Amenable absorption in amalgamated free product von Neumann algebras

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Introduction and statement of our main theorem

What is... an AFP von Neumann algebra?

For each $i \in \{1, 2\}$, let $B \subset M_i$ be an inclusion of von Neumann algebras with faithful normal conditional expectation $\mathrm{E}_i : M_i \to B$.

Definition (Voiculescu 1985)

The **amalgamated free product** von Neumann algebra

$$(M, \mathrm{E}) = (M_1, \mathrm{E}_1) *_B (M_2, \mathrm{E}_2)$$

is the unique pair consisting of a von Neumann algebra M with a faithful normal conditional expectation $\mathrm{E} : M \to B$ such that:

- for each $i \in \{1, 2\}$, $M_i \subset M$ is a von Neumann subalgebra
- $M = M_1 \vee M_2$
- M_1 and M_2 are $*$-**free** with respect to E, meaning that

$$\mathrm{E}(x_1 \cdots x_n) = 0$$

whenever $n \geq 1$, $x_j \in \ker(\mathrm{E}_{\iota_j})$, $\iota_j \in \{1, 2\}$ and $\iota_1 \neq \cdots \neq \iota_n$

What is... an AFP von Neumann algebra?

Once conditional expectations are fixed, we simply write

$$M = M_1 *_B M_2$$

When $B = \mathbf{C}1$, $\mathrm{E}_i = \varphi_i 1$ and $\mathrm{E} = \varphi 1$, we simply say that

$$(M, \varphi) = (M_1, \varphi_1) * (M_2, \varphi_2)$$

is the **free product** von Neumann algebra.

Example (Group von Neumann algebras)

Let $G = G_1 *_H G_2$ be an amalgamated product of discrete groups. Then we have

$$\mathrm{L}(G) = \mathrm{L}(G_1) *_{\mathrm{L}(H)} \mathrm{L}(G_2)$$

with respect to canonical conditional expectations.

Recall that $\mathrm{L}(G) = \lambda_G(G)''$ where $\lambda_G(g)\delta_h = \delta_{gh}$ for all $g, h \in G$.

Popa's maximal amenability result in $\mathrm{L}(\mathbf{F}_2)$

It is well known that any II_1 factor N contains a copy of the hyperfinite II_1 factor R. Kadison asked the following question.

Problem (Kadison 1967)

Let N be any II_1 factor and $x = x^* \in N$ any selfadjoint element. Does there always exist an intermediate hyperfinite subfactor $R \subset N$ such that $x \in R$?

Popa answered negatively the above question by proving the following pioneering result.

Theorem (Popa 1983)

Write $\mathbf{F}_2 = \langle a, b \rangle$. The generator subalgebra $A := \{\lambda_{\mathbf{F}_2}(a)\}''$ is **maximal amenable** *inside $M := \mathrm{L}(\mathbf{F}_2)$, that is, whenever Q is amenable and $A \subset Q \subset M$ then $A = Q$.*

Popa introduced a powerful method to prove maximal amenability known as **asymptotic orthogonality property (AOP)**.

Maximal amenability in free product von Neumann algebras

Popa's method and result have been recently extended by H–Ueda to **arbitrary** free product von Neumann algebras.

A von Neumann subalgebra $Q \subset M$ is **with expectation** if there exists a faithful normal conditional expectation $\mathrm{E}_Q : M \to Q$.

Theorem (H–Ueda 2015)

For each $i \in \{1, 2\}$, let (M_i, φ_i) be any von Neumann algebra endowed with any faithful normal state. Denote by $M = M_1 * M_2$ the free product von Neumann algebra.

Let $Q \subset M$ be any amenable von Neumann subalgebra with expectation such that $Q \cap M_1$ is diffuse and with expectation. Then $Q \subset M_1$.

In particular, if M_1 is diffuse and amenable, then M_1 is **maximal amenable** *(with expectation) inside M.*

See also Ozawa (2015) for a different proof in **tracial** free products.

Boutonnet–Carderi's maximal amenability result

Recently, Boutonnet–Carderi found a new way to prove maximal amenability results in **tracial** von Neumann algebras.

Their idea is to exploit amenability via the existence of non normal central states and use C^*-algebraic tools.

Theorem (Boutonnet–Carderi 2014)

Let $\Lambda < \Gamma$ be an inclusion of infinite discrete groups. Assume that Λ is amenable and that there exists a compact Γ-space X for which whenever $\mu \in \mathrm{Prob}_\Lambda(X)$ and $g \in \Gamma \setminus \Lambda$, we have $g_*\mu \perp \mu$.

