

The categorifications of $\mathbb{Z}G$ are completely classified by $[\omega] \in H^3(G, \mathbb{C}^\times)$.

Near-group categories

Definition (Siehler 2003)

Let G be a finite group.
A near-group category with G is a fusion category $\mathcal{C}$ with $\mathrm{Irr}(\mathcal{C}) = G \sqcup \{\rho\}$.

The possible fusion rules are

$$[g][h] = [gh], \quad g, h \in G,$$

$$[g][\rho] = [\rho][g] = [\rho],$$

$$[\rho]^2 = \sum_{g \in G} [g] \oplus m[\rho], \quad m = 0, 1, 2, \ldots.$$

We denote by $R(G, m)$ the corresponding based ring.

Examples

Example

$\mathfrak{S}_3$= the symmetric group of degree 3.
$\mathrm{Irr}(\mathfrak{S}_3) = \{1, \varepsilon, \rho\}$.

$$\varepsilon \otimes \varepsilon \cong 1,$$

$$\varepsilon \otimes \rho \cong \rho \otimes \varepsilon \cong \rho,$$

$$\rho \otimes \rho \cong 1 \oplus \varepsilon \oplus \rho.$$

$\mathrm{Rep}(\mathfrak{S}_3)$ is a categorification of $R(\mathbb{Z}_2, 1)$.

$\mathrm{Rep}(\mathfrak{A}_4)$ is a categorification of $R(\mathbb{Z}_3, 2)$.
$\mathrm{Rep}(D_8)$ and $\mathrm{Rep}(Q_8)$ are categorifications of $R(\mathbb{Z}_2 \times \mathbb{Z}_2, 0)$.

Ising model is a categorification of $R(\mathbb{Z}_2, 0)$.
Even part of WZW model with $SU(2)_3$ is a categorification of $R(\{e\}, 1)$.

Even part of the E_6 subfactors are categorifications of $R(\mathbb{Z}_2, 2)$.

Category $\mathrm{End}(M)$

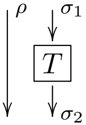
Let M be a type III factor.
The set of unital endomorphisms $\mathrm{End}(M)$ is a tensor category with

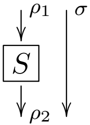
$$\rho \otimes \sigma = \rho \circ \sigma,$$

$$\mathrm{Hom}_{\mathrm{End}(M)}(\rho, \sigma) = \{T \in M;\ T\rho(x) = \sigma(x)T\} =: (\rho, \sigma).$$

For $S \in (\rho_1, \rho_2)$ and $T \in (\sigma_1, \sigma_2)$, $S \otimes T \in (\rho_1 \circ \sigma_1, \rho_2 \circ \sigma_2)$ is given by

$$S \otimes T := S\rho_1(T) = \rho_2(T)S.$$

In particular, 

$= 1_\rho \otimes T = \rho(T)$, while 

$= S \otimes 1_\sigma = S$.

$\mathrm{End}(M)$ is a C^* category, that is, $(\rho, \sigma)^* = (\sigma, \rho)$ and for $T \in (\rho, \sigma)$,

$$\|T^* \circ T\| = \|T\|^2.$$

G-kernels

Assume that $\mathcal{C} \subset \mathrm{End}(M)$ is a categorification of $\mathbb{Z}G$.
We can choose $\alpha_g \in \mathrm{Aut}(M)$ such that $\{[\alpha_g]\}_{g\in G} = \mathrm{Irr}(\mathcal{C}) \cong G$ and

$$[\alpha_g][\alpha_h] = [\alpha_{gh}].$$

Such a map $\alpha : G \to \mathrm{Aut}(M)$ is called a G-kernel.

$\exists U_{g,h} \in \mathcal{U}(M)$ satisfying $\alpha_g \circ \alpha_h = \mathrm{Ad}\, U_{g,h} \circ \alpha_{gh}$.
Associativity $(\alpha_g \circ \alpha_h) \circ \alpha_k = \alpha_g \circ (\alpha_h \circ \alpha_k)$ implies

$$\mathrm{Ad}(U_{g,h}U_{gh,k}) \circ \alpha_{ghk} = \mathrm{Ad}(\alpha_g(U_{h,k})U_{g,hk}) \circ \alpha_{ghk},$$

and there exists $\omega \in Z^3(G, \mathbb{T})$ satisfying

$$\alpha_g(U_{h,k})U_{g,hk} = \omega(g,h,k)U_{g,h}U_{gh,k}.$$

The cohomology class $[\omega] \in H^3(G, \mathbb{T})$ does not depend on the choice of $U_{g,h}$.

G-kernels (continued)

When $[\omega] = 0$, we may choose $U_{g,h}$ satisfying the 2-cocycle relation:

$$\alpha_g(U_{h,k})U_{g,hk} = U_{g,h}U_{gh,k}.$$

The pair $(\alpha, \{U_{g,h}\})$ is a cocycle action.

For a finite group, every cocycle action on a factor is known to be equivalent to an genuine action, i.e. $\exists V_g \in \mathcal{U}(M)$ satisfying

$$U_{g,h} = \alpha_g(V_h^{-1})V_g^{-1}V_{gh},$$

and $\beta_g \circ \beta_h = \beta_{gh}$, where $\beta_g = \operatorname{Ad} V_g \circ \alpha_g$.

