Noncommutative Spectral Invariants and Black Hole Entropy

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Abstract

We consider an intrinsic entropy associated with a local conformal net $\mathcal{A}$ by the coefficients in the expansion of the logarithm of the trace of the "heat kernel" semigroup. In analogy with Weyl theorem on the asymptotic density distribution of the Laplacian eigenvalues, passing to a quantum system with infinitely many degrees of freedom, we regard these coefficients as noncommutative geometric invariants. Under a natural modularity assumption, the leading term of the entropy (noncommutative area) is proportional to the central charge c, the first order correction (noncommutative Euler characteristic) is proportional to $\log \mu_{\mathcal{A}}$, where $\mu_{\mathcal{A}}$ is the global index of $\mathcal{A}$, and the second spectral invariant is again proportional to c.

We give a further general method to define a mean entropy by considering conformal symmetries that preserve a discretization of S^1 and we get the same value proportional to c.

We then make the corresponding analysis with the proper Hamiltonian associated to an interval. We find here, in complete generality, a proper mean entropy proportional to $\log \mu_{\mathcal{A}}$ with a first order correction defined by means of the relative entropy associated with canonical states.

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By considering a class of black holes with an associated conformal quantum field theory on the horizon, using in part existing analysis, one gets a possible way to link the noncommutative area with the Bekenstein-Hawking classical area description of entropy.

1 Introduction

This paper essentially deals with chiral conformal Quantum Field Theory, but our motivations primarily concern black hole thermodynamics; the basic link to this subject is through QFT on a curved spacetime and the idea, that has appeared from different and independent viewpoints in recent literature, that the restriction of the quantum field to the black hole horizon should give rise to a conformal QFT. Combined with the well known Bekenstein interpretation of the area of the horizon as proportional to the black hole entropy, this suggests that a geometric definition of the entropy of conformal QFT should play a relevant rôle in black hole thermodynamics. To this end we shall define an intrinsic entropy associated to a conformal QFT, with a noncommutative geometrical point of view. We will regard a local conformal net as a noncommutative manifold or, more precisely, a QFT manifold (i.e. a noncommutative manifold with infinitely many degrees of freedom) and shall be guided in our analysis by the classical equivalent, most importantly from Weyl's asymptotic for the trace of the heat kernel. One could say that in our framework back reaction effects of the quantum fields on the classical spacetime are negligible, but do affect the geometry of the associated noncommutative manifold.

Our paper is organized as follows:

- Here below we recall a number of ideas about black hole physics that have motivated our work, yet we refer to the literature (see e.g. [53]) for basics facts on black hole thermodynamics as Hawking effect, generalized second low, etc..

- We then recall Weyl's theorem that motivates our "QFT ellipticity" assumption on the conformal Hamiltonian, i.e. on the asymptotic of logarithmic of the characters (elementary motivations are contained in Appendix B). This assumption holds in all computed cases. We shall show that it holds for all modular local conformal nets, namely nets with the usual rational behavior (see Sect. 3.2) and it turns out to hold in particular in all models with central charge less than one, that are classified in [29, 30]. Indeed one has the asymptotic formula for a modular net $\mathcal{A}$

$$\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12}\frac{1}{t} + \log\frac{d(\rho)}{\sqrt{\mu_{\mathcal{A}}}} - \frac{\pi c}{12}t \ , \quad \text{as } t \to 0^+ \tag{1}$$

  where c is the central charge, $L_{0,\rho}$ and $d(\rho)$ are the conformal Hamiltonian and the DHR dimension of the representation ρ, and $\mu_{\mathcal{A}}$ is equal to the global index $\sum_i d(\rho_i)^2$, the sum of the indices of all DHR charges [31, 38] (see Sect. 3.2).

- Our basic object is a local conformal net $\mathcal{A}$ of von Neumann algebras, namely the family of local operator algebras maximally generated by smeared fields (basic notions can be found in Appendix A); this is our noncommutative manifold and we use (temporarily) the QFT ellipticity/modularity assumption for our analysis. In analogy with Weyl's theorem we define the noncommutative geometric spectral invariants $\{a_i\}$ of a conformal net (the coefficients in the above asymptotic (1,13)), in particular the noncommutative area and the noncommutative Euler characteristic. Indeed as we are in the QFT setting (thus with infinitely many degrees of freedom) $\log \mathrm{Tr}(e^{-2\pi t L_0})$, rather than $\mathrm{Tr}(e^{-2\pi t L_0})$, provides the asymptotic of the corresponding finite-dimensional system, see Appendix B.

  For the Physics viewpoint, $\log \mathrm{Tr}(e^{-2\pi t L_0})$ counts logarithmically the number of possible states and so determines the microscopic entropy $S_{\mathcal{A}}$ of the system, therefore we put

$$S_{\mathcal{A}} \equiv a_0 \ .$$

  The following table summarizes the the value and the meaning of the spectral invariants (up to proportionality constants):

Invariant	*Value*	*Geometry*	*Physics*
a_0	$\pi c/12$	Noncommutative area	Entropy
a_1	$-\frac{1}{2}\log \mu_{\mathcal{A}}$	Noncommmutative Euler characteristic	1st order entropy
a_2	$-\pi c/12$	2nd spectral invariant	2nd order entropy

  Note that $a_2 = -a_0$, that is a consequence of the modular symmetry.

  The analog of the Kac-Wakimoto formula [35], and more generally the quantum index formula in [37], can now be read as an expression that the incremental free energy (adding/removing DHR charges [16]) is proportional to the increment of the noncommutative Euler characteristic (Sect. 3.3).

- We shall show that, for a conformal net on the two-dimensional Minkowski spacetime, an expansion analog to (1) holds, where a_0 duplicates. At this point we look for a direct connections with black hole thermodynamics. In the paper [12] (following [48]) on black holes one finds computations that fit well with our results. There $c/12 = A/8\pi$ so one immediatly gets that $S_{\mathcal{A}}$ has the Bekenstein behavior

$$S_{\mathcal{A}} = A/4 \ ,$$

  where A is the classical area of the black hole horizon.

- We then provide a general analysis where we do not any longer use the modularity assumption. We first recall how the n-cover $\mathrm{Diff}^{(n)}(S^1)$ of $\mathrm{Diff}(S^1)$ acts on S^1, see [38]. The generator of the corresponding rotation one-parameter group is viewed as a conformal Hamiltonian associated with a discretization of S^1, namely to a partition of S^1 in n intervals, where n is then supposed to tend to infinity.

If one subtracts from the corresponding entropy (logarithm of partition function) the naive entropy associated with $1/n$ times the original conformal Hamiltonian, the resulting entropy should take into account the noncommutative geometrical complexity. We thus give in this way a general definition of mean free energy and it turns out immediately that

$$F_{\text{mean}} = \pi c/12 \ ,$$

that agrees with the above found value for the entropy a_0, hence again $F_{\text{mean}} = A/4$ in the above setting.

- At this point we get in the second part of the paper, where we study the "local" version of the above structure, namely we consider the operator algebra associated with a given interval and the associated proper dynamics with a one-parameter group of special conformal transformations. We consider the generators of this "dilatation" group in $\mathrm{Diff}(S^1)$ and in $\mathrm{Diff}^{(n)}(S^1)$ as Hamiltonians and we attempt to compute the associated noncommutative spectral invariants. Only conformal symmetries and the split property play a rôle here and results are very general.

  We then extend to the general model independent setting a formula by Schroer and Wiesbrock [47] for the Tomita-Takesaki modular group of the von Neumann algebra associated with n separated intervals; in other words we prove the KMS thermal equilibrium property, for above proper dynamics associated with the discretization os S^1, with respect to a canonical state, in any representation. This is one of our main tools for the sequel.

- With this proper Hamiltonian, in analogy of the previous analysis, we define the partition function Z_n associated with this discretization of S^1 with n-intervals and then the μ-free energy $F_{\text{mean},\mu}$ as the $\lim_{n\to\infty} -\beta^{-1}\log Z_n(\beta)/n$ at inverse equilibrium temperature β (Hawking temperature). It turns out that, in any irreducible representation,

  $$F_{\text{mean},\mu} = \frac{1}{2}\log \mu_{\mathcal{A}} \ ,$$

  where $\mu_{\mathcal{A}}$ is the μ-index of the net, namely the Jones index of the 2-interval inclusion of von Neumann algebras in the vacuum sector [31] (Sect. 3.2).

  Pursuing the above analogy we interpret the first noncommutative local spectral invariants. It turns out that the 0^{th} invariant $a_{0,\mu}$, equal by definition to the proper noncommutative area, is proportional to the mean entropy. The first spectral invariant $a_{1,\mu}$, equal by definition to the proper noncommutative Euler characteristic, turns out to be proportional to the proper mean entropy $S_{\mathrm{mean},\mu}$. (Locally the μ-index seems to play the rôle of the central charge globally, but we have no definite interpretation of this fact.)

- Our mathematical methods concern Jones' index [25], as extended by Kosaki [32], and Connes-Haagerup noncommutative measure theory, see [51]. We have put our mathematical results in Appendix C, in order not to interrupt the main theme of the paper. A quick introduction to Operator Algebras and Conformal Field Theory can be found in [27].

2 On black hole entropy

We now recall a few motivational items concerning black hole physics.

The holographic principle [24, 50]. The celebrated Bekenstein formula [3] for the entropy of a black hole is

$$S = \alpha A$$

where A is the area of the black hole horizon and α is a constant. This formula was initially motivated by consistency arguments and the area theorem. One of the most surprising fact is that it sets the entropy to be proportional to the area, rather than to the volume, as an intuitive picture of the entropy as logarithmic counting of the number of possible states would suggest.