Then $\mathrm{L}(\Lambda)$ is **maximal amenable** *inside $\mathrm{L}(\Gamma)$.*

The above theorem applies in particular to hyperbolic groups and amalgamated product groups.

Amenable absorption in AFP von Neumann algebras

Our main result is:

Theorem (Boutonnet–H 2016)

For each $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of arbitrary von Neumann algebras with expectation. Denote by $M = M_1 *_B M_2$ the amalgamated free product von Neumann algebra.

Let $Q \subset M$ be any von Neumann subalgebra with expectation satisfying the following two conditions:

1. *Q is amenable relative to M_1 inside M (e.g. Q is amenable).*
2. *$Q \cap M_1 \subset M_1$ is with expectation and $Q \cap M_1 \npreceq_{M_1} B$.*

Then $Q \subset M_1$.

Our result completely settles the questions of maximal amenability and amenable absorption in AFP von Neumann algebras.

It simultaneously generalizes results by Popa (1983), Boutonnet–Carderi (2014), Leary (2014) and H–Ueda (2015).

Strategy of proof

- Popa's **AOP** approach cannot work to prove amenable absorption in general AFP von Neumann algebras.
- Instead, we use Boutonnet–Carderi's approach and exploit amenability via the existence of **conditional expectations**.
- For this approach to work, we need to extend Popa's intertwining theory to **arbitrary** von Neumann algebras.
- We prove a useful intertwining criterion for subalgebras with expectation in terms of the existence of non vanishing **bimodular completely positive maps** defined on Jones basic construction.
- We will then sketch the proof of our main theorem using the intertwining criterion.

Popa's intertwining theory in arbitrary von Neumann algebras

Jones basic construction

Let M be any von Neumann algebra and $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$ the standard form of M.

Let $B \subset M$ be any von Neumann subalgebra with faithful normal conditional expectation $\mathrm{E}_B : M \to B$. Then denote by $e_B : \mathrm{L}^2(M) \to \mathrm{L}^2(B)$ the corresponding Jones projection.

Definition (Jones basic construction)

Denote by $\langle M, B\rangle := (JBJ)' \cap \mathbf{B}(\mathrm{L}^2(M))$ the **basic construction** of M with respect to B.

Observe that $\langle M, B\rangle$ is also the von Neumann subalgebra of $\mathbf{B}(\mathrm{L}^2(M))$ generated by M and e_B.

Denote by $\mathrm{T}_M : \langle M, B\rangle_+ \to \widehat{M_+}$ the canonical faithful normal semifinite operator valued weight satisfying $\mathrm{T}_M(e_B) = 1$.

Popa's intertwining theory: terminology

Let M be any σ-finite von Neumann algebra and $A, B \subset M$ any von Neumann subalgebras with expectation.

Definition

We say that A **embeds with expectation into** B **inside** M and write $A \preceq_M B$ if there exist projections $e \in A$, $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that:

- the unital inclusion $\theta(eAe) \subset fBf$ is **with expectation** and
- $av = v\theta(a)$ for all $a \in eAe$.

Transitivity property: H–Isono (2015)

If $A \preceq_M B$ and $D \subset A$ is a (unital) von Neumann subalgebra with expectation, then $D \preceq_M B$.

Popa's intertwining theory: characterization in tracial case

When (M, τ) is tracial, the weight $\mathrm{Tr} := \tau \circ \mathrm{T}_M$ defines a faithful normal semifinite trace on the basic construction $\langle M, B \rangle$.

Theorem (Popa 2001, Ozawa–Popa 2008)

*Let M be any **tracial** von Neumann algebra and $A, B \subset M$ any von Neumann subalgebras. TFAE:*

1. $A \preceq_M B$.
2. *There exists a nonzero projection* $p \in A' \cap \langle M, B \rangle$ *such that* $\mathrm{Tr}(p) < +\infty$.
3. *There exists an* A*-central normal state* φ *on* $\langle M, B \rangle$.
4. *There exists an* A*-central state* ψ *on* $\langle M, B \rangle$ *such that*
$$\psi|_M \text{ is normal} \quad \text{and} \quad \psi|_{\mathrm{C}^*(Me_BM)} \neq 0.$$

Popa's intertwining theory: characterization in general case

We generalize Popa's intertwining theory to **arbitrary** von Neumann algebras as follows.

