Dual Temperley Lieb basis, Quantum Weingarten and a conjecture of Jones

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Overview

Joint work with Mike Brannan (TAMU) – (trailer... cf next Monday's arXiv)

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1. Temperley Lieb algebra and main result
2. Free orthogonal quantum group O_d^+ and Weingarten calculus.
3. Outline of proof and main result.

Temperley Lieb algebra

Let $d \in \mathbb{C}^*$ and $k \in \mathbb{N}$ be fixed parameters. The *Temperley-Lieb algebra* $TL_k(d)$ is the unital associative algebra generated by elements $1, u_1, \ldots, u_{k-1}$ subject to the following relations

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- $u_i u_j = u_j u_i$ when $|i - j| \geq 2$
- $u_i u_{i+1} u_i = u_i$
- $u_i^2 = du_i$

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- Consider circles with $2k$ points and non-crossing pair partitions $NC_2(2k)$ on it. There are also C_k such elements, and they can be seen as a basis of the Temperley Lieb algebra.
- $\mathbb{C}[NC_2(2k)] = TL_k(d)$.

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- This is known to extend to a faithful scalar product on $\mathbb{C}[NC_2(2k)] = TL_k(d)$ if $d \in [2, \infty)$
- This scalar product comes from the Markov trace on the Temperley Lieb algebra.

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- The *dual basis* associated to B is the unique linear basis $\hat{B} = \{\hat{x}_1, \ldots, \hat{x}_n\}$ of E such that

$$\langle x_i, \hat{x}_j \rangle = \delta_{ij}. \qquad (1 \leq i, j \leq n).$$

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- For $TL_k(d)$ (with $k \in \mathbb{N}, d \in [2, \infty)$) and the Markov trace Hilbert structure, we consider the canonical diagram basis $B = \{D_p\}_{p \in NC_2(2k)}$ and the corresponding dual basis $\hat{B} = \{\hat{D}_p\}_{p \in NC_2(2k)}$.

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- Theorem (main result)

Yes, for any $d \in \mathbb{R} - (-2, 2)$.

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- Partial results/confirmations of Ocneanu by Reznikoff, Morrison, Frenkel-Khovanov.

A particular case of interest: the JW projection

- Let $d \in [2, \infty)$ and $k \in \mathbb{N}$. Then there exists a unique non-zero self-adjoint projection $q_k \in TL_k(d)$, the *Jones-Wenzl* projection, with the property that

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- Since $d^{-1/2} u_i$ is a projection for $1 \leq i \leq k-1$, we can abstractly define q_k via the formula

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- There are recursion relations for defining q_k.

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2. The dual basis elements of Temperley Lieb algebra (and similar elements – permutation algebra, partition algebra, Brauer algebra) are well studied for the purpose of computing Haar measures over (quantum) groups: *Weingarten calculus*.

Quantum groups

- The *algebra of polynomial functions on the free orthogonal quantum group* is the universal unital $*$-algebra

$$O_d^+ := * - \mathrm{alg}\big((u_{ij})_{1\leq i,j\leq d} \mid U = [u_{ij}] \text{ unitary in } M_d(O_d^+) \& U = \bar{U}\big). \tag{1}$$

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- It is an example of Woronowicz's *compact quantum group* (Wang)

Quantum groups

- *Coproduct*: a unital $*$-homomorphism $\Delta : O_d^+ \to O_d^+ \otimes O_d^+$ determined by

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- It satisfies *co-associativity*: $(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta$.
- There exists a unique *Haar integral*. That is, a faithful state $\mu = \mu : O_d^+ \to \mathbb{C}$, left and right invariant

$$(\mu \otimes \iota)\Delta = (\iota \otimes \mu)\Delta = \mu(\cdot)1. \tag{2}$$

Quantum Weingarten

Theorem (Weingarten formula; Banica, C)

If l is odd,

$$\mu(u_{i(1)j(1)}u_{i(2)j(2)}\dots u_{i(l)j(l)}) = 0$$

otherwise,

$$\mu(u_{i(1)j(1)}u_{i(2)j(2)}\dots u_{i(l)j(l)}) = \sum_{\substack{p,q\in NC_2(l)\\ \ker j\geq p,\ \ker i\geq p}} Wg_d(p,q),$$

