Subfactors and the Brauer-Picard group

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August 5, 2016

A (II_1) subfactor is a unital inclusion $N \subseteq M$ of Type II_1 factors.

The Jones index is the Murray-von Neumann dimension $[M : N] = \dim_N L^2(M)$.

Jones' Index Theorem: $[M : N] \in \{4\cos^2 \frac{\pi}{k}\}_{k=3,4,5\ldots} \cup [4, \infty]$.

Let $N \subseteq M$ be a finite index subfactor.

$\mathcal{N}$ - category of N-N bimodules $\otimes$-generated by

$$ {}_N M_N \cong {}_N M_M \otimes_M {}_M M_N $$

$\mathcal{M}$ - category of M-M bimodules $\otimes$-generated by

$$ {}_M M_N \otimes_N {}_N M_M $$

$\mathcal{N}$ and $\mathcal{M}$ are $\mathbb{C}$-linear semisimple rigid monoidal categories.

The subfactor has **finite depth** if $\mathcal{N}$ and $\mathcal{M}$ have finitely many simple objects. In this case $\mathcal{N}$ and $\mathcal{M}$ are (unitary/C*) **fusion categories**.

Let $N \subseteq M$ be a finite index subfactor with associated tensor categories $\mathcal{N}$ (consisting of N-N bimodules) and $\mathcal{M}$ (consisting of M-M bimodules).

The category $\mathcal{K}$ of N-M bimodules generated by ${}_N M_M$ and $\mathcal{N}$ is a **bimodule category** over $\mathcal{N}$ and $\mathcal{M}$, which is **invertible**:

$$({}_{\mathcal{N}}\mathcal{K}_{\mathcal{M}}) \boxtimes_{\mathcal{M}} ({}_{\mathcal{M}}\mathcal{K}^{op}{}_{\mathcal{N}}) \cong {}_{\mathcal{N}}\mathcal{N}_{\mathcal{N}}$$

and

$$({}_{\mathcal{M}}\mathcal{K}^{op}{}_{\mathcal{N}}) \boxtimes_{\mathcal{N}} ({}_{\mathcal{N}}\mathcal{K}_{\mathcal{M}}) \cong {}_{\mathcal{M}}\mathcal{M}_{\mathcal{M}}$$

An invertible bimodule category is called a **Morita equivalence** (Mueger)

Let $N \subseteq M$ be a finite depth subfactor, with associated fusion categories $\mathcal{N}$, $\mathcal{M}$, and Morita equivalence ${}_{\mathcal{N}}\mathcal{K}_{\mathcal{M}}$.

- The generating object $A = {}_N M_N$ in $\mathcal{N}$ is an **algebra** (Frobenius algebra, division algebra, Longo's Q-system).
- The category ${}_{\mathcal{N}}\mathcal{K}$ is equivalent to the category mod-A of right A-modules in $\mathcal{N}$.
- Then $\mathcal{M} \cong \mathrm{End}({}_{\mathcal{N}}\mathcal{K})$ is the **dual category** of ${}_{\mathcal{N}}\mathcal{K}$, consisting of module endofunctors, and is equivalent to A-mod-A, the category of A-A bimodules in $\mathcal{N}$.
- The generating object ${}_M M_N \otimes_N {}_N M_M$ is similarly an algebra in $\mathcal{M}$ (corresponding to the algebra $A \otimes A$ in A-mod-A.)

This point of view is developed by Mueger, Yamagami.

Give a finite depth subfactor $N \subset M$, we have two fusion categories $\mathcal{N}$ and $\mathcal{M}$, with distinguished algebras, and a Morita equivalence between them.

Natural questions:

- What are all fusion categories in the Morita equivalence class of $\mathcal{N}$ and $\mathcal{M}$?
- What are all Morita equivalences between such categories?
- What are all Q-systems in these categories / all subfactors which realize these categories?