This dimensional reduction has more recently led to the formulation of the holographic principle according to which, in a theory combining quantum theory and gravity, the degrees of freedom of a three dimensional world can be stored in a two dimensional projection. One of the argument is that "one can't hide behind a black hole": if black hole projects itself on a screen, due to gravity a second black hole can't eclipse its image on the screen [50].

Hawking temperature. Fixing the proportionality constant. Let's recall how the proportionality constant can be fixed as $\alpha = 1/4$ by considering quantum effects (cf. [52]). As shown by Hawking, a black hole emits a thermal radiation with inverse temperature

$$\beta = \frac{2\pi}{\kappa} \ ,$$

where κ is the surface gravity. Let's consider the Schwarzschild spacetime with radius R, thus describing a black hole of mass $M = 2R$. In this case $\kappa = \frac{1}{4M}$, thus $\beta = 8\pi M$. As

$$S = \alpha A = \alpha 4\pi R^2 = \alpha 16\pi M^2$$

we have

$$\mathrm{d}S = \alpha 32\pi M \mathrm{d}M.$$

On the other hand by the generalized second principle of thermodynamics

$$\mathrm{d}S = \beta \mathrm{d}H = \beta \mathrm{d}M \ ,$$

where $H = M$ is the energy, so $\beta = 8\pi M = \alpha 32\pi M$ yielding

$$\alpha = 1/4.$$

Limit of information. Discretization of the horizon [4]. Consider the horizon to be made by cells of area $\sim \ell^2$, where ℓ is the Planck length. Thus

$$A = n\ell^2.$$

Now say that each cell has k degrees of freedom: in the simplest example each cell is occupied by a particle with spin up/spin down and so $k = 2$. The total number of degrees of freedom are then

$$\text{Degrees of freedom} = k^n; \tag{2}$$

thus

$$\text{Entropy} = Cn \log k = C\frac{A}{\ell^2} \log k \tag{3}$$

where C is a constant, namely the entropy is proportional to the area A of the black hole.

It follows that the increment of entropy by adding a particle to the black hole

$$\mathrm{d}S = C \log k \tag{4}$$

is proportional to the logarithm of an integer. More generally if there are distinct particles $p_1, p_2, \ldots p_s$ and p_i has k_i degrees of freedom we have

$$\text{Degrees of freedom} = k_1^{n_1} k_2^{n_2} \cdots k_s^{n_s}, \tag{5}$$

where $n = n_1 + n_2 + \ldots n_s$, so

$$\text{Entropy} = C \log k_1^{n_1} k_2^{n_2} \cdots k_s^{n_s} = C \sum_i n_i \log k_i \ . \tag{6}$$

The conformal horizon of a black hole. The horizon of a black hole is the boundary of the no escape region of the spacetime where signals can enter, but cannot get out. There is no particular physical phenomena occurring on the horizon, an observer can cross it without feeling anything, yet it is a codimension one submanifold where certain parameters (coordinates) pick critical values. For this reason it is thus natural to expect the horizon to exhibit further symmetries acquainted at these critical values.

This point, related to the above mentioned holographic principle, is well expressed in the holography that holds in the anti-de Sitter spacetime [39]. Here the algebraic approach gives a natural "coordinate free" description [43]. More recently a general algebraic holography has been realized in the two-dimensional de Sitter spacetime by means of local conformal (pseudo)-nets of von Neumann algebras on S^1 [20].

There is an apparent conflict between the discretization of the boundary and conformal invariance: our point of view is that the conformal symmetries that respect the discretization are the physically relevant ones. One should think of conformal QFT on the boundary as a noncommutative manifold, and we shall soon be back on this point. The corresponding structure will be explained later on.

Entropy from conformal boundary. This point of view has emerged in recent years in different works as in [48, 12, 2] where conformal symmetries on the horizon are used to compute black hole entropy.

For example, in the reference [12] by Carlip the black hole is described, in particular, by a spacetime with a (local) Killing horizon; a natural set of boundary conditions leads to a representation of the Virasoro algebra with central charge c and it turns out that, in normalized units,

$$\frac{c}{12} = \frac{A}{8\pi} \ , \tag{7}$$

where A is the area of a cross section of the horizon (the black hole area). One then use a heuristic formula derived with certain assumption by Cardy

$$\rho(\lambda) \sim \exp\left(2\pi\sqrt{\tfrac{1}{6}c(\lambda - \tfrac{1}{24}c)}\right) \quad \text{as } \lambda \to +\infty \tag{8}$$

on the number of states $\rho(\lambda)$ corresponding to the eigenvalue λ of the (two-dimensional) conformal Hamiltonian. One computes the boundary term of the energy (that turns out to be equal to $= A/8\pi$), inserts this and the value of c in eq. (8) one gets the expected Bekenstein behavior

$$\log \rho \sim \frac{A}{4} \ .$$

Operator algebras and conformal boundary. Quantum index theorem. Recall now the work in [21, 36] in the context of black holes described by a curved spacetime with a bifurcate Killing horizon. $\mathcal{A}$ is a conformal net arising on the horizon. By applying a general theorem by Wiesbrock, $\mathcal{A}$ is a Möbius covariant net (cf. [49, 36, 37, 46]); moreover $\mathcal{A}$ is expected to be diffeomorphism covariant and the diffeomorphism symmetry uniquely determined (see [11]); for example this is the case when the quantum field is free, as $\mathcal{A}$ is then isomorphic to the net associated with the $U(1)$-current algebra, see [21] (this fact has been noticed again in [40]). We thus assume $\mathcal{A}$ to be diffeomorphism covariant.

In [35, 36, 37] we have obtained a general, model independent formula for a black hole with a bifurcate Killing horizon (assuming the KMS property for geodesic observers):

$$\mathrm{d}F = \frac{2\pi}{\kappa}\big(\log d(\rho) - \log d(\sigma)\big) \tag{9}$$

where $\mathrm{d}F$ is the incremental free energy by adding/removing DHR charges ρ, σ localizable in bounded regions ([16]), $\kappa/2\pi$ is the Hawking temperature with κ the surface gravity, $d(\rho)$ is the Doplicher-Haag-Roberts statistical dimension of ρ, that turns out to be equal to the square root of the Jones index of ρ [33]. Recall that, in a n-dimensional spacetimes, $n \geq 3$, we have $d(\rho) \in \mathbb{N}\cup\infty$. The above formula holds also for finitely many charges, and we regard (3) as a physical description of (9). It can be read as a quantum index theorem (or, more appropriately, "QFT index theorem" as it concerns infinitely many degrees of freedom) where the quantum Fredholm index $\log d(\rho) - \log d(\sigma)$ is expressed in terms of $\mathrm{d}F$ and the geometric quantity κ. A good illustration of this point is provided by the topological sectors in [38].

3 QFT, heat kernel asymptotic and entropy

3.1 Weyl's theorem and ellipticity

Let M be a compact oriented Riemann manifold and Δ the Laplace operator on $L^2(M)$. The eigenvalues of M can be thought as "resonant frequencies" of M and capture most of the geometry of M [26].

At the root of this analysis is the famous Weyl's theorem on the asymptotic density distribution of such eigenvalues. This can be stated as an asymptotic formula for the heat kernel, see [45]. One has the following asymptotic expansion as $t \to 0^+$:

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots) \tag{10}$$

and thus, by Tauberian theorems (see [6]), the asymptotic formula as $\lambda \to +\infty$

$$N(\lambda) \sim \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma((n/2)+1)}\lambda^{n/2}$$

for the number $N(\lambda)$ of eigenvalues of Δ less than λ, where Γ is Euler Gamma-function.

In (10) the spectral invariants n and $a_0, a_1, \dots$ encode geometric information and in particular

$$a_0 = \mathrm{vol}(M), \qquad a_1 = \frac{1}{6}\int_M \kappa(m)d\mathrm{vol}(m),$$

where κ is the scalar curvature, thus in particular if $n = 2$ the a_1 is proportional to the Euler characteristic equal $\frac{1}{2\pi}\int_M \kappa(m)d\mathrm{vol}(m)$ by Gauss-Bonnet theorem. Motivated by the Weyl asymptotic (10) and Lemma 24 we now make the following definition.

Now, having in mind a "second quantized" Hamiltonian (see Sect. B), we give the following definition to capture the asymptotic associated with the (here undefined) "one-particle Hamiltonian".

A positive linear operator H on a Hilbert space is *QFT elliptic* if there exists $n > 0$ and $a_i \in \mathbb{R}$, $a_0 \neq 0$, such that

$$\log \operatorname{Tr}(e^{-tH}) \sim \frac{1}{t^{n/2}}(a_0 + a_1 t + \cdots) \quad \text{as } t \to 0^+ \ . \tag{11}$$

Then

$$n = -2 \lim_{t \to 0^+} \frac{\log \log \operatorname{Tr}(e^{-tH})}{\log t}$$

is called the *dimension* of H and $a_i \equiv a_i(H)$ the i^{th} *spectral invariant* of H. The following is obvious.

Lemma 1. *Let H, H' be QFT elliptic operators with dimension n and n' and spectral invariants a_i and a'_i. If*

$$\lim_{t \to 0^+} \frac{\operatorname{Tr}(e^{-tH})}{\operatorname{Tr}(e^{-tH'})} = \lambda \neq 0$$

exists, then $n = n'$ and $a_i = a'_i$, $i = 0, 1, 2, \ldots, m-1$, $m \equiv \lfloor n/2 \rfloor$; if m is an integer $\log \lambda = a_m - a'_m$.