Theorem (Boutonnet–H 2016)

Let M be any von Neumann algebra and $A, B \subset M$ any von Neumann subalgebras with expectation. TFAE:

1. $A \preceq_M B$.
2. *There exists a nonzero projection* $p \in A' \cap \langle M, B \rangle$ *such that* $\mathrm{T}_M(p) \in M_+$.
3. *There exists a nonzero normal A-bimodular completely positive map* $\Phi : \langle M, B \rangle \to A$.
4. *There exists an A-bimodular completely positive map* $\Psi : \langle M, B \rangle \to A$ *such that*

$$\Psi|_M \text{ is normal} \quad \text{and} \quad \Psi|_{\mathrm{C}^*(Me_B M)} \neq 0.$$

Our intertwining criterion further generalizes H–Isono (2015).

Proof of the intertwining criterion

(4) $\Rightarrow$ (3). Assume that there exists an A-bimodular completely positive map $\Psi : \langle M, B\rangle \to A$ such that

$$\Psi|_M \text{ is normal} \quad \text{and} \quad \Psi|_{\mathrm{C}^*(Me_B M)} \neq 0.$$

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Following Ozawa–Popa, take an approximate unit $(f_\lambda)_{\lambda \in \Lambda}$ in $\mathrm{C}^*(Me_B M)$ that is quasi-central for the C^*-algebra $M + \mathrm{C}^*(Me_B M)$, i.e. $\lim_\lambda \|xf_\lambda - f_\lambda x\| = 0$ for every $x \in M$.

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Choose a cofinal ultrafilter $\mathcal{U}$ on the directed set Λ and define

$$\Phi(T) := \sigma\text{-weak} \lim_{\lambda\to\mathcal{U}} \Psi(f_\lambda T f_\lambda)$$

Using properties of $(f_\lambda)_{\lambda\in\Lambda}$, we have $\Phi|_{\mathrm{C}^*(Me_B M)} = \Psi|_{\mathrm{C}^*(Me_B M)}$ and Φ is A-bimodular.

Then we show that $\Phi : \langle M, B\rangle \to A$ is **normal**.

Proof of the intertwining criterion

(3) $\Rightarrow$ (2). Assume that there exists a nonzero normal A-bimodular completely positive map $\Phi : \langle M, B \rangle \to A$.

We regard $\Phi : \langle M, B \rangle_+ \to \widehat{A_+}$ as a nonzero normal **bounded** operator valued weight.

Proof of the intertwining criterion

(3) $\Rightarrow$ (2). Assume that there exists a nonzero normal A-bimodular completely positive map $\Phi : \langle M, B\rangle \to A$.

We regard $\Phi : \langle M, B\rangle_+ \to \widehat{A_+}$ as a nonzero normal **bounded** operator valued weight.

Denote by $\mathrm{E}_A : M \to A$ a faithful normal conditional expectation. Then $\mathrm{T}_A := \mathrm{E}_A \circ \mathrm{T}_M : \langle M, B\rangle_+ \to \widehat{A_+}$ is a faithful normal semifinite operator valued weight.

Proof of the intertwining criterion

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Since Φ is bounded on $A' \cap \langle M, B \rangle$, T_A is not purely infinite on $A' \cap \langle M, B \rangle$ by Haagerup's theorem (1979).

Then we can find a nonzero projection $p \in A' \cap \langle M, B \rangle$ such that $\mathrm{T}_M(p) \in M_+$.

Proof of the intertwining criterion

(2) $\Rightarrow$ (1). Assume that there exists a nonzero projection $p \in A' \cap \langle M, B \rangle$ such that $\mathrm{T}_M(p) \in M_+$.

Then there exists a nonzero projection $q \in \mathcal{Z}(A)p$ such that $Aq \subset q\langle M, B \rangle q$ is with expectation by Haagerup's result (1979).

(2) $\Rightarrow$ (1). Assume that there exists a nonzero projection $p \in A' \cap \langle M, B\rangle$ such that $\mathrm{T}_M(p) \in M_+$.

Then there exists a nonzero projection $q \in \mathcal{Z}(A)p$ such that $Aq \subset q\langle M, B\rangle q$ is with expectation by Haagerup's result (1979).

Using that e_B has central support equal to 1 in $\langle M, B\rangle$ and $e_B\langle M, B\rangle e_B = Be_B$, the **transitivity property** implies that $Aq \preceq_{\langle M,B\rangle} Be_B$.

There exist projections $e \in A$, $f \in B$, a nonzero partial isometry $V \in eq\langle M, B\rangle fe_B$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that $aV = V\theta(a)$ for every $a \in A$.

Proof of the intertwining criterion

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There exist projections $e \in A$, $f \in B$, a nonzero partial isometry $V \in eq\langle M, B\rangle fe_B$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that $aV = V\theta(a)$ for every $a \in A$.

