Connective C*-algebras

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in celebration of Professor Sakai's seminal work

K-homology: $K^0(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\iota : C(\mathbb{T}^2) \to \mathbb{C}$ and by a "discrete asymptotic morphism" $\varphi_n : C(\mathbb{T}^2) \to M_n(\mathbb{C})$, $\varphi_n(z_1) = u_n$ and $\varphi_n(z_2) = v_n$

Voiculescu's ('83) almost commuting unitaries

$\|v_n u_n - u_n v_n\| = |e^{2\pi i/n} - 1| \to 0.$

$$u_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad v_n = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda^3 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix} \quad \lambda = e^{2\pi i/n}$$

$$K^0(A) = KK(A, \mathbb{C}) \to \mathrm{Hom}(K_0(A), \mathbb{Z})$$

$$(\varphi_n)_\sharp : K_0(C(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}\beta \to \mathbb{Z}$$

$$(\varphi_n)_\sharp(\beta) \equiv 1$$

A separable C*-algebra.

A is **residually finite dimensional** (RFD) if $\exists$ $*$-mono $A \hookrightarrow \prod_{n=1}^{\infty} M_{k(n)}$.

A is **quasidiagonal** if $\exists$ $*$-mono

$$A \hookrightarrow \frac{\prod_{n=1}^{\infty} M_{k(n)}}{\bigoplus_{n=1}^{\infty} M_{k(n)}}$$

liftable to cpc map $A \to \prod_{n=1}^{\infty} M_{k(n)}$.

Equivalently, $\exists$ $\{\varphi_n : A \to M_{k(n)}\}_n$ cpc maps
such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ and $\|\varphi_n(a)\| \to \|a\|$.

QD algebras more abundant, better permanence properties than RFD.

Ozawa-Rørdam-Sato: G elementary amenable group $\Rightarrow$ $C^*(G)$ is QD.
Tikuisis-White-Winter: separable and nuclear C*-algebras in the UCT class with faithful trace are QD. $C^*(G)$ is QD for all amenable groups.

The cone: $CB(H) = C_0[0,1) \otimes B(H)$

A is **connective** if $\exists$ $*$-mono

$$A \hookrightarrow \frac{\prod_{n=1}^{\infty} CB(H_n)}{\bigoplus_{n=1}^{\infty} CB(H_n)}$$

liftable to cpc map $A \to \prod_{n=1}^{\infty} CB(H_n)$.

Connectivity of a separable C*-algebra has three consequences:

- absence of nonzero projections (Cohen, Choi)
- quasidiagonality (Voiculescu)
- allows de-suspension in E-theory for nuclear algebras (D-Pennig)

First examples

- If $\exists\ \pi : A \hookrightarrow B(H)$ null homotopic then A is connective. Indeed any $A \subset C_0(0,1] \otimes B(H) =: CB(H)$ is connective.

If G discrete countable group, let $I(G)$ be the augumentation ideal:

$$0 \to I(G) \to C^*(G) \xrightarrow{\iota} \mathbb{C} \to 0$$

Any representation of $\mathbb{F}_n$ (Choi) or $\mathbb{F}_n \times \mathbb{F}_m$ (Brown-Ozawa) is homotopic to a multiple of the trivial representation ι. Thus:

- $I(\mathbb{F}_n)$ and $I(\mathbb{F}_n \times \mathbb{F}_m)$ are connective.

Hahn-Mazurkiewicz theorem: $\forall\ X$ connected, locally connected, compact metrizable space is a quotient of $[0,1]$, i.e. X is a Peano space.

- X Peano space $\Rightarrow C_0(X \setminus x_0) \subset C_0(0,1]$ is connective

Remark

- A connective $\Leftrightarrow \exists$ a sequence $\{\varphi_n : A \to CB(H_n)\}_{n\in\mathbb{N}}$ of cpc maps :

$$\|\varphi_n(a)\varphi_n(a) - \varphi_n(ab)\| \to 0, \quad \|\varphi_n(a)\| \to \|a\|$$

- May arrange all H_n finite dimensional using contractibility $\Rightarrow$ quasidiagonality

Non Peano spaces

X Hausdorff compact and $x_0 \in X$,
$C_0(X \setminus x_0)$ connective $\Leftrightarrow X$ connected
$\Leftrightarrow X \setminus x_0$ has no compact open subsets $\neq \emptyset$.