Proof. We have

$$\begin{aligned}
\log \lambda = \lim_{t \to 0^+} \log \frac{\operatorname{Tr}(e^{-tH})}{\operatorname{Tr}(e^{-tH'})} &= \lim_{t \to 0^+} \big(\log \operatorname{Tr}(e^{-tH}) - \log \operatorname{Tr}(e^{-tH'}) \big) \\
&= \lim_{t \to 0^+} \left(\frac{1}{t^{n/2}}(a_0 + a_1 t + a_2 t^2 + \ldots) - \frac{1}{t^{n'/2}}(a'_0 + a'_1 t + a'_2 t^2 + \ldots) \right)
\end{aligned} \tag{12}$$

which is possible only in the stated case. □

3.2 Spectral invariants associated with L_0

The asymptotic of the character $\operatorname{Tr}(e^{-2\pi t L_0})$ as $t \to 0^+$ is known for an irreducible representation of the Virasoro algebra [52], but is unknown for a general reducible representation, in particular for the representation associated with an arbitrary local conformal net. Cardy has provided an argument based on modular invariance that implies

$$\log \operatorname{Tr}(e^{-2\pi t L_0}) \sim \text{const.} \frac{1}{t} \quad \text{as} \ \ t \to 0^+$$

where the constant depends on the central charge c only.

Motivated by Weyl's theorem and the above expansion, we shall define a local conformal net $\mathcal{A}$ to be two-dimensional *QFT elliptic* if its conformal Hamiltonian L_0 is QFT elliptic with dimension 2, see Section 3.1, namely

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots) \quad \text{as} \quad t \to 0^+ \tag{13}$$

QFT ellipticity is essentially the *nuclearity condition* of Buchholz and Wichmann [5] (and we fix the dimension).

We shall then regard $\mathcal{A}$ as a 2-dimensional noncommutative manifold, where L_0 corresponds to the Laplacian and the spectral invariants of L_0 are noncommutative geometric invariants for $\mathcal{A}$. In particular $a_0 \equiv a_0(2\pi L_0)$ is $1/4\pi$ times the *noncommutative area* of $\mathcal{A}$ and $12a_1$ is the *noncommutative Euler characteristic* of $\mathcal{A}$. [1] Of course $a_0, a_1, \ldots$ have a priori no classical geometric interpretation, but are defined in analogy with classical invariants.

We now explain how to obtain a more precise form of the asymptotic (13) under a general condition. Let $\mathcal{A}$ be a completely rational local conformal field net on S^1. For a DHR sector ρ, we consider the specialized character $\chi_\rho(\tau)$ for complex numbers τ with $\mathrm{Im}\,\tau > 0$ as follows:

$$\chi_\rho(\tau) = \mathrm{Tr}\left(e^{2\pi i \tau (L_{0,\rho} - c/24)}\right).$$

Here the operator $L_{0,\rho}$ is conformal Hamiltonian in the representation ρ and c is the central charge. We assume that the above Trace converges, which in particular means each eigenspace of $L_{0,\rho}$ is finite dimensional. On one hand, it is known in many cases that we have an action of $SL(2,\mathbb{Z})$ on the linear span of these specialized characters through change of variables τ as follows:

$$\begin{aligned} \chi_\rho(-1/\tau) &= \sum_\nu S^\chi_{\rho,\nu} \chi_\nu(\tau), \\ \chi_\rho(\tau+1) &= \sum_\nu T^\chi_{\rho,\nu} \chi_\nu(\tau). \end{aligned} \tag{14}$$

On the other hand, we have a unitary representation of the group $SL(2,\mathbb{Z})$ on the space spanned by the sector ρ's arising from the nondegenerate braiding as in Rehren [42], in particular we have the associated matrices $(S_{\rho,\nu})$ and $(T_{\rho,\nu})$. It has been conjectured, e.g. Fröhlich-Gabbiani [18, page 625], that these two representations coincide, that is, we have $S^\chi = S$, $T^\chi = T$. Note that we always have $T^\chi = T$ by the spin-statistics theorem [19], so in order to verify these identities, it is enough to show that the fusion rules dictated by S^χ and the fusion rules dictated by composition of DHR-sectors coincide. Such identification of the two fusion rules have been verified in many examples including all local conformal nets with central charge less than 1

[1] For simplicity we do not put a factor $1/4\pi$ in defining the asymptotic (13).

classified in [29]. Also note that if these two representations of $SL(2,\mathbb{Z})$ coincide, we have the following Kac-Wakimoto formula, as explained in [18, page 626].

$$d(\rho) = \frac{S_{\rho,0}}{S_{0,0}} = \frac{S^{\chi}_{\rho,0}}{S^{\chi}_{0,0}} = \lim_{\tau\to i\infty} \frac{\sum_\nu S^{\chi}_{\rho,\nu}\chi_\nu(\tau)}{\sum_\nu S^{\chi}_{0,\nu}\chi_\nu(\tau)} = \lim_{\tau\to 0} \frac{\chi_\rho(\tau)}{\chi_0(\tau)}. \tag{15}$$

Here we denote the vacuum sector by 0 and $d(\rho)$ is the statistical dimension of ρ. (Note that we have $h_\rho > 0$ for $\rho \neq 0$, where h_ρ is the lowest eigenvalue of the operator $L_{0,\rho}$, see Lemma 21.)

We shall say that $\mathcal{A}$ is *modular* if $\mathcal{A}$ the μ-index $\mu_{\mathcal{A}} < \infty$ (see Sect. 6.1), the modular symmetries (14) hold (in particular the characters are defined, namely $\mathrm{Tr}(e^{-tL_{0,\rho}}) < \infty$) and the above two representations of $SL(2,\mathbb{Z})$ are identical. Note that a modular net is completely rational.

Modularity holds in all computed rational case, cf. [54]. The $SU(N)_k$ nets and the Virasoro nets Vir_c with $c < 1$ are both modular. Furthermore, we have the following.

Proposition 2. *Let $\mathcal{A}$ be a modular local conformal net and $\mathcal{B}$ an irreducible extension of $\mathcal{A}$. Then $\mathcal{B}$ is also modular.*

Proof. Since $\mathcal{A}$ is completely rational, the extension has finite index and $\mathcal{B}$ is also completely rational. We denote the S-matrices for $\mathcal{A}$ and $\mathcal{B}$ arising from the braiding as in [42] by S and $\tilde{S}$, respectively. For irreducible DHR sectors ρ and σ of $\mathcal{A}$ and $\mathcal{B}$, respectively, we put $b_{\sigma,\rho} = \dim(\alpha_\rho, \sigma)$, where α_ρ is α-induction. This $b_{\sigma,\rho}$ is equal to the multiplicity of ρ in the representation σ restricted to $\mathcal{A}$. Then we have $\sum_{\sigma'} \tilde{S}_{\sigma,\sigma'} b_{\sigma',\rho} = \sum_{\rho'} b_{\sigma,\rho'} S_{\rho',\rho}$ by [7, Theorem 6.5]. We now have

$$\begin{aligned}
\chi_\sigma(-1/\tau) &= \sum_\rho b_{\sigma,\rho}\chi_\rho(-1/\tau) \\
&= \sum_{\rho,\rho'} b_{\sigma,\rho} S_{\rho,\rho'}\chi_{\rho'}(\tau) \\
&= \sum_{\sigma',\rho'} \tilde{S}_{\sigma,\sigma'} b_{\sigma',\rho'}\chi_{\rho'}(\tau) \\
&= \sum_{\sigma'} \tilde{S}_{\sigma,\sigma'}\chi_{\sigma'}(\tau).
\end{aligned}$$

This shows that the matrix $\tilde{S}$ arising from the braiding for $\mathcal{B}$ also gives a transformation matrix for the characters. □

Proposition 3. *Assume that $\mathcal{A}$ is modular. Then the following asymptotic formula holds:*

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{\pi c}{12}\frac{1}{t} - \frac{1}{2}\log \mu_{\mathcal{A}} - \frac{\pi c}{12} t \qquad \text{as } t \to 0^+ \ .$$

Proof. We first have

$$\mathrm{Tr}(e^{-2\pi t L_0}) = e^{-c\pi t/12} \sum_{\nu} S_{0,\nu} e^{c\pi/(12t)} \,\mathrm{Tr}(e^{-2\pi L_{0,\nu}/t}).$$

Then in this finite summation, the terms for $\nu \neq 0$ are exponentially smaller than the term for $\nu = 0$. This gives

$$\mathrm{Tr}(e^{-2\pi t L_0}) \sim S_{00} e^{-\frac{\pi c}{12}(t-1/t)} \ ,$$

therefore

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim -\frac{c\pi}{12} t + \log S_{00} + \frac{c\pi}{12}\frac{1}{t},$$

and we know that $S_{00} = \mu_{\mathcal{A}}^{-1/2}$ (e.g. [42]), so we get the above statement. □

In particular, in the case $c < 1$, two-dimensional QFT ellipticity can be proved for all local conformal nets. We give also an independent proof of this corollary as follows.