In the above $U'_{g,h} = \xi(g,h)U_{g,h}$ with $\xi \in Z^2(G,\mathbb{T})$ also works.

The liftings of a G-kernel to actions are parametrized by $H^2(G,\mathbb{T})$.

Group Actions

Let $\beta : G \to \mathrm{Aut}(M)$ be an action satisfying $\beta_g \notin \mathrm{Inn}(M)$ for $g \neq e$.
Assume that an inner perturbation $\mathrm{Ad}\, W_g \circ \beta_g$ is an action too.
Then

$$\mathrm{Ad}(W_g \beta_g(W_h)) \circ \beta_{gh} = \mathrm{Ad}\, W_{gh} \circ \beta_{gh},$$

and $\exists \xi \in Z^2(G, \mathbb{T})$ satisfying $W_g \beta_g(W_h) = \xi(g,h) W_{gh}$.

When $[\xi] = 0$ in $H^2(G, \mathbb{T})$, we may choose $\{W_g\}$ to form a β-cocycle, $W_g \beta_g(W_h) = W_{gh}$, which is known to be a coboundary, that is, $\exists S \in \mathcal{U}(M)$ satisfying $W_g = S\alpha_g(S^{-1})$.
Thus $\mathrm{Ad}\, W_g \circ \beta_g = \mathrm{Ad}\, S \circ \beta_g \circ \mathrm{Ad}\, S^{-1}$.

Summary:

- A G-kernel α can be lifted to an action if and only if $[\omega] \in H^3(G, \mathbb{T})$ is trivial.
- When $[\omega] = 0$, the inner conjugacy classes of the liftings of α to actions are parametrized by $H^2(G, \mathbb{T})$.

$\mathrm{End}_0(M)$

For $\rho \in \mathrm{End}(M)$, set

$$d(\rho) = [M : \rho(M)]_0^{1/2},$$

and call it the (statistical) dimension of ρ.

Let

$$\mathrm{End}_0(M) = \{\rho \in \mathrm{End}(M);\ d(\rho) < \infty\}.$$

$\mathrm{End}_0(M)$ is a rigid C^* tensor category, i.e., each object ρ has its conjugate object $\overline{\rho}$ with $R_\rho \in (\mathrm{id}, \overline{\rho}\rho)$, $\overline{R_\rho} \in (\mathrm{id}, \overline{\rho}\rho)$ satisfying

$$\overline{R_\rho}^* \rho(R_\rho) = R_\rho^* \overline{\rho}(\overline{R_\rho}) = 1,$$

$$R_\rho^* R_\rho = \overline{R_\rho}^* \overline{R_\rho} = d(\rho).$$

Popa's uniqueness theorem

Theorem

Let M be the hyperfinite type III_1 factor.
Every C^* fusion category is uniquely embedded into $\mathrm{End}_0(M)$.

Definition

A monoidal functor from a strict tensor category $\mathcal{C}$ to another strict tensor category $\mathcal{D}$ is a pair (F, L) of a functor $F : \mathcal{C} \to \mathcal{D}$ and natural isomorphisms $L_{\rho,\sigma}$, $\rho, \sigma \in \mathcal{C}$, with

$$L_{\rho,\sigma} \in \mathrm{Hom}_{\mathcal{D}}(F(\rho) \otimes F(\sigma), F(\rho \otimes \sigma))$$

$$L_{\rho\otimes\sigma,\tau} \circ (L_{\rho,\sigma} \otimes I_{F(\tau)}) = L_{\rho,\sigma\otimes\tau} \circ (I_{F(\rho)} \otimes L_{\sigma,\tau}).$$

We may and do assume $F(\mathbf{1}_{\mathcal{C}}) = \mathbf{1}_{\mathcal{D}}$ and $L_{\mathbf{1}_{\mathcal{C}},\rho} = L_{\rho,\mathbf{1}_{\mathbb{C}}} = I_{F(\rho)}$.
When $\mathcal{C}$ and $\mathcal{D}$ are C* categories, we further assume that $L_{\rho,\sigma}$ is a unitary.

Popa's uniqueness theorem (continued)

Theorem

Let M and P be hyperfinite type III$_1$ factors, and let $\mathcal{C}$ and $\mathcal{D}$ be C^* fusion categories embedded in $\mathrm{End}_0(M)$ and $\mathrm{End}_0(P)$ respectively.

Let (F, L) be a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ that is an equivalence of the two C^* fusion categories $\mathcal{C}$ and $\mathcal{D}$.
Then there exists a surjective isomorphism $\Phi : M \to P$ and unitaries $V_\rho \in P$ for each object $\rho \in \mathcal{C}$ satisfying

$$F(\rho) = \mathrm{Ad}\, V_\rho \circ \Phi \circ \rho \circ \Phi^{-1},$$

$$F(X) = V_\sigma \Phi(X) V_\rho^*, \quad X \in (\rho, \sigma),$$

$$L_{\rho,\sigma} = V_{\rho\circ\sigma} \Phi \circ \rho \circ \Phi^{-1}(V_\sigma^*) V_\rho^* = V_{\rho\circ\sigma} V_\rho^* F(\rho)(V_\sigma^*).$$

To classify C^* fusion categories $\mathcal{C}$, we may always assume $\mathcal{C} \subset \mathrm{End}_0(M)$.