Definition (Etingof-Nikshych-Ostrik)

The Brauer-Picard groupoid of a fusion category $\mathcal{C}$ has:

- Objects - fusion categories which are Morita equivalent to $\mathcal{C}$
- (1-) Morphisms - Morita equivalences

- There are also 2 and 3 morphisms but we will not be concerned with these.
- There are finitely many objects (up to tensor autoequivalence) and finitely many morphisms (up to equivalence of bimodule categories). (**Ocneanu rigidity)**
- E-N-O introduced this 3-groupoid to describe the graded extension theory of fusion categories using homotopy theory

Definition (Etingof-Nikshych-Ostrik)

The Brauer-Picard group of $\mathcal{C}$ is the group of Morita auto-equivalences of $\mathcal{C}$ (modulo equivalence of bimodule categories).

The group

$$\mathrm{Out}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C})/\mathrm{Inn}(\mathcal{C})$$

of outer tensor auto-equivalences of $\mathcal{C}$ is a subgroup of the Brauer-Picard group via

$$\alpha \mapsto {}_{\mathcal{C}}\mathcal{C}_{\alpha(\mathcal{C})}$$

While the Brauer-Picard group is an invariant of Morita equivalence, $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Out}(\mathcal{C})$ are **not**.

The Drinfeld center of $Z(\mathcal{C})$ a monoidal category $\mathcal{C}$ is the category of half braidings of $\mathcal{C}$. An object in $Z(\mathcal{C})$ is an object X in $\mathcal{C}$, together with a collection of isomorphisms $X \otimes Y \to Y \otimes X$, natural in $Y \in \mathcal{C}$, satisfying the "hexagon" relation. The Drinfeld center is a *braided* monoidal category.

For a (spherical) fusion category, the Drinfeld center is a modular tensor category (Mueger).

The Drinfeld center is an invariant of Morita equivalence.

For a finite depth subfactor $N \subseteq M$, the Drinfeld center of $\mathcal{N}$ and $\mathcal{M}$ is one of the even parts of Ocneanu's asymptotic inclusion / Longo-Rehren inclusion.

Theorem (Etingof-Nikshych-Ostrik)

The Brauer-Picard group of a fusion category $\mathcal{C}$ is isomorphic to the group of braided tensor auto-equivalences of the Drinfeld center of $\mathcal{C}$

Basic strategy to compute the Brauer-Picard group/groupoid of a fusion category $\mathcal{C}$:

- Look for algebras in $\mathcal{C}$
- Describe the module categories over these algebras
- Find the dual categories over these module categories
- Look for outer automorphisms
- Repeat, until groupoid is filled in

To “describe” module and bimodule categories, it is sometimes helpful to *decategorify* (throw away morphisms and consider isomorphism classes of objects, product is tensor, sum is direct sum, formal negatives)

- Fusion category $\Longrightarrow$ based ring (**Grothendieck ring**)
- Module category $\Longrightarrow$ based module over Grothendieck ring
- Bimodule category $\Longrightarrow$ based bimodule Grothendieck rings

Improtant to note:

- For a given based module over the Grothendieck ring of a fusion category, there may be many or no corresponding module categories
- The based bimodule of the relative tensor product of two bimodule categories over fusion categories is not necessarily the same as the relative tensor product of the based bimodules. **But** there is a map in one direction, giving combinatorial obstructions.

Let G be a finite group acting by outer automorphisms on a II_1 factor M, and let $N = M^G \subseteq M$.

Then

$$\mathcal{N} \cong \mathrm{Rep}_G, \quad \mathrm{M} \cong \mathrm{Vec}_G$$

- Algebras in Vec_G are parametrized by pairs (H, ω), where H is a subgroup and $\omega \in H^2(H, \mathbb{C}^*)$, modulo conjugacy.
- $\mathrm{Out}(\mathrm{Vec}_G) \cong H^2(G, \mathbb{C}^*) \rtimes \mathrm{Out}(G)$

Ex: Let Let $\mathcal{C} = \mathrm{Vec}_G$ for $G = \mathbb{Z}_3$.