Corollary 4. *Let $\mathcal{A}$ be a local conformal net with $c < 1$. Then $\mathcal{A}$ is two-dimensional QFT elliptic with noncommutative area $a_0 = 2\pi c/24$, thus*

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{c}{24}\frac{2\pi}{t} \quad \text{as} \quad t \to 0^+ \ .$$

Proof. The Virasoro net Vir_c with a central charge $c < 1$ is completely rational and $\mathcal{A}$ is a finite index extension of Vir_c [29]. Hence the conformal Hamiltonian L_0 of $\mathcal{A}$ is a finite direct sum of conformal Hamiltonians associated with irreducible representations of Vir_c. As the stated asymptotic is valid for all these conformal Hamiltonians [52, Prop. 6.14], the proposition holds true. □

Corollary 5. *Let $\mathcal{A}$ be modular and ρ a representation of $\mathcal{A}$. The following asymptotic formula holds:*

$$\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12}\frac{1}{t} + \frac{1}{2}\log\frac{d(\rho)^2}{\mu_{\mathcal{A}}} - \frac{\pi c}{12} t \qquad \text{as } t \to 0^+ \ .$$

Proof. We can assume $d(\rho) < \infty$ as otherwise both members of the asymptotic equality are infinite.

By using Prop. 3 and the Kac-Wakimoto formula (15,17), we have

$$\begin{aligned}
\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) &= \log\left(\mathrm{Tr}(e^{-2\pi t L_0})\frac{\mathrm{Tr}(e^{-2\pi t L_{0,\rho}})}{\mathrm{Tr}(e^{-2\pi t L_0})}\right)\\
&= \log \mathrm{Tr}(e^{-2\pi t L_0}) + \log\frac{\mathrm{Tr}(e^{-2\pi t L_{0,\rho}})}{\mathrm{Tr}(e^{-2\pi t L_0})}\\
&\sim \frac{\pi c}{12}\frac{1}{t} - \frac{1}{2}\log\mu_{\mathcal{A}} - \frac{\pi c}{12} t + \log d(\rho)
\end{aligned}$$

hence the corollary follows. □

We note explicitly that the information on the normalized index is contained in the spectral density of the Hamiltonian:

$$\log d(\rho) - \frac{1}{2}\log \mu_{\mathcal{A}} = \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} t \log \mathrm{Tr}(e^{-tL_{0,\rho}}) \ .$$

Recall now the following particular case of Kohlbecker's Tauberian theorem [6, Th. 4.12.1]. Let m be a Borel measure on $[0, \infty)$ finite on compact sets. The logarithm of the Laplace transform has the asymptotic behavior

$$\log \int e^{-t\lambda} \mathrm{d}m(\lambda) \sim C\frac{1}{t} \quad \text{as } t \to 0^+$$

$C > 0$, if and only if

$$\log m[0, \lambda] \sim 2\sqrt{C\lambda}, \quad \text{as } \lambda \to +\infty \ . \tag{16}$$

As a further corollary, we then have an asymptotic formula which is, in part, a version of Cardy's formula (8). Notice however that fomula (8) concerns CFT on a two-dimensional spacetime, while we deal with conformal nets on S^1.

Corollary 6. *Let $\mathcal{A}$ be a modular local conformal net on S^1 and ρ an irreducible representation of $\mathcal{A}$. Then*

$$\log N(\lambda) \sim 2\pi\sqrt{\frac{c}{6}\lambda} \quad \textit{as } \lambda \to \infty$$

where $N(\lambda)$ is the number of eigenvalues (with multiplicity) of $L_{0,\rho}$ that are $\leq \lambda$.

Proof. By Cor. 4 we have $\log \mathrm{Tr}(e^{-tL_{0,\rho}}) \sim C/t$ with $C = \pi^2 c/6$. As

$$\mathrm{Tr}(e^{-tL_{0,\rho}}) = \int e^{-t\lambda} \mathrm{d}m(\lambda)$$

where $m[0, \lambda] = N(\lambda)$, (16) reads $\log N(\lambda) \sim 2\sqrt{2\pi^2 c/12}\sqrt{\lambda} = 2\pi\sqrt{c\lambda/6}$. □

From the physics viewpoint it is natural to define $S_{\mathcal{A}}$, the *entropy of* $\mathcal{A}$, as the leading coefficient of the expansion (13) of $\log \mathrm{Tr}(e^{-2\pi t L_0})$, thus

$$a_0 = S_{\mathcal{A}} \ ,$$
$$a_1, a_2, \cdots = \text{higher order corrections to } S_{\mathcal{A}} \ .$$

Note that, by definition, the entropy is proportional to the noncommutative area: it is just a matter of reading the same formula from different point of views.

3.3 The incremental free energy in [35] (increment of the first spectral invariant)

Let $\mathcal{A}$ be a local conformal net and ρ, σ a DHR representation of $\mathcal{A}$ (see Sect. A). The above mentioned Kac-Wakimoto formula

$$\lim_{t\to 0^+} \frac{\mathrm{Tr}(e^{-tL_{0,\rho}})}{\mathrm{Tr}(e^{-tL_{0,\sigma}})} = \frac{d(\rho)}{d(\sigma)} \ . \tag{17}$$

has been tested in wide generality and always holds true, see [54], and we have just seen to hold true if $\mathcal{A}$ is modular.

Proposition 7. *If $\mathcal{A}$ is modular, then*

$$\log d(\rho) - \log d(\sigma) = a_1(2\pi L_{0,\rho}) - a_1(2\pi L_{0,\sigma}) \equiv \tfrac{1}{12}(\chi_\rho - \chi_\sigma)$$

where $\chi_\rho - \chi_\sigma$ is the increment of the noncommutative Euler characteristic by adding the charge ρ and removing the charge σ.

Proof. This is an immediate corollary of Prop. 5. □

Recall now the work in [21, 36] in the context of black holes described by a curved spacetime with a bifurcate Killing horizon. There $\mathcal{A}$ is a local conformal net canonically arising on the horizon. According to the general analysis (by using Wiesbrock's theorem) $\mathcal{A}$ is a Möbius covariant net, but $\mathcal{A}$ is expected to be diffeomorphism covariant too [11] for example this is the case when the quantum field is free, as $\mathcal{A}$ is then isomorphic to the net associated with the $U(1)$-current algebra [21] (see also [40]).

The incremental free energy $\mathrm{d}F$ by adding the charge ρ and removing the charge σ (in the Hartle-Hawking states) in [35] or, more generally, its symmetrization, see [37, Thm. 5.4], is defined and turns out to be given by

$$\mathrm{d}F = \beta\big(\log d(\rho) - \log d(\sigma)\big) = \frac{2\pi}{\kappa}\big(\log d(\rho) - \log d(\sigma)\big) \tag{18}$$

where κ is the surface gravity and $\beta \equiv 2\pi/\kappa$ of the Hawking temperature.

We thus assume $\mathcal{A}$ to be diffeomorphism covariant and that Prop. 7 holds. Recall that, in higher dimensional spacetimes, $d(\rho) \in \mathbb{N} \cup \infty$ [16]. We then have:

Corollary 8. *With the above assumptions, the the incremental free energy by adding the DHR charge ρ and removing the charge σ is proportional to the increment of the noncommutative Euler characteristic*

$$\mathrm{d}F = \frac{\pi}{6\kappa}\big(\chi_\sigma - \chi_\rho\big). \tag{19}$$

Adding a charge is proportional to the logarithm of an integer.

Proof. The proof is immediate from the above discussion. □

The above formulas (18,19) are consistent with the interpretation of the entropy by logarithmic counting states and the fact that it is proportional to an integer as in eq. (3).

Compared with the work [35], the above corollary expresses the incremental free energy by a true difference of global entropies $\log \mathrm{Tr}(e^{-tL_{0,\rho}})$ and $\log \mathrm{Tr}(e^{-tL_{0,\sigma}})$ by Prop. 7.

3.4 Relation to black hole entropy. I

A microscopic derivation of black hole entropy and its relation to conformal symmetries and central charge is discussed in [48]. The potentiality of our discussion in relation to black hole entropy and Bekenstein classical area description is well examplified by using the reference [12] recalled in Sect. 2. Yet we use here only the value of the central charge (eq. (7)) and not Cardy's formula nor the boundary term of the energy. We shall make here the assumption that the associated local conformal net $\mathcal{A}$ is modular. (Later we shall introduce the mean free energy and put it in relation to Bekenstein entropy, on the same lines, without the modularity assumption.)

Corollary 9. *For a black hole in the above class [12], we have*

$$S_{\mathcal{A}} = A/4$$

where A is the area of the black hole horizon.

Proof. Immediate from the relation $c/12 = A/8\pi$ (7) and the value $S_{\mathcal{A}} = 2\pi c/12$ of the entropy for modular nets on the two-dimensional Minkowski spacetime. □

We have therefore the picture in the following diagram:

$$\begin{tikzcd}
\text{Entropy} \arrow[r, "\text{physics}"] & a_0 \arrow[d, "\text{modular}"', "\text{nets}"] & 4\pi \cdot \text{Noncommutative area} \arrow[l, "\text{geometry}"'] \\
& 2\pi c/12 \arrow[d, "\text{black hole}"', "\text{models}"] & \\
& A/4 &
\end{tikzcd}$$

4 Discretization and conformal invariance

There is an apparent conflict in regarding the horizon of a black hole both having a discrete essence and a conformal group of symmetries. In the sequel we take simultaneously account of both pictures by considering the n-cover $\mathrm{Diff}^{(n)}(S^1)$ of $\mathrm{Diff}(S^1)$

acting on S^1 and respecting the cell partitioning of S^1. Thus the conformal Hamiltonian becomes the generator of the rotation group for the unitary action of $\mathrm{Diff}^{(n)}(S^1)$. We then consider mean quantities, as entropy, as n tends to infinity.