Dual coefficients with Weingarten

Theorem

Dual basis element $\hat{D}_p$ associated to to a diagram $D_p \in TL_k(d)$ is given by

$$\hat{D}_p = \sum_{q \in NC_2(2k)} Wg_d(p,q) D_q,$$

Dual coefficients with Weingarten

Theorem

For $d \in [2, \infty)$, the kth Jones-Wenzl projection $q_k \in TL_k(d)$ is given by

$$q_k = \sum_{q \in NC_2(2k)} \frac{\mathrm{Wg}_d(\mathbf{1}, q)}{\mathrm{Wg}_d(\mathbf{1}, \mathbf{1})} D_q,$$

Back to the initial problem

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Question:
Show that $\mathrm{Wg}_d(p, q)$ is never zero, for any $p, q \in NC_2(2k)$ and $d \geqslant 2$.

Classical decay estimates

This question admits a generic answer in the classical case.

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In the U_n and O_n case:

Theorem (C, Śniady)

$\mathrm{Wg}(p, q) \sim Moeb(1, p \vee q/2) d^{-k-|p \vee q|/2}$.

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1. For classical groups, it is difficult to go beyond the 'generic' case with the techniques developed by C, Śniady, Novak, Matsumoto, etc...

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Remarks:

1. For classical groups, it is difficult to go beyond the 'generic' case with the techniques developed by C, Śniady, Novak, Matsumoto, etc...
2. Much less was known so far about the asymptotics of the free orthogonal group Weingarten function.

Quantum decay estimates

- The results of Banica, Curran, Speicher provide estimates of the form

$$\mathrm{Wg}_d(p, q) = \begin{cases} O(d^{-2k+|p \vee q|}), & p \neq q \\ d^{-k} + O(d^{-k-2}), & p = q \end{cases} \qquad (d \to \infty).$$

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- In Curran Speicher 2011, a sharper result in a few more cases (including some explicit asymptotics for some $p \neq q$).
- Unfortunately these prior results are far from covering all values. As a simple low rank example to illustrate this, we have the following example.

Quantum decay estimates

- Let $k = 4$, $p = \{1,6\}\{2,5\}\{3,4\}\{7,8\}$, and $q = \{1,2\}\{3,8\}\{4,7\}\{5,6\}$.

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- Then $|p \vee q| = 2$, and the results of Banica, Curran, Speicher predict $\mathrm{Wg}_d(p,q) = O(d^{-6})$.
- But in fact the leading order turns out to be much smaller: one actually has $m_0(p,q) = 1$, $L(p,q) = 8$, and our main results actually yields

$$\mathrm{Wg}_d(p,q) = d^{-8} + O(d^{-10}).$$

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- $\sum_i u_{1i}^2 = 1$ translates into $d\,\mathrm{Wg}_d(\{(1,2)\},\{(1,2)\}) = 1$.
- similarly
$$\mu(\sum_i u_{1i}u_{2i}u_{21}u_{11}) = 0$$
translates into
$$d\,\mathrm{Wg}_d(\{(14)(23)\},\{(12)(34)\}) + Wg_d(\{(14)(23)\},\{(14)(23)\}) = 0$$

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- We consider *them all*. This gives many equations on Wg, or the form

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where $***$ are NC pair partitions, and xxx are integers, and the sum is finite.
- Quantum groups theory shows that it holds for all d integers ≥ 2. By rationality, it holds for all generic complex numbers.

Classical case

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Theorem (Weingarten)

There are enough orthogonality relations to determine Wg *in large enough dimension in the classical setup.*

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- 'Weingarten calculus' is a series of techniques to compute Wg without orthogonality relations...
- ...however, at this point, we have no option but use Weingarten's original orthogonality idea in the quantum case to obtain satisfactory (and indeed, optimal) estimates.

Weingarten graph

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- The vertex set is given by

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where by convention we define $NC_2(0) \times NC_2(0) = \{(\emptyset, \emptyset)\}$.

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where by convention we define $NC_2(0) \times NC_2(0) = \{(\emptyset, \emptyset)\}$.
- $((p, q)(p', q'))$ is an edge if there exists an *orthogonality relation* for which (p', q') appears in the decomposition of (p, q).