- H^2 is trivial
- Vec_G has 2 module categories (subgroups $\{0\}$ and G)
- $\mathrm{Rep}_G \cong \mathrm{Vec}_G$
- $\mathrm{Out}(G) = \mathbb{Z}_2$

Thus the Brauer-Picard group of $\mathcal{C}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$

Ex: Let $G = A_4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

- 5 subgroups up to conjugacy ($\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, G)
- $H^2(G, \mathbb{C}^*) = H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^*) = \mathbb{Z}_2$
- Vec_G has 7 module categories
- The algebras for $\mathbb{Z}_2 \times \mathbb{Z}_2$ give Morita auto-equivalences
- The algebras for G and $\mathbb{Z}_3$ give Morita equivalences to Rep_G
- The algebra for $\mathbb{Z}_2$ gives a Morita equivalence to a third category $\mathcal{D}$
- $\mathrm{Out}(G) = \mathbb{Z}_2 \Longrightarrow \mathrm{Out}(\mathrm{Vec}_G) = \mathbb{Z}_2 \times \mathbb{Z}_2$

This is enough to deduce that the Brauer-Picard groupoid has exactly three objects and the Brauer-Picard group has order 12.

In fact the group is D_6 (Nikshych-Riepel)

Haagerup subfactor

Unitary fusion category with simple objects

$$\{g\}_{g\in\mathbb{Z}_3} \cup \{gX\}_{g\in\mathbb{Z}_3}$$

and fusion rules

$$gX = Xg^{-1},\ X^2 = 1 + \sum_{g\in\mathbb{Z}_3} gX$$

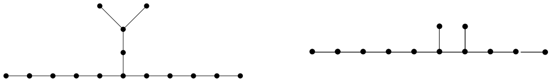
Q-systems for $1 + gX$ (permuted by *inner* automorphisms)

Index: $\dfrac{5+\sqrt{13}}{2}$ (smallest index above 4 for finite depth)

Brauer-Picard groupoid: (G-Snyder)

- 3 module categories $(1,\ 1+gX,\ \sum_{g\in\mathbb{Z}_3} g)$
- dual categories all distinct
- Brauer-Picard group trivial

Principal graphs:



(These bipartite graphs give fusion rules of the bimodule object ${}_N M_M$ - even vertices correspond to simple objects in the fusion categories $\mathcal{N}$ and $\mathcal{M}$, and odd vertices correspond to simple objects in the bimodule category ${}_{\mathcal{N}}\mathcal{K}_{\mathcal{M}}$).

Index: $\dfrac{5+\sqrt{17}}{2}$

$\mathrm{Out}(\mathcal{N})$ and $\mathrm{Out}(\mathcal{M})$ are trivial.

Brauer-Picard group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ (G-Snyder) - proof uses combinatorics of modules and bimodules over based rings.

Theorem (G-Izumi-Snyder)

There is another fusion category in the Morita equivalence class of the Asaeda-Haagerup categories with simple objects

$$\{g\}_{g \in \mathbb{Z}_4} \cup \{gX\}_{g \in \mathbb{Z}_4}$$

and fusion rules

$$gX = Xg^{-1}, \; X^2 = 1 + 2 \sum_{g \in \mathbb{Z}_4} gX$$

and 2 different Q-systems for $1 + gX$ for each g.

- The 8 Q-systems for $1 + gX$ have 4 distinct orbits under inner automorphisms, but the automorphism group of the category acts transtively on these 8 Q-systems - the Brauer-Picard group consists of outer automorphisms on this category
- Thre are 6 fusion categories in the Morita equivalence class

Some consequences of realizing Asaeda-Haagerup categories in Izumi's Cuntz algebra framework

- There are $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded extensions of the AH fusion categories by the Brauer-Picard group
- Modular data of Drinfeld center can be computed and seems to fit into a series (analogous to Evans-Gannon description for Haagerup subfactor)

Izumi-Haagerup subfactors

C^* fusion category with simple objects

$$\{g\}_{g\in G} \cup \{gX\}_{g\in G}, \ G \text{ a finite Abelian group}$$

and fusion rules

$$gX = Xg^{-1}, \ X^2 = 1 + \sum_{g\in G} gX,$$

and a Q-systems for $1 + X$.