Note that in the sequel of this paper we shall not any longer need the modularity or QFT-ellipticity assumptions.

4.1 The action of the n-cover of $\mathrm{Diff}(S^1)$

We recall now some facts on $\mathrm{Diff}^{(n)}(S^1)$ and its canonical embedding into $\mathrm{Diff}(S^1)$, see [38].

The Virasoro algebra is the infinite dimensional Lie algebra generated by elements $\{L_n \mid n \in \mathbb{Z}\}$ and c with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{20}$$

and $[L_n, c] = 0$. It is the (complexification of) the unique, non-trivial one-dimensional central extension of the Lie algebra of $\mathrm{Vect}(S^1)$.

The elements L_{-1}, L_0, L_1 of the Virasoro algebra are clearly a basis of $s\ell(2,\mathbb{C})$. The Virasoro algebra contains infinitely many further copies of $s\ell(2,\mathbb{C})$: for every fixed $n > 0$ we get a copy generated by the elements $L^{(n)}_{-1}, L^{(n)}_0, L^{(n)}_1$, where

$$L^{(n)}_{\pm 1} \equiv \frac{1}{n} L_{\pm n} \ , \tag{21}$$

$$L^{(n)}_0 \equiv \frac{1}{n} L_0 + \frac{c}{24}\frac{(n^2-1)}{n} \ . \tag{22}$$

We have indeed

$$[L^{(n)}_1, L^{(n)}_{-1}] = 2L^{(n)}_0, \qquad [L^{(n)}_{\pm 1}, L^{(n)}_0] = \pm L^{(n)}_{\pm 1} \tag{23}$$

that are the relations for the usual generators in $s\ell(2,\mathbb{C})$.

It follows that, setting for a fixed $n > 0$

$$L^{(n)}_m \equiv \frac{1}{n} L_{nm} \ , \quad m \neq 0 \ , \tag{24}$$

and $L^{(n)}_0$ as in (22), the map

$$\begin{cases} L_m \mapsto L^{(n)}_m \\ c \mapsto nc \ , \end{cases}$$

gives an embedding of the Virasoro algebra into itself. There corresponds an embedding of $\mathrm{Diff}^{(n)}(S^1)$, the n-cover of $\mathrm{Diff}(S^1)$, into $\mathrm{Diff}(S^1)$ as stated in the following.

Proposition 10. *[38] There is a unique continuous isomorphism $M^{(n)}$ of $\mathrm{Diff}^{(n)}(S^1)$ into $\mathrm{Diff}(S^1)$ such that for all $g \in \mathrm{Diff}^{(n)}(S^1)$ the following diagram commutes*

$$\begin{array}{ccc} S^1 & \xrightarrow{M_g^{(n)}} & S^1 \\ {\scriptstyle z^n}\downarrow & & \downarrow{\scriptstyle z^n} \\ S^1 & \xrightarrow{M_{\underline{g}}} & S^1 \end{array} \tag{25}$$

i.e. $M_g^{(n)}(z)^n = M_{\underline{g}}(z^n)$ *for all* $z \in S^1$.

Here $\underline{g}$ *is the element of* $\mathrm{Diff}(S^1)$ *corresponding to* g *and* $M_{\underline{g}}$ *is the obvious action of* $\underline{g}$ *on* S^1.

$\mathsf{M\ddot{o}b}^{(n)} \equiv \{g \in \mathrm{Diff}^{(n)}(S^1) : \ \underline{g} \in \mathsf{M\ddot{o}b}\}$ *is the* n*-cover of* $\mathsf{M\ddot{o}b}$ *and* $M^{(n)}$ *restricts to an embedding of* $\mathsf{M\ddot{o}b}^{(n)}$ *into* $\mathrm{Diff}(S^1)$.

Clearly the embedding $\mathsf{M\ddot{o}b}^{(n)} \hookrightarrow \mathrm{Diff}(S^1)$ corresponds to the embedding $L_m \mapsto L_m^{(n)}$, $m = -1, 0, 1$, of $s\ell(2,\mathbb{C})$ into the Virasoro algebra.

4.2 The mean free energy (topological increment of the second spectral invariant)

Let $\mathcal{A}$ be a local conformal net on S^1 (in any representation). We divide S^1 into n equally spaced cells, namely we consider the n-interval $E_n \equiv \sqrt[n]{S^+}$, where S^+ is the upper semicircle. To each interval I_k contains minimal information (as the cells of Planck length). There is a canonical evolution associated with E_n corresponding to the rotations on the full S^1, namely the rescaled rotations $R(\frac{1}{n}\vartheta)$ giving rise of two rescaled conformal Hamiltonians: one is $\hat{L}_0^{(n)} \equiv \frac{1}{n}L_0$ comes by purely rescaling the Hamiltonian, the other is the one associated with the representation $U^{(n)}$ of $\mathrm{Diff}^{(n)}(S^1)$, namely $L_0^{(n)} = \frac{1}{n}L_0 + \frac{c}{24}\frac{(n^2-1)}{n}$, takes care of "boundary effects". The geometrical complexity should be encoded in the difference between the two terms.

We define the free energy associated with the above partition of S^1 as the difference of the free energy associated by the corresponding partition functions at infinite temperature:

$$F_n \equiv t^{-1}\log\mathrm{Tr}(e^{-t2\pi L_0^{(n)}}) - t^{-1}\log\mathrm{Tr}(e^{-t2\pi \hat{L}_0^{(n)}}) \ .$$

(one could generalize the definition of F_n without the existence of characters, but we do not dwell on this point). Clearly $F_n = \frac{c}{24}\frac{(n^2-1)}{n}2\pi$ hence we get the following model independent formula for the mean free energy associated to the "discretization of S^1".

Theorem 11. *let* $\mathcal{A}$ *be a local conformal net. We have*

$$F_{\mathrm{mean}} = 2\pi\frac{c}{24} \tag{26}$$

Proof. Obviously $F_{\rm mean} \equiv \lim_{n\to\infty} \frac{1}{n} F_n = 2\pi c/24$. $\square$

Note that we clearly have the relation

$$a_2(2\pi L_0^{(n)}) - a_2(2\pi \hat{L}_0^{(n)}) = F_n$$

thus also $F_{\rm mean}$ has a noncommutative geometrical meaning.

Concerning a two-dimensional conformal QFT, both chiral components contribute to the topological entropy thus, assuming the central charge to be equal for both components, the physical topological entropy duplicates:

$$F_{\rm mean} = 2\pi \frac{c}{12} \ , \tag{27}$$

we shall explain this point in Sect. 7.

4.3 Relation to black hole entropy. II

As noted, the derivation of the value $F_{\rm mean} = 2\pi c/12$ is model independent and general, essentially it follows only by diffeomorphism invariance.

As the value of $F_{\rm mean}$ coincides with the value of $S_{\mathcal{A}}$ (for modular nets), we now have a link with the classical area restriction, just as in Sect. 3.4, without any modularity assumption on $\mathcal{A}$.

For a black hole as in the Corollary 9, we have indeed

$$F_{\rm mean} = A/4$$

where A is the area of the black hole horizon. This is immediate from the relation $c/12 = A/8\pi$ (7) and the found value $F_{\rm mean} = 2\pi c/12$ of the two-dimensional free energy (27).

5 The modular group of a n-interval von Neumann algebra

Here we extend to the general model independent setting, and in an arbitrary representation, a formula (announced in [38]) for the modular group discussed by Schroer and Wiesbrock [47] in the context of the $U(1)$-current algebra local conformal net.

Let E be a symmetric n-interval of S^1, thus $E \equiv \sqrt[n]{I}$ for some $I \in \mathcal{I}$, i.e. $E = \{z \in S^1 : z^n \in I\}$. Let $I_0, I_1, \cdots I_{n-1}$ be the n connected components of E; we may assume that $I_k = R(2\pi k/n) I_0$, where R is the rotation subgroup of $\textsf{Möb}$.

Let $\mathcal{A}$ be a local conformal net on S^1 with the split property, in a irreducible representation. By the split property we have a natural isomorphism

$$\chi_E : \mathcal{A}(E) \equiv \mathcal{A}(I_0) \vee \mathcal{A}(I_1) \vee \cdots \vee \mathcal{A}(I_{n-1}) \to \mathcal{A}(I_0) \otimes \mathcal{A}(I_1) \otimes \cdots \otimes \mathcal{A}(I_{n-1}) \ .$$

A product state φ is a state on $\mathcal{A}(E)$ of the form

$$\varphi \equiv (\varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_{n-1}) \cdot \chi_E \ ,$$

where φ_k is a normal faithful state on $\mathcal{A}(I_k)$ and $\varphi_k = \varphi_0 \cdot \mathrm{Ad}U(R(2k\pi/n))$ is called a *rotation invariant product state.*

We now exhibit a modular group of $\mathcal{A}(E)$ having a geometrical meaning.

Let Φ_k be the isomorphism between $\mathcal{A}(I_k)$ and $\mathcal{A}(I)$ associated with the function z^n, namely

$$\Phi_k(x) \equiv U(h_k)xU(h_k)^*, \quad x \in \mathcal{A}(I_k)$$

where h_k is any element of $\mathrm{Diff}(S^1)$ such that $h_k(z) = z^n$, $z \in I_k$, (by locality the definition of Φ_k is independent of the choice of h_k).

Let φ_k be the state on $\mathcal{A}(I_k)$ given by $\varphi_k \equiv \omega_I \cdot \Phi_k$, where ω is the vacuum state, and let φ_E the product state on $\mathcal{A}(E)$ that restricts to φ_k on $\mathcal{A}(I_k)$. Clearly φ_E is a rotation invariant product state.