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- For any (p, q) there is a path to $(\emptyset, \emptyset)$.
- Therefore, there exists a *shortest distance* $L(p, q)$, from (p, q) to $(\emptyset, \emptyset)$.
- The diagram has a parity property.

Weingarten sub-diagram

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We fix once and for all such a Weingarten subgraph $H \subset G$.
H exists but is not uniquely defined.

Main result

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Theorem

Then the Weingarten function $d \mapsto \mathrm{Wg}_d(p,q)$ admits the following absolutely convergent Laurent series expansion

$$\mathrm{Wg}_d(p,q) = (-1)^{|p \vee q| + k} \sum_{r \geq 0} m_r(p,q) d^{-L(p,q)-2r}$$

for $\left(|d| > 2\cos\left(\frac{\pi}{k+1}\right)\right)$.

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- The leading order term of $\mathrm{Wg}_d(p,q)$ is given by

$$\mathrm{Wg}_d(p,q) \sim m_0(p,q)(-1)^{k+|p\vee q|} d^{-L(p,q)} \neq 0 \qquad (|d| \to \infty).$$

Sign of coefficients

We rescale the sign of Wg as follows:

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The interest of this notation is that $\widetilde{Wg}_d(p,q)$ will always be positive.

Paths on the subgraph

- The orthogonality relation at (p, q) yields $\widetilde{Wg}_d(p, q) = d^{-1} \sum_{(p_1, q_1)} \widetilde{Wg}_d(p_1, q_1)$ (partial sum)

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$$\widetilde{Wg}_d(p, q) = \sum_{(p', q') \in V - \{\emptyset, \emptyset\}} c_{(p', q'), 1} \widetilde{Wg}_d(p', q') + K_1,$$

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where K_1 is 0 unless (p, q) was the element of $NC_2(2)^2$, and all coefficients $c_{(p', q'), 1}$ are zero unless $((p, q), (p', q'))$ is an edge of H (in which case it is $1/d$).

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l.h.s: what we want to evaluate
r.h.s: a (stationary) sequence that we will need to evaluate the l.h.s

Convergence as a power series

The important things to note are

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1. $c_{(p',q'),s}$ is of the form pd^{-s} where p is a natural number (a number of paths of length s).
2. K_s is a polynomial in d^{-1} with natural numbers as coefficients. Viewing K_s as a sequence or polynomials, the induced sequence of coefficients of degree l becomes steady as soon as $k \geqslant l$.

Consequently,

$$\sum_{(p',q')\in V-\{\emptyset,\emptyset\}} c_{(p',q'),s}\widetilde{Wg}_d(p',q') + K_s$$

converges as a power series in d^{-1}.

Convergence as a sequence

- Getting back to

$$\widetilde{Wg}_d(p,q) = \sum_{(p',q')\in V-\{\emptyset,\emptyset\}} c_{(p',q'),s}\widetilde{Wg}_d(p',q') + K_s,$$

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- Therefore,

$$\sum_{(p',q')\in V-\{\emptyset,\emptyset\}} c_{(p',q'),s}\widetilde{Wg}_d(p',q') + K_s$$

  converges also as a sequence when $d > k+1$.

Radius of convergence

- In addition, $\widetilde{Wg}_d(p,q)$ is a rational fraction in d with poles $|d| \leqslant 2\cos\left(\frac{\pi}{k+1}\right)$ [Crámer + a theorem by di Francesco]

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Remark: this large convergence radius (and the independence on the choice of H) is not obvious at all from the combinatorics of the proof....

Summary

Theorem

The Weingarten function $d \mapsto \mathrm{Wg}_d(p,q)$ *admits the following absolutely convergent Laurent series expansion*

$$\mathrm{Wg}_d(p,q) = (-1)^{|p \vee q|+k} \sum_{r \geq 0} m_r(p,q) d^{-L(p,q)-2r}$$

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As a corollary:

Theorem

given $p \in NC_2(2k)$ and d generic, let $f_{pq,d}$ be defined by the equation

$$\hat{D}_p = \sum f_{pq,d} D_q.$$

Then $f_{pq,d} \neq 0$ for any $d \in \mathbb{R} - (-2,2)$.

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