Cocycle superrigidity for translations actions of product groups

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Orbit equivalence

Let $\Gamma \curvearrowright (X, \mu)$ be an action of a countable group Γ on a probability space (X, μ) which is assumed

- **measure preserving:** $\mu(g \cdot A) = \mu(A)$, for all $g \in \Gamma$ and $A \subset X$,
- **free:** $\{x \mid g \cdot x = x\}$ has measure 0, for all $g \neq \mathrm{id}$, and
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Definition

Two actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (Y, \nu)$ are orbit equivalent if there is an isomorphism $\alpha : X \to Y$ satisfying $\Gamma \cdot x = \Gamma \cdot y \Leftrightarrow \Delta \cdot \alpha(x) = \Delta \cdot \alpha(y)$, almost everywhere. (that is, the equivalence relations are isomorphic.)

Examples of free ergodic measure preserving actions

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  1. the irrational rotation $\mathbb{Z} \curvearrowright^\alpha \mathbb{T}$, by an angle $\alpha \notin 2\pi\mathbb{Q}$.
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     where $\Gamma < \mathrm{SO}(n+1)$ is a countable dense subgroup.
  3. the left translation action $\Gamma \curvearrowright G$,
     where Γ is residually finite and $G = \varprojlim \Gamma/\Gamma_n$ is a profinite completion.
  4. the left-right translation action $\Gamma \times \Lambda \curvearrowright G$: $(g, h) \cdot x = gxh^{-1}$,
     where G is a compact group containing Γ, Λ densely.

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$\Longrightarrow$ the classification of crossed product algebras $L^\infty(X) \rtimes \Gamma$ can be divided into two problems:

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2. Classify all actions that are orbit equivalent to $\Gamma \curvearrowright (X, \mu)$.

The amenable/nonamenable dichotomy

Definition

A group Γ is **amenable** if it admits a sequence $\{F_n\}$ of finite subsets such that $\dfrac{|gF_n \Delta F_n|}{|F_n|} \to 0$, for all $g \in \Gamma$. **Examples**: abelian and solvable groups.
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Dye (1959), Ornstein-Weiss (1980), Connes-Feldman-Weiss (1981):
If Γ and Λ are infinite amenable, then any free ergodic actions
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Epstein (2007): If Γ is not amenable, then there exist uncountably many non-orbit equivalent free ergodic actions $\Gamma \curvearrowright (X, \mu)$.

- **Gaboriau-Popa (2003)**: same if Γ is a free group.
- **I (2006)**: same if Γ contains $\mathbb{F}_2$ as a subgroup.
- **Gaboriau-Lyons (2007)**: any non-amenable group Γ contains $\mathbb{F}_2$ as a "measurable subgroup".

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Two cocycles $w_1, w_2 : \Gamma \times X \to \Delta$ are **cohomologous** if there exists a measurable map $\varphi : X \to \Delta$ such that $w_1(g,x) = \varphi(g \cdot x) w_2(g,x) \varphi(x)^{-1}$.

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- **Kida (2006)**: all actions of most mapping class groups.
- **I (2008)**: profinite actions of property (T) groups.

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Theorem (Popa, 2005-2006)

Let Γ be a property (T) group (e.g. $\Gamma = SL_n(\mathbb{Z}), n \geq 3$) or a product group $\Gamma = \Gamma_1 \times \Gamma_2$, with Γ_1 infinite and Γ_2 non-amenable.
Let $\Gamma \curvearrowright (X, \mu)$ a Bernoulli action and Δ any U_{fin} group (e.g. countable).

Then any cocycle $w : \Gamma \times X \to \Delta$ is cohomologous to a homomorphism.

Cocycle superrigidity for compact actions, I

Theorem (I, 2008)

Let G be a profinite group and Γ be a dense property (T) subgroup. Let Δ be a countable group and $w : \Gamma \times G \to \Delta$ a cocycle for the left translation action $\Gamma \curvearrowright G$.

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Cocycle superrigid actions

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- **Popa-Vaes (2008)**: the usual action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$, for $n \geq 5$.
- **Peterson-Sinclair (2009)**: Bernoulli actions of L^2-rigid groups.
- **Tucker-Drob (2014)**: Bernoulli actions of inner amenable non-amenable groups.
- **I (2014)**: $\mathrm{SL}_n(\mathbb{Z}[\frac{1}{p}]) \curvearrowright \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, for $n \geq 3$ and p prime.
- **Drimbe (2015):** any co-induced action $\Gamma \curvearrowright (X_0,\mu_0)^{\Gamma/\Lambda}$, where Γ has property (T) and Λ is an infinite index subgroup.