Proof: For any $x \in X$ find discrete path $x_1, x_2, ..., x_n$ joining x with x_0 then linearly interpolate ev_{x_i}.

E-theory and deformations

An asymptotic morphism $(\varphi_t)_{t\in[0,\infty)}$ is a family of maps $\varphi_t \colon A \to B$ parametrized by $t \in [0,\infty)$ such that $t \mapsto \varphi_t$ is pointwise continuous and the axioms for $*$-homomorphisms are satisfied *asymptotically* for $t \to \infty$.
Homotopy classes of asymptotic morphisms from the suspension of A to the stabilization of the suspension of B provide a model for **E-theory**:
Connes-Higson ('90): $E(A,B) = [[SA, SB \otimes \mathcal{K}]]$.
H. Larsen-Thomsen: $KK(A,B) \cong [[SA, SB \otimes \mathcal{K}]]^{\mathrm{cpc}}$.
The suspensions and the stabilization of B are necessary to obtain a natural abelian group structure on $E(A,B)$.
Equally important: SA becomes **quasidiagonal**, hence $\exists$ a large supply of almost multiplicative maps $SA \to \mathcal{K}$.
A deformation $\varphi_t : A \to B \otimes \mathcal{K}$ contains in principle more geometric information. We are confronted with the dilemma of understanding $[[A, B \otimes \mathcal{K}]]$, while only $E(A,B) = [[SA, SB \otimes \mathcal{K}]]$ is computable.
Best case scenario: the monoid homomorphism $[[A, B \otimes \mathcal{K}]] \to E(A,B)$ induced by the suspension map is an isomorphism.

Unsuspending in E-theory

Theorem (D-Loring 94) TFAE

- $[[A, B \otimes \mathcal{K}]] \to E(A, B)$ is an isomorphism $\forall\ B$
- $[[A, A \otimes \mathcal{K}]]$ is a group
- A is homotopy symmetric, i.e. $[[\mathrm{id}_A]] \in [[A, A \otimes \mathcal{K}]]$ invertible.

Fact: X Hausdorff compact and connected and $x_0 \in X \Rightarrow C_0(X \setminus x_0)$ is homotopy symmetric (Dad 94). This explains Voiculescu's example.

Theorem (D-Pennig)

A nuclear is homotopy symmetric $\Leftrightarrow$ A is connective.

Corollary

A nuclear connective $\Rightarrow [[A, B \otimes \mathcal{K}]] \cong KK(A, B)$

$[[A,B]]_{\mathbb{N}}$ =homotopy classes of discrete asymptotic homs.
We showed $[[A, SB \otimes \mathcal{K}]]_{\mathbb{N}}$ and $[[A, B \otimes \mathcal{K}]]_{\mathbb{N}}$ are groups if A is nuclear & connective. If B unital the arguments are reminiscent of $Ext(A,B)$ is a group for A nuclear. Extend to nonunital case using Puppe-sequences.
Thomsen: $\exists$ exact sequence of pointed sets

$$[[A, SB \otimes \mathcal{K}]]_{\mathbb{N}} \xrightarrow{\alpha} [[A, B \otimes \mathcal{K}]] \xrightarrow{\beta} [[A, B \otimes \mathcal{K}]]_{\mathbb{N}} \xrightarrow{1-\sigma} [[A, B \otimes \mathcal{K}]]_{\mathbb{N}}.$$

σ is the shift map $\sigma[[\psi_n]] = [[\psi_{n+1}]]$, β is the natural restriction map and α is defined by stringing together the components of a discrete asymptotic morphism $\{\varphi_n : A \to C_0(0,1) \otimes B \otimes \mathcal{K}\}_n$ to form a continuous asymptotic morphism $\{\Phi_t : A \to B \otimes \mathcal{K}\}_{t\in[0,\infty)}$.
$(1-\sigma)$ is a morphism of groups and both α and β are monoid homomorphisms. The exact sequence of pointed monoids

$$[[A, SB \otimes \mathcal{K}]]_{\mathbb{N}} \to [[A, B \otimes \mathcal{K}]] \to \ker(1-\sigma) \to 0$$

implies that $[[A, B \otimes \mathcal{K}]]$ is a group. (H-spaces analogy).