By Pimsner–Popa **push down lemma** (Izumi–Longo–Popa 1996), we have $\mathrm{T}_M(V) \neq 0$. Moreover $a\,\mathrm{T}_M(V) = \mathrm{T}_M(V)\,\theta(a)$ for every $a \in A$. Taking the polar decomposition $\mathrm{T}_M(V) = v\,|\mathrm{T}_M(V)|$, we have $v \in M$, $v \neq 0$ and $av = v\theta(a)$ for every $a \in A$.

Proof of our main theorem

Amenable absorption in AFP von Neumann algebras

Recall the statement of our main theorem:

Theorem (Boutonnet–H 2016)

For each $i \in \{1, 2\}$, let $B \subset M_i$ be any inclusion of arbitrary von Neumann algebras with expectation. Denote by $M = M_1 *_B M_2$ the amalgamated free product von Neumann algebra.

Let $Q \subset M$ be any von Neumann subalgebra with expectation satisfying the following two conditions:

1. *Q is amenable relative to M_1 inside M (e.g. Q is amenable).*
2. *$Q \cap M_1 \subset M_1$ is with expectation and $Q \cap M_1 \npreceq_{M_1} B$.*

Then $Q \subset M_1$.

We will sketch the proof in the case when Q is **amenable**.

Statement of the vanishing lemma

Since $Q \subset M$ is amenable and with expectation, we may choose and we fix a **conditional expectation** $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to Q$ such that $\Phi|_M$ is faithful and normal.

Denote by $\mathrm{E} : M \to B$ the canonical conditional expectation. Choose a faithful state $\varphi \in M_*$ such that $\varphi \circ \mathrm{E} = \varphi$.

Denote by $\mathcal{K}$ the closure of the linear span in $\mathrm{L}^2(M)$ of all the reduced words $x_1 \cdots x_n \xi_\varphi$ where $n \geq 1$ and $x_1 \in M_2 \ominus B$.
Denote by $P_{\mathcal{K}} : \mathrm{L}^2(M) \to \mathcal{K}$ the orthogonal projection.

Vanishing lemma

We have $\Phi(uP_{\mathcal{K}}u^*) = 0$ for every $u \in \mathcal{U}(M_1)$.

Therefore, for every $u \in \mathcal{U}(M_1)$, the projection $1 - uP_{\mathcal{K}}u^*$ belongs to the multiplicative domain of Φ and $\Phi(1 - uP_{\mathcal{K}}u^*) = 1$.

Equivalent formulation of the vanishing lemma

Recall that $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to Q$ with $\Phi|_M$ faithful and normal and $\mathcal{K} = \overline{\mathrm{span}}\,\{x_1 \cdots x_n \xi_\varphi : n \geq 1, x_1 \cdots x_n \text{ is reduced}, x_1 \in M_2 \ominus B\}$.

By definition, $\mathcal{K}$ is naturally a B-M_1-bimodule and as M_1-M_1-bimodules, we have the following isomorphism

$$\mathrm{L}^2(M) \ominus \mathrm{L}^2(M_1) \cong \mathrm{L}^2(M_1) \otimes_B \mathcal{K}$$

Since $Q \cap M_1 \subset M$ is with expectation, there exists a faithful normal conditional expectation $\mathrm{E}_{Q \cap M_1} : M \to Q \cap M_1$.

Put $\Theta := \mathrm{E}_{Q \cap M_1} \circ \Phi : \mathbf{B}(\mathrm{L}^2(M)) \to Q \cap M_1$. Since $\mathrm{E}_{Q \cap M_1}$ is faithful, in order to prove the **vanishing lemma**, it suffices to show that $\Theta(u P_{\mathcal{K}} u^*) = 0$ for every $u \in \mathcal{U}(M_1)$.

Proof of the vanishing lemma using intertwining theory

Vanishing lemma (Θ version)

We have $\Theta(uP_{\mathcal{K}}u^*) = 0$ *for every* $u \in \mathcal{U}(M_1)$.

Proof of the vanishing lemma.

Key observation: $\mathrm{L}^2(M_1)$ is a $\langle M_1, B\rangle$-B-bimodule. Then

$$\mathrm{L}^2(M) \ominus \mathrm{L}^2(M_1) = \mathrm{L}^2(M_1) \otimes_B \mathcal{K}$$

is naturally endowed with a structure of $\langle M_1, B\rangle$-M_1-bimodule.