Theorem 12. *Let $\mathcal{A}$ be a local conformal net in a irreducible representation and U the covariance unitary representation of* $\mathrm{Diff}(S^1)$*. With $E = \sqrt[n]{I}$ an n-interval as above, the canonical rotation invariant product state φ_E on $\mathcal{A}(E)$ has modular group σ^{φ_E} given by*

$$\sigma_t^{\varphi_E} = \mathrm{Ad}U^{(n)}(\Lambda_I(-2\pi t))\restriction_{\mathcal{A}(E)}$$

where Λ_I is the lift to $\mathsf{M\ddot{o}b}^{(n)}$ of the one-parameter subgroup of $\mathsf{M\ddot{o}b}$ of generalized dilatation associated with I (see Appendix A) and $U^{(n)} = U \cdot M^{(n)}$ is the unitary representation of $\mathsf{M\ddot{o}b}^{(n)}$ associated with U.

Proof. Since both $\sigma_t^{\varphi_E}$ and $\mathrm{Ad}U^{(n)}(\Lambda_I(-2\pi t))\restriction_{\mathcal{A}(E)}$ are tensor product of their restrictions to the components $\mathcal{A}(I_k)$, by rotation invariance it suffices to prove the formula on each $\mathcal{A}(I_k)$.

We have

$$\begin{aligned} \sigma_t^{\varphi_E}\restriction_{\mathcal{A}(I_k)} &= \sigma_t^{\omega\cdot\Phi_k}\restriction_{\mathcal{A}(I_k)} = \Phi_k^{-1} \cdot \sigma_t^{\omega_I} \cdot \Phi_k \\ &= \Phi_k^{-1} \cdot \mathrm{Ad}U(\Lambda_I(-2\pi t)) \restriction_{\mathcal{A}(I)} \cdot \Phi_k = \mathrm{Ad}U^{(n)}(\Lambda_I(-2\pi t)) \restriction_{\mathcal{A}(I_k)} \ . \end{aligned} \tag{28}$$

□

Corollary 13. *In the above proposition, setting $V(t) \equiv U^{(n)}(\Lambda_I(-2\pi t))$, we have:*

$$\mathrm{Ad}V(t) \restriction_{\mathcal{A}(E)} = \sigma_t^{\varphi_E}, \qquad \mathrm{Ad}V(-t) \restriction_{\mathcal{A}(E')} = \sigma_t^{\varphi_{E'}} \ .$$

Proof. The first equality has been already shown. Since $E' = \sqrt[n]{I'}$, to get the second equality we just have to show that $V(-t) = U^{(n)}(\Lambda_{I'}(-2\pi t))$, which is clearly the case since $\Lambda_{I'}(-t) = \Lambda_I(t)$. □

Note that the abstract results in Appendix C now apply.

6 Entropy and global index with the proper Hamiltonian

In this section we pursue the above point of view, but we replace the conformal Hamiltonian L_0 with the "local" Hamiltonian

$$K_1 \equiv i(L_1 - L_{-1}) \ ,$$

the generator of the one-parameter dilatation unitary group associated with the upper semicircle S^+ (see Appendix A). This dynamics satisfies the equilibrium condition at Hawking temperature and is natural to be considered, see e.g. [22, 53, 35, 37]. As above we will consider the corresponding dynamics for the action of $\mathrm{Diff}^{(n)}(S^1)$ and compute noncommutative spectral invariants. It turns out the analysis below can be done in complete generality: it is only based on conformal invariance ad the split property (recall that the latter follows automatically from the existence of characters).

6.1 μ-index

Let $\mathcal{A}$ be a local conformal net with the split property in the vacuum representation and $E \subset S^1$ a 2-interval, namely E and its complement E' are union of two proper intervals. The *μ-index* of $\mathcal{A}$ is defined as

$$\mu_{\mathcal{A}} \equiv [\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$$

where the brackets denote the Jones index and $\hat{\mathcal{A}}(E) \equiv \mathcal{A}(E')'$. It turns out that $\mu_{\mathcal{A}}$ does not depends on E and

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2$$

sum over the indices of all irreducible DHR charges, namely $\mu_{\mathcal{A}}$ coincides with the *global index* of $\mathcal{A}$. If $\mathcal{A}$ is in a the representation ρ the μ-index is $\mu^{\rho}_{\mathcal{A}} = d(\rho)^2 \mu_{\mathcal{A}}$. More generally, if E_n is an n-interval, and in the representation ρ we have

$$\mu^{\rho}_{\mathcal{A},n} \equiv [\hat{\mathcal{A}}(E_n) : \mathcal{A}(E_n)] = d(\rho)^2 \mu_{\mathcal{A}}^{n-1} \ .$$

Note that the formula

$$\mu_{\mathcal{A}} = \lim_{n\to\infty} \sqrt[n]{[\hat{\mathcal{A}}(E_n) : \mathcal{A}(E_n)]}$$

gives the μ-index in any irreducible representation. Indeed we have:

Proposition 14. *Let $\mathcal{A}$ be a split, local Möbius covariant net in a irreducible representation ρ. Given an interval I, both $\mu_{\mathcal{A}}$ and $d(\rho)$ can be measured in I.*

Proof. Fix be an interval I and divide I in $2n-1$ contiguous intervals $I_1 < J_1 < I_2 < J_2 < \cdots < J_{n-1} < I_n$, where $<$ denotes the counter-clockwise order. Then $\vee_{i=1}^{n}\mathcal{A}(I_i) \subset \big(\vee_{i=1}^{n-1}\mathcal{A}(J_i)\big)' \cap \mathcal{A}(I)$ is an n-interval inclusion, thus its index is equal to $d(\rho)^2\mu_{\mathcal{A}}^{n-1}$ and we have

$$\lim_{n\to\infty}[\big(\vee_{i=1}^{n-1}\mathcal{A}(J_i)\big)' \cap \mathcal{A}(I) : \vee_{i=1}^{n}\mathcal{A}(I_i)]^{\frac{1}{n}} = \lim_{n\to\infty}\big(d(\rho)^2\mu_{\mathcal{A}}^{n-1}\big)^{\frac{1}{n}} = \mu_{\mathcal{A}}$$

showing that $\mu_{\mathcal{A}}$ can be detected within the interval I and so is the case also for $d(\rho)^2 = [\big(\vee_{i=1}^{n-1}\mathcal{A}(J_i)\big)' \cap \mathcal{A}(I) : \vee_{i=1}^{n}\mathcal{A}(I_i)]/\mu_{\mathcal{A}}^{n-1}$ (for instance with $n=2$). $\square$

As is known, the central charge may also be measured locally, as it appears locally in the commutation relations with the stress-energy tensor.

6.2 μ-entropy and spectral invariants for the proper Hamiltonian

Let $\mathcal{A}$ be a local conformal net on S^1 with the split property in an irreducible representation ρ. Let $I = S^+$ be the upper semicircle and $E \equiv E_n = \sqrt[n]{I}$ the associated n-interval and K_n the infinitesimal generator of $V^{(n)}$, where $V^{(n)}(t) = U^{(n)}(\Lambda_I(-2\pi t))$ as in Cor. 13. Note that

$$K_n \equiv i(L_n^{(n)} - L_{-n}^{(n)}) = \tfrac{i}{n}(L_n - L_{-n}) \ .$$

The complement E_n' of E_n is the n-interval $E_n' = \sqrt[n]{I'}$. Let φ_{E_n} be the rotation-invariant product state on $\mathcal{A}(E_n)$ defined in Prop. 12 and $\xi_n \equiv \xi_{E_n}$ a cyclic separating vector for $\mathcal{A}(E_n)$ implementing φ_{E_n}. We have:

Theorem 15. *We have*

$$(e^{-2\pi K_n}\xi_n, \xi_n) = d(\rho)\mu_{\mathcal{A}}^{\frac{n-1}{2}}$$

thus

$$\log(e^{-\frac{2\pi i}{n}(L_n - L_{-n})}\xi_n, \xi_n) = \tfrac{n-1}{2}\log\mu_{\mathcal{A}} + \log d(\rho) = \tfrac{n-1}{2}\log(\sum_i d(\rho_i)^2) + \log d(\rho)$$

Proof. The unitary $U(R(2\pi/n))$ implements an isomorphism between $\mathcal{A}(E_n)$ and $\mathcal{A}(E_n')$ mapping $\varphi_{E_n'}$ to φ_{E_n} and K_n to $-K_n$. Hence, if ξ_n' is a cyclic and separating vector for $\mathcal{A}(E_n')$ implementing the state $\varphi_{E_n'}$, we have $(e^{-2\pi K_n}\xi_n, \xi_n) = (e^{2\pi K_n}\xi_n', \xi_n')$, thus by Cor. 29

$$(e^{-2\pi K_n}\xi_n, \xi_n)^2 = \mu_{\mathcal{A},n}^{\rho} \equiv [\hat{\mathcal{A}}(E_n) : \mathcal{A}(E_n)] = d(\rho)^2\mu_{\mathcal{A}}^{n-1} \ .$$

$\square$

If the μ-index is finite, we shall denote by

$$\hat{\varphi}_{E_n} = \varphi_{E_n} \cdot \varepsilon_{E_n}$$

the state on $\hat{\mathcal{A}}(E_n)$ obtained by extending φ_{E_n} by the conditional expectation $\varepsilon_{E_n} : \hat{\mathcal{A}}(E_n) \to \mathcal{A}(E_n)$. The state $\hat{\varphi}_{E'_n}$ on $\hat{\mathcal{A}}(E'_n)$ is defined analogously. If $\mu_{\mathcal{A}} = \infty$ there exists an operator-valued weight $\varepsilon_{E_n} : \hat{\mathcal{A}}(E_n) \to \mathcal{A}(E_n)$ by Haagerup theorem and Prop. 12, but for our purposes here we can stay in the finite μ-index case.