Permanence properties of connectivity yield large classes of homotopic symmetric C*-algebras

Theorem (D-Pennig)

(a) *homotopy symmetry passes to nuclear subalgebras.*

(b) *If* $(A_n)_n$ *connective and* $A \subset \prod_n A_n / \bigoplus_n A_n \Rightarrow A$ *is homotopy symmetric.*

(c) $(A_n)_n$ *nuclear homotopy symmetric* $\Rightarrow \operatorname{inj\,lim}_n A$ *homotopy symmetric.*

(d) *If* $0 \to J \to A \to B \to 0$ *exact sequence.*
J, B *nuclear, homotopy symmetric* $\Rightarrow A$ *homotopy symmetric.*

(e) *The class of homotopy symmetric* C^**-algebras is closed under tensor products by separable* C^**-algebras and under (asymptotic) homotopy equivalence.*

(f) *The class of separable nuclear homotopy symmetric* C^**-algebras is closed under crossed products by second countable compact groups.*

Corollary

- *If A sep. nuclear, then A connective $\Leftrightarrow$ $A \otimes O_2$ connective.*
- *Let A, B sep. nuclear with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$.*
  Then A connective $\Leftrightarrow$ B connective.

Second part uses Kirchberg's results.

Corollary

A sep. nuclear continuous field over compact connected metrizable X.
If some fiber $A(x_0)$ homotopy symmetric $\Rightarrow$ A homotopy symmetric.

Proof: Embed A in $E := \{f \in C(X, O_2) \colon f(x_0) \in A(x_0)\}$ (Blanchard)

$$0 \to C_0(X \setminus x_0) \otimes O_2 \to E \to A(x_0) \to 0$$

Discrete groups G

Recall augmentation ideal $I(G)$: $0 \to I(G) \to C^*(G) \xrightarrow{\iota} \mathbb{C} \to 0$

A discrete countable group G is connective if $I(G)$ is connective.

Corollary

Let $1 \to N \to G \to H \to 1$ be a central extension of discrete countable amenable groups where N is torsion free. H connective $\Rightarrow$ G connective.

Proof: $C^*(G)$ continuous field over $\widehat{N}$ with one fiber isomorphic to $C^*(H)$.

Theorem (D-Pennig)

If G is a countable torsion free nilpotent group, then G is connective hence $K^0(I(G)) \cong [[I(G), \mathcal{K}]]$.

If $s \in G$, $s^n = 1$, $n > 1$, then G is not connective.
$1 - \frac{1}{n}(1 + s + \cdots + s^{n-1})$ nonzero projection in $I(G)$.

G, H countable discrete groups and let J be a set with a left action of H. Wreath product: $G \wr H = (\bigoplus_J G) \rtimes H$.

Theorem (D-Pennig-Schneider)

G, H connective $\Rightarrow$ $G \wr H$ connective.

Corollary

G, H connective and $\alpha \colon H \to \mathrm{Aut}(G)$ periodic $\Rightarrow$ $G \rtimes_\alpha H$ connective.

Examples

The free solvable groups $S_{r,n}$ on r generators of derived length n.

- $S_{r,1} \cong \mathbb{Z}^r$
- $S_{r,2}$ is the free metabelian group on r generators.

$S_{r,2}$ has universal property: it maps surjectively onto any other metabelian group with r generators.

Obstructions to connectivity of A

(1) existence of nonzero projections in $A \otimes B$.
(2) non-quasidiagonality

Prop. (Pasnicu-Rørdam)

If $\mathrm{Prim}(A)$ has a non-empty compact-open subset $\Rightarrow A \otimes O_\infty$ contains non-zero projections.

Thus, if $\mathrm{Prim}(A)$ has a non-empty compact-open subset $\Rightarrow A$ not connective.

Prop.

If A is nuclear and $\mathrm{Prim}(A)$ is Hausdorff, then A is connective $\Leftrightarrow$ if $\mathrm{Prim}(A)$ does not contain a non-empty compact-open subset.

$\pi \in \widehat{A}$ is shielded, if $\nexists$ eventually non-constant sequence in $\widehat{A}$ convergent solely to π.

Lemma

If π is a shielded and closed point of $\widehat{A} \Rightarrow \ker \pi$ not connective.