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Let G be a profinite group and Γ, Λ be finitely generated dense subgroups. Consider the left-right translation action $\Gamma \times \Lambda \curvearrowright G$. Let Δ be a countable group and $w : (\Gamma \times \Lambda) \times G \to \Delta$ be a cocycle.

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Remark. Any compact action $\Gamma \times \Lambda \curvearrowright X$ such that Γ, Λ act freely and ergodically is isomorphic to a **left-right translation** action $\Gamma \times \Lambda \curvearrowright G$.

Strong ergodicity, I

Definition

An action $\Gamma \curvearrowright (X, \mu)$ is called **strongly ergodic** if there are no non-trivial asymptotically invariant measurable sets $A_n \subset X$:

$$\mu(g \cdot A_n \Delta A_n) \to 0, \forall g \in \Gamma \quad \Longrightarrow \quad \mu(A_n)(1 - \mu(A_n)) \to 0.$$

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Examples

- **Selberg (1965)** $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{SL}_2(\mathbb{Z}_p)$ is strongly ergodic, for prime p.
- **Bourgain-Varjú (2010)** $\Gamma \curvearrowright \overline{\Gamma} < \mathrm{SL}_2(\mathbb{Z}_p)$ is strongly ergodic, for any non-amenable subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$.

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Step 3 (I, 2007). The uniformity condition of Step 2 implies that the restriction of w to $\Lambda \times G$ is "virtually" cohomologous to a homomorphism.

Orbit equivalence superrigidity

Theorem (Gaboriau-I-Tucker-Drob, 2016)

Let G be a profinite group and Γ, Λ be finitely generated dense subgroups. Let Δ be a countable group and $w : (\Gamma \times \Lambda) \times G \to \Delta$ a cocycle. Assume that $\Gamma \curvearrowright G$ is strongly ergodic.

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Then the actions are **virtually conjugate***: there exist an open subgroup $G_0 < G$, a finite index subgroup $\Delta_0 < \Delta$, and a Δ_0-ergodic component $Y_0 \subset Y$ such that $(\Gamma \cap G_0) \times (\Lambda \cap G_0) \curvearrowright G_0$ is conjugate to $\Delta_0 \curvearrowright Y_0$.*

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- applies to $\mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathrm{PSL}_2(\mathbb{Z}_p)$, for any prime p.
- more generally, applies to $\Gamma \times \Gamma \curvearrowright \overline{\Gamma}$, where $\Gamma < \mathrm{PSL}_2(\mathbb{Z})$ is a non-amenable subgroup, and $\overline{\Gamma}$ is the closure of Γ in $\mathrm{PSL}_2(\mathbb{Z}_p)$.

W*-superrigidity

An action $\Gamma \curvearrowright (X, \mu)$ is called W*-superrigid if any action $\Delta \curvearrowright (Y, \nu)$ such that $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Delta$ is conjugate to $\Gamma \curvearrowright (X, \mu)$.

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- **Peterson (2009)** existence of virtually W*-superrigid actions
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Ozawa-Popa (2007) $L^\infty(\mathrm{PSL}_2(\mathbb{Z}_p)) \rtimes (\mathrm{PSL}_2(\mathbb{Z}) \times \mathrm{PSL}_2(\mathbb{Z}))$ has a unique Cartan subalgebra, up to unitary conjugacy.

Cocycle superrigidity for compact actions, III

Theorem (Gaboriau-I-Tucker-Drob, 2016)

Let G be a simply connected simple Lie group, and Γ, Λ dense subgroups. Assume that $\Gamma \curvearrowright G$ is strongly ergodic, and $(\star)$ Λ contains an infinite cyclic subgroup with compact closure. Let Δ be a countable group.

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Remark. $(\star)$ holds for any dense subgroup $\Lambda < \mathrm{SL}_2(\mathbb{R})$, but fails for some dense subgroups $\Lambda < \mathrm{SL}_n(\mathbb{R})$, if $n \geq 3$.

Strong ergodicity, II

Theorem

Let G be a connected simple Lie group. Let $\Gamma < G$ be a countable dense subgroup such that $Ad(\Gamma)$ consists of matrices with algebraic entries. Here, $Ad : G \to GL(\mathfrak{g})$ denotes the adjoint representation of G.

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- **Boutonnet-I-Salehi-Golsefidy (2015)**
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