Corollary 16. *We have*

$$\begin{aligned} K_n \equiv \tfrac{i}{n}(L_n - L_{-n}) &= -\frac{1}{2\pi}\Big(\log \big(\frac{\mathrm{d}\varphi_{E_n}}{\mathrm{d}\hat{\varphi}_{E'_n}}\big) + \tfrac{n-1}{2} \log \mu_{\mathcal{A}} + \log d(\rho)\Big) \\ &= -\frac{1}{2\pi}\Big(\log \big(\frac{\mathrm{d}\hat{\varphi}_{E_n}}{\mathrm{d}\varphi_{E'_n}}\big) + \tfrac{n-1}{2} \log \mu_{\mathcal{A}} + \log d(\rho)\Big) \ . \end{aligned}$$

Proof. The von Neumann algebra $\mathcal{A}(I)$ associated to an interval is a factor [9] hence, by the spit property, also the von Neumann algebra $\mathcal{A}(E_n)$ associated with the n-interval E_n is a factor. As both $V^{(n)}(t)$ and $\big(\frac{\mathrm{d}\hat{\varphi}_{E_n}}{\mathrm{d}\varphi_{E'_n}}\big)^{it}$ implement $\sigma_t^{\hat{\varphi}_{E_n}}$ on $\mathcal{A}(E_n)$ and $\sigma_{-t}^{\varphi_{E'_n}}$ on $\hat{\mathcal{A}}(E_n)$, we have that $-2\pi K_n$ is equal to $\log(\mathrm{d}\hat{\varphi}_{E_n}/\mathrm{d}\varphi_{E'_n})$ plus a constant term (see Appendix C). Such constant is fixed by Th. 15 to be $\frac{n-1}{2}\log \mu_{\mathcal{A}} + \log d(\rho)$.
□

The quantity

$$Z_n(t) \equiv (e^{-tK_n}\xi_n, \xi_n)$$

is the geometric partition function associated to the symmetric n-interval partition of S^1, thus by Th. 15

$$F_{n,\mu} \equiv -t^{-1} \log Z_n(t)|_{t=2\pi} = -\tfrac{n-1}{4\pi} \log \mu_{\mathcal{A}} - \tfrac{1}{2\pi} \log d(\rho) \tag{29}$$

is the associated n-free energy, that we call the *n-μ-free energy*. The n-μ-free energy divided by the numbers of cells (intervals) gives asymptotically the *mean μ-free energy*.

Corollary 17. *The mean μ-free energy is given by*

$$F_{\text{mean},\mu} = -\tfrac{1}{4\pi} \log \mu_{\mathcal{A}} \ .$$

Proof. Immediate by eq. (29) we have

$$\begin{aligned} F_{\text{mean},\mu} \equiv \lim_{n\to\infty} \tfrac{1}{n} F_{n,\mu} &= -\lim_{n\to\infty} \tfrac{1}{2\pi n} \log(e^{-2\pi K_n}\xi_n, \xi_n) \\ &= -\lim_{n\to\infty} \big(\tfrac{n-1}{4\pi n}\log \mu_{\mathcal{A}} + \tfrac{1}{2\pi n}\log d(\rho)\big) = -\tfrac{1}{4\pi} \log \mu_{\mathcal{A}} \ . \end{aligned} \tag{30}$$

□

In analogy with Sect. 3.2 the 0^{th} *and* 1^{st} *spectral invariants* are then defined by

$$a_{0,\mu} \equiv \lim_{n\to\infty} \frac{t\log Z_n(t)}{n}\Big|_{t=2\pi} \tag{31}$$

$$a_{1,\mu} \equiv \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\frac{t\log Z_n(t)}{n}\Big|_{t=2\pi} \tag{32}$$

Note that $-\frac{\mathrm{d}}{\mathrm{d}t}\log Z_n(t)$ is the $n-\mu$-*energy* $H_{n,\mu}$ associated with $Z_n(t)$. Due to the thermodynamical relation

$$\text{Free energy} = T\cdot\text{Entropy} - \text{Energy}$$

where T is the temperature, we thus define the mean $n-\mu$-*entropy* by

$$S_{n,\mu} = t(F_{n,\mu} + H_{n,\mu})\ .$$

We have:

$$\frac{\mathrm{d}}{\mathrm{d}t} t\log Z_n(t) = \log Z_n(t) + t\frac{\frac{\mathrm{d}}{\mathrm{d}t}Z_n(t)}{Z_n(t)} \tag{33}$$

$$= -t\big(F_{n,\mu} + H_{n,\mu}\big) \tag{34}$$

$$= -S_{n,\mu} \tag{35}$$

thus the mean μ-*entropy* at Hawking inverse temperature 2π is given by

$$S_{\text{mean},\mu} = \lim_{n\to\infty} S_{n,\mu}/n = -\lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{t\log Z_n(t)}{n}\Big|_{t=2\pi}\ .$$

Proposition 18. *$S_{n,\mu} = S(\hat\varphi_{E_n}|\varphi_{E'_n})$, where the latter is Araki relative entropy between the states $\hat\varphi_{E_n}$ and $\varphi_{E'_n}$.*

Proof. We fix a natural cone $L^2(\mathcal{A}(E_n))_+$ (that is unique up to unitary equivalence); for example, in the vacuum representation, we can take the natural cone with respect to the vacuum vector Ω.

The derivative of $\log Z_n(t)$ at $t = 2\pi$ is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\log(e^{-tK_n}\xi_n,\xi_n)|_{t=2\pi} &= -\,\frac{(K_n e^{-tK_n}\xi_n,\xi_n)}{(e^{-tK_n}\xi_n,\xi_n)}\Big|_{t=2\pi}\\
&= -\,\mu^{\rho}_{\mathcal{A},n}\frac{(K_n\Delta^{1/2}\xi_n,\Delta^{1/2}\xi_n)}{(e^{-tK_n}\xi_n,\xi_n)}\Big|_{t=2\pi}\\
&= -\,(K_n JJ\Delta^{1/2}\xi_n, JJ\Delta^{1/2}\xi_n)\\
&= -\,(K_n\hat\xi'_n,\hat\xi'_n)\\
&= \tfrac{1}{2\pi}\big((\log\Delta\hat\xi'_n,\hat\xi'_n) + \tfrac12\log\mu^{\rho}_{\mathcal{A},n}\big)\\
&= t^{-1}\big(-S(\hat\varphi_{E_n}|\varphi_{E'_n}) + \tfrac12\log\mu^{\rho}_{\mathcal{A},n}\big)|_{t=2\pi}
\end{aligned}$$

where $\Delta \equiv \Delta_{\hat{\xi}'_n,\xi_n}$ is Araki relative modular operator between the vectors $\xi_n, \hat{\xi}'_n \in L^2(\mathcal{A}(E_n))_+$ implementing the states φ_{E_n} on $\mathcal{A}(E_n)$ and $\hat{\varphi}_{E'_n}$ on $\hat{\mathcal{A}}(E'_n)$, and J is the corresponding modular conjugation. Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} t \log(e^{-tK_n}\xi_n, \xi_n)|_{t=2\pi} =& \log(e^{-tK_n}\xi_n, \xi_n)|_{t=2\pi} + t\frac{\mathrm{d}}{\mathrm{d}t}\log(e^{-tK_n}\xi_n, \xi_n)|_{t=2\pi} \\
=& \log(e^{-tK_n}\xi_n, \xi_n)|_{t=2\pi} - S(\hat{\varphi}_{E_n}|\varphi_{E'_n}) + \tfrac{1}{2}\log\mu^\rho_{\mathcal{A},n} \\
=& \tfrac{1}{2}\log\mu^\rho_{\mathcal{A},n} - S(\hat{\varphi}_{E_n}|\varphi_{E'_n}) + \tfrac{1}{2}\log\mu^\rho_{\mathcal{A},n} \\
=& - S(\hat{\varphi}_{E_n}|\varphi_{E'_n}) + \log\mu^\rho_{\mathcal{A},n} \\
=& - S(\hat{\varphi}_{E_n}|\varphi_{E'_n}) + (n-1)\log\mu_{\mathcal{A}} + \log d(\rho)
\end{aligned}$$

which gives the thesis. □

Corollary 19. *We have*

$$\begin{aligned}
a_{0,\mu} =& \tfrac{1}{2}\log\mu_{\mathcal{A}} \ , \\
a_{1,\mu} =& - S_{\mathrm{mean},\mu} = \log\mu_{\mathcal{A}} - \lim_{n\to\infty}\tfrac{1}{n}S(\hat{\varphi}_{E_n}|\varphi_{E'_n}) \ .
\end{aligned}$$

Proof. Immediate by the above discussion. □

By definition the *μ-noncommutative Euler characteristic* $\chi_{\mathcal{A},\mu}$ is defined, in analogy with the previous sections, to be equal to 12 times the first spectral invariant. Thus we have:

$$\chi_{\mathcal{A},\mu} \equiv 12a_{1,\mu} = -12S_{\mathrm{mean},\mu} \ .$$

7 CFT on a two-dimensional spacetime

Here we give the version of the considered asymptotic expansion in the case of a conformal QFT on a two-dimensional spacetime. The extension of the rest of our analysis is then immediate and we do not make it explicitly.