Proof: $\mathrm{Prim}(\ker \pi)$ is compact-open.

Corollary

Let G be a countable discrete group. If the trivial representation $\iota \in \widehat{G}$ is shielded, then G is not connective.

Proof: $I(G) = \ker \iota \Rightarrow \mathrm{Prim}(I(G))$ is compact-open.

The Hantzsche-Wendt group

There are precisely 10 closed flat 3-dimensional manifolds.
J.H. Conway and J.P. Rossetti call these manifolds platycosms ("flat universes").
The Hantzsche-Wendt manifold (called didicosm) is the only platycosm with finite homology. Its fundamental group G:

$$G = \langle x, y \colon x^2 y x^2 = y, \quad y^2 x y^2 = x \rangle$$

G is a torsion free 3-dim crystallographic group (Bieberbach group).

$$1 \to \mathbb{Z}^3 \to G \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1.$$

D-Pennig

The Hantzsche-Wendt group G is not connective.

Amenable Lie groups

We discuss the connectivity of (reduced) C*-algebras associated to Lie groups

If G is a solvable Lie group whose center contains a noncompact closed connected subgroup, then $C^*(G)$ is connective.

Let G be a (real or complex) linear connected nilpotent Lie group. Then $C^*(G)$ is connective if and only if G is not compact.

G linear connected reductive group G if

$$G \subset GL_n(\mathbb{R}) \quad \text{or} \quad G \subset GL_n(\mathbb{C})$$

is a selfadjoint closed connected group.
G semisimple if it has finite center.

Reductive Lie groups

(complex case)

If G is a linear connected complex reductive Lie group, then $C_r^*(G)$ is connective if and only if G is not compact.

G complex: $C_r^*(G) \cong C_0(\widehat{G}_r, \mathcal{K})$. (Lipsman, Penington-Plymen)
$\widehat{G}_d=$ discrete series reps $=$ equivalence classes of square-integrable reps.

(real case)

Let G be a linear connected real reductive Lie group. TFAE:

(i) $C_r^*(G)$ is connective
(ii) $\widehat{G}_d = \emptyset$
(iii) G does not have a compact Cartan subgroup
(iv) there are no nonzero projections in $C_r^*(G)$.

Harish-Chandra, J. Arthur, A. Wassermann, Higson-Clare-Crisp:

$$C_r^*(G) \hookrightarrow \bigoplus_{\sigma \in \widehat{G}_d} K(H_\sigma) \oplus \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, K(H_\sigma)),$$

where the second direct sum involves proper parabolic subgroups and hence $\dim(\widehat{A}_P) > 0$ (A_P are real vector groups).
$C_r^*(\mathrm{SL}_n(\mathbb{R}))$ is not connective since $\mathrm{SL}_n(\mathbb{R})$ has discrete series representations, $n \geq 2$.
$C_r^*(\mathrm{SO}(p,q))$ is connective $\Leftrightarrow$ pq=odd.

Let G be a linear connected real reductive Lie group. Then $\bigcap_{\pi \in \widehat{G}_d} \ker(\pi) \subset C_r^*(G)$ is always a connective C*-algebra.

Proof: $\bigcap_{\pi \in \widehat{G}_d} \ker(\pi) \subset \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, K(H_\sigma))$.

Full C*-algebras of Lie groups

G semisimple, the adjoint group $\mathrm{Ad}(G) = G/Z(G) =$ direct product of simple Lie groups.

Using results of Valette:

Let G be a complex connected semisimple Lie group. The following assertions are equivalent.

(i) $C^*(G)$ is connective.

(ii) $C^*(G)$ has no nontrivial projections.

(iii) The adjoint group $\mathrm{Ad}(G)$ of G has at least one simple factor which is isomorphic to a Lorentz group $\mathrm{PSL}_2(\mathbb{C})$.

$C^*(\mathrm{SL}_2(\mathbb{C}))$ is connective but $C^*(\mathrm{SL}_3(\mathbb{C}))$ is not. However

$$I(\mathrm{SL}_3(\mathbb{C})) = \ker\left(\iota : C^*(\mathrm{SL}_3(\mathbb{C}) \to \mathbb{C}\right) \text{ connective}$$

(Using results of Fell and Francois Pierrot)