- For $G = \mathbb{Z}_3$, this is the Haagerup subfactor.
- For $G = \mathbb{Z}_4$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, exist unique I-H subfactors (Izumi, Morrison-Penneys,Penneys-Peters)
- The Asaeda-Haagerup category above is an *orbifold* of even part of an I-H subfactor for $G = \mathbb{Z}_4 \times \mathbb{Z}_2$
- Numerical evidence for existence for $G = \mathbb{Z}_{2n+1}$ up to $n = 9$ (Evans-Gannon)

What do Brauer-Picard groups look like for Izumi-Haagerup subfactors?

We have seen that the BP group for the Haagerup subfactor is trivial.

For $G = \mathbb{Z}_4$, the BP group is $\mathbb{Z}_2$.

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, it is more interesting.

Unlike other known examples of I-H subfactors, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ I-H subfactor is self-dual - the Q-system for $1 + X$ gives a (non-trivial) Morita autoequivalence.

But there are many others.

Equivariantization: Let G be a finite group acting by tensor automorphisms on a fusion category $\mathcal{C}$. The equivariantization $\mathcal{C}^G$ is the tensor category of "fixed points" of G.

Here a "fixed point" means an orbit under the G-action which is linked by a collection of isomorphisms $g(X) \to X$ satisfying a coherence relation.

There is also a crossed product construction $\mathcal{C} \rtimes G$, and

$$\mathcal{C}^G \overset{\text{M.E.}}{\cong} \mathcal{C} \rtimes G$$

Ex: For the trivial action of G on *Vec*,

$$\mathcal{C}^G \cong \mathrm{Rep}_G \text{ and } \mathcal{C} \rtimes G \cong \mathrm{Vec}_G$$

For the I-H subfactor for $\mathbb{Z}_2 \times \mathbb{Z}_2$, there is a $\mathbb{Z}_3$ action which fixes X and cyclically permutes the non-trivial invertible objects. (Izumi)

The equivariantization of the I-H Q-system for $1 + X$ gives the "4442" subfactor.

Let $\mathcal{C}$ be the fusion category for the I-H subfactor for $\mathbb{Z}_2 \times \mathbb{Z}_2$. We have

$$\mathrm{Rep}_{A_4} \subset \mathcal{C}^{\mathbb{Z}_3} \longleftrightarrow \mathcal{C} \rtimes \mathbb{Z}_3 \supset \mathrm{Vec}_{A_4}$$

There are Morita autoequivalences of $\mathcal{C} \rtimes \mathbb{Z}_3$ given by algebras for $\mathbb{Z}_2 \times \mathbb{Z}_2$ (order 3) and $1 + X$ (order 2).

The algebra for $\mathbb{Z}_2$ gives a M.E. to a third category, and the algebras for $1 + gX$, $g \neq 0$, gives a M.E. to a fourth category.

Theorem

The Brauer-Picard groupoid for $\mathcal{C}^{\mathbb{Z}_3}$ has four objects and the BP group is S_3.

From the BP groupoid for $\mathcal{C}^{\mathbb{Z}_3}$, we can deduce the BP groupoid of $\mathcal{C}$.

Theorem

Lett $\mathcal{C}$ be the fusion category associated to the I-H subfactor for $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are 30 module categories over $\mathcal{C}$, all of which give Morita autoequivalences. Since Out($\mathcal{C}$) has order 12, the Brauer-Picard has order 360.

The Drinfeld center of $\mathcal{C}$ has rank 40, and decomposes as a tensor product of a rank 4 and rank 10 modular tensor categories (G-Izumi). What are its braided autoequivalences?