Proof of the vanishing lemma using intertwining theory

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is naturally endowed with a structure of $\langle M_1, B\rangle$-M_1-bimodule.

Thus there is a normal $*$-homomorphism $\pi : \langle M_1, B\rangle \to \mathbf{B}(\mathrm{L}^2(M))$ such that $\pi(e_B) = P_{\mathcal{K}}$ and $\pi(x) = x\, e_{M_1}^{\perp}$ for every $x \in M_1$.

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Then $\Psi := \Theta \circ \pi : \langle M_1, B\rangle \to Q \cap M_1$ is a $Q \cap M_1$-bimodular completely positive map such that $\Psi|_{M_1}$ is normal.

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Since $Q \cap M_1 \not\preceq_{M_1} B$, we have $\Psi|_{\mathrm{C}^*(M_1 e_B M_1)} = 0$ by the **intertwining theorem**. This implies that for every $u \in \mathcal{U}(M_1)$,
$\Theta(uP_{\mathcal{K}}u^*) = \Theta(ue_{M_1}^{\perp}\, P_{\mathcal{K}}\, (ue_{M_1}^{\perp})^*) = \Psi(ue_B u^*) = 0.$ $\square$

Compatibility between Φ and E_{M_1}

Recall that $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to Q$ with $\Phi|_M$ faithful and normal and $\mathcal{K} = \overline{\mathrm{span}}\,\{x_1 \cdots x_n \xi_\varphi : n \geq 1, x_1 \cdots x_n \text{ is reduced}, x_1 \in M_2 \ominus B\}$.

Claim

We have $\Phi(x) = \Phi(\mathrm{E}_{M_1}(x))$ *for every* $x \in M$.

Proof of the claim.

By normality of Φ on M, it suffices to prove that $\Phi(x) = 0$ for a word of the form $x = u\, x_1 \cdots x_p\, v \in M \ominus M_1$ where $u, v \in \mathcal{U}(M_1)$ and $x_1 \cdots x_p$ is a reduced word with $x_1, x_p \in M_2 \ominus B$.

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By **construction of the AFP** and the definition of $\mathcal{K}$, we have

$$(1 - uP_{\mathcal{K}}u^*)\, x\, (1 - v^* P_{\mathcal{K}} v) = u\, (1 - P_{\mathcal{K}})\, x_1 \cdots x_p\, (1 - P_{\mathcal{K}})\, v = 0$$

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By **construction of the AFP** and the definition of $\mathcal{K}$, we have

$$(1 - uP_{\mathcal{K}}u^*)\, x\, (1 - v^* P_{\mathcal{K}} v) = u\,(1 - P_{\mathcal{K}})\, x_1 \cdots x_p\, (1 - P_{\mathcal{K}})\, v = 0$$

Applying Φ and using the **vanishing lemma**, we finally have

$$\Phi(x) = \Phi((1 - uP_{\mathcal{K}}u^*)\, x\, (1 - v^* P_{\mathcal{K}} v)) = 0 \qquad \square$$

Proof of the main theorem

Recall that $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to Q$ with $\Phi|_M$ faithful and normal.

Proof of the main theorem.

Fix a faithful state $\psi \in M_*$ such that $\psi \circ (\Phi|_M) = \psi$. The previous **claim** implies that

$$\psi \circ \mathrm{E}_{M_1} = \psi \circ (\Phi|_M) \circ \mathrm{E}_{M_1} = \psi \circ (\Phi|_M) = \psi$$

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Then for every $x \in Q$, the previous **claim** also implies that

$$\|x\|_\psi = \|\Phi(x)\|_\psi = \|\Phi(\mathrm{E}_{M_1}(x))\|_\psi \leq \|\mathrm{E}_{M_1}(x)\|_\psi \leq \|x\|_\psi$$

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Then for every $x \in Q$, the previous **claim** also implies that

$$\|x\|_\psi = \|\Phi(x)\|_\psi = \|\Phi(\mathrm{E}_{M_1}(x))\|_\psi \leq \|\mathrm{E}_{M_1}(x)\|_\psi \leq \|x\|_\psi$$

Then $\|x\|_\psi = \|\mathrm{E}_{M_1}(x)\|_\psi$ and since $\psi \circ \mathrm{E}_{M_1} = \psi$, we have

$$\|x - \mathrm{E}_{M_1}(x)\|_\psi^2 = \|x\|_\psi^2 - \|\mathrm{E}_{M_1}(x)\|_\psi^2 = 0$$

This implies that $x = \mathrm{E}_{M_1}(x) \in M_1$. Therefore $Q \subset M_1$. $\square$