The model independent structure of conformal quantum field theory on the two-dimensional Minkowski spacetime M_2 is naturally described by a local, diffeomorphism covariant net $\mathcal{A}$ of von Neumann algebras $\mathcal{A}(\mathcal{O})$ associated with double cones $\mathcal{O}$ of M_2, see e.g. [30].

Denoting with (x,t) the space and time coordinates of a point of M_2, the restriction of $\mathcal{A}$ to the light axis $x \pm t = 0$ gives rise to two local chiral conformal nets $\mathcal{A}_\pm$ on $\mathbb{R}$ that, by conformal invariance, extend to local conformal nets on S^1.

Given the double cone

$$\mathcal{O} = \{(x,t) : x \pm t \in I_\pm\}$$

associated with the intervals I_+ and I_- of the light axis, denote by $\mathcal{A}_0(\mathcal{O})$ the von Neumann algebra

$$\mathcal{A}_0(\mathcal{O}) = \mathcal{A}_+(I_+) \vee \mathcal{A}(I_-) \simeq \mathcal{A}_+(I_+) \otimes \mathcal{A}(I_-);$$

then $\mathcal{A}_0$ is a local conformal subnet of $\mathcal{A}$. In the rational case one expects the subnet to have finite Jones index:

$$[\mathcal{A}(\mathcal{O}) : \mathcal{A}_0(\mathcal{O})] < \infty \ .$$

This is the case if $\mathcal{A}_0$ is completely rational, namely if $\mathcal{A}_\pm$ are completely rational, which is automatic for example if the central charge(s) of $\mathcal{A}$ (i.e. of $\mathcal{A}_\pm$) are less than one.

The classification of all local conformal nets on M_2 with central charge $c < 1$ has been obtained in [30].

We shall say that $\mathcal{A}$ is *modular* if both $\mathcal{A}_+$ and $\mathcal{A}_-$ are modular.

Rehren describes the structure of the inclusion $\mathcal{A}_0(\mathcal{O}) \subset \mathcal{A}(\mathcal{O})$ in terms of modular invariants [44].

The restriction to $\mathcal{A}_0$ of the identity representation of $\mathcal{A}$ decomposes as $\bigoplus Z_{ij}\rho_i^+ \otimes \rho_j^-$ with $\{\rho_i^+\}$ and $\{\rho_i^-\}$ irreducible sectors of $\mathcal{A}_+$ and $\mathcal{A}_-$. Accordingly, the conformal Hamiltonian H of $\mathcal{A}$ (the generator of the rotation one-parameter group in the time direction), has a decomposition

$$e^{-tH} = \bigoplus_{i,j} Z_{ij} e^{-tL^+_{0,i}} \otimes e^{-tL^-_{0,j}}$$

where $L^\pm_{0,i}$ is the conformal Hamiltonian of $\mathcal{A}_\pm$ in the representation $\rho_i^\pm$.

Proposition 20. *Let $\mathcal{A}$ be a modular local conformal net on the two-dimensional Minkowski spacetime. We have the expansion as $t \to 0^+$:*

$$\log \operatorname{Tr}(e^{-2\pi t H}) \sim \frac{2\pi c}{12}\frac{1}{t} - \frac{1}{2}\log \mu_{\mathcal{A}} - \frac{2\pi c}{12} t \ ,$$

where $c \equiv (c_+ + c_-)/2$ is the average of the central charges $c_\pm$ of $\mathcal{A}_\pm$.

Proof. We have the asymptotic equality as $t \to 0^+$:

$$\begin{aligned}
\operatorname{Tr}(e^{-2\pi t H}) =& \sum_{i,j} Z_{ij} \operatorname{Tr}(e^{-2\pi t L^+_{0,i}}) \operatorname{Tr}(e^{-2\pi t L^-_{0,j}}) \\
\sim& \sum_{i,j} Z_{ij} d(\rho_i^+) d(\rho_j^-) \operatorname{Tr}(e^{-2\pi t L^+_0}) \operatorname{Tr}(e^{-2\pi t L^-_0}) \\
=& [\mathcal{A} : \mathcal{A}_0] \operatorname{Tr}(e^{-2\pi t L^+_0}) \operatorname{Tr}(e^{-2\pi t L^-_0}) \ ,
\end{aligned}$$

where we have used the Kac-Wakimoto formula in the first equality, while the identity $[\mathcal{A}:\mathcal{A}_0]=\sum_{i,j}Z_{ij}d(\rho_i^+)d(\rho_j^-)$ follows because $\bigoplus_{i,j}Z_{ij}\rho_i^+\otimes\rho_j^-$ is equivalent to the canonical endomorphism of $\mathcal{A}_0\subset\mathcal{A}$, thus

$$[\mathcal{A}:\mathcal{A}_0]=d(\sum_{i,j}Z_{ij}\rho_i^+\otimes\rho_j^-)=\sum_{i,j}Z_{ij}d(\rho_i^+)d(\rho_j^-)\ .$$

By [31, Prop. 24] we have the equality

$$[\mathcal{A}:\mathcal{A}_0]=\sqrt{\mu_{\mathcal{A}_0}/\mu_{\mathcal{A}}}\ . \tag{36}$$

Note that the above μ-indices are two-dimensional, while the formula in [31] concerns nets on S^1, but the same argument entails the equality (36).

Therefore we have

$$\log\mathrm{Tr}(e^{-2\pi tH})\sim\frac{1}{2}\big(\log\mu_{\mathcal{A}_0}-\log\mu_{\mathcal{A}}\big)+\log\mathrm{Tr}(e^{-2\pi tL_0^+})+\log\mathrm{Tr}(e^{-2\pi tL_0^-})\ .$$

By Prop. 3 we then obtain

$$\begin{aligned}&\log\mathrm{Tr}(e^{-2\pi tH})\\ &\sim\frac{1}{2}\big(\log\mu_{\mathcal{A}_0}-\log\mu_{\mathcal{A}}\big)+\frac{\pi c_+}{12t}-\frac{1}{2}\log\mu_{\mathcal{A}_+}-\frac{\pi c_+ t}{12}+\frac{\pi c_-}{12t}-\frac{1}{2}\log\mu_{\mathcal{A}_-}-\frac{\pi c_- t}{12}\\ &\sim\frac{1}{2}\big(\log\mu_{\mathcal{A}_0}-\log\mu_{\mathcal{A}}\big)+\frac{2\pi c}{12t}-\frac{1}{2}\log\mu_{\mathcal{A}_0}-\frac{2\pi ct}{12}\\ &=\frac{2\pi c}{12t}-\frac{1}{2}\log\mu_{\mathcal{A}}-\frac{2\pi ct}{12}\ ,\end{aligned}$$

where we have made use of the identity $\mu_{\mathcal{A}_0}=\mu_{\mathcal{A}_+}\mu_{\mathcal{A}_-}$. □

In the physical context, the expansion (20) is natural to be considered, rather than the one for the chiral components in Prop. 3.

Note also that a modular net $\mathcal{A}$ on the two-dimensional Minkowski space is maximal if and only if $\log\mu_{\mathcal{A}}=0$ [30]. This is consistent with the appearance of $\log\mu_{\mathcal{A}}$ only as a first order correction to the entropy.

A Appendix. Conformal nets on S^1

We recall here some basic facts and results about conformal nets in the form needed in the paper.

We denote by $\mathcal{I}$ the family of proper intervals of S^1. A *net* $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I\in\mathcal{I}\to\mathcal{A}(I)\subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

The net $\mathcal{A}$ is called *local* if it satisfies:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where the brackets denote the commutator.

The net $\mathcal{A}$ is called *Möbius covariant* if it satisfies in addition the following properties **C,D,E**:

C. *Möbius covariance.* There exists a strongly continuous unitary representation U of of the Möbius group $\mathsf{Möb}$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \;=\; \mathcal{A}(gI), \quad g \in \mathsf{Möb},\; I \in \mathcal{I}.$$

Here $\mathsf{Möb}$ acts on S^1 by Möbius transformations.

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and Ω is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

Let $\mathcal{A}$ be a Möbius covariant net. By the Reeh-Schlieder theorem the vacuum vector Ω is cyclic and separating for each $\mathcal{A}(I)$. The Bisognano-Wichmann property then holds, see [9]: the Tomita-Takesaki modular operator Δ_I and conjugation J_I associated with $(\mathcal{A}(I), \Omega)$, $I \in \mathcal{I}$, are given by

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it},\; t \in \mathbb{R}, \qquad U(r_I) = J_I\;. \tag{37}$$

Here Λ_I is the one-parameter subgroup of $\mathsf{Möb}$ of special conformal transformations preserving I (also called dilatations associated with I): by identifying the upper semicircle S^1 with $\mathbb{R} \cup \{\infty\}$ via the stereographic map, thus S^+ with $\mathbb{R}^+$, $\Lambda_{S^+}(t)$ is the map $x \mapsto e^{-t}x$ on $\mathbb{R} \cup \{\infty\}$. Then $\Lambda_I(t)$ is defined for any $I \in \mathcal{I}$ by conjugation by an element of $\mathsf{Möb}$. $U(r_I)$ implements a geometric action on $\mathcal{A}$ corresponding to the Möbius reflection r_I on S^1 mapping I onto I', i.e. fixing the boundary points of I, see [9]. Here I' denotes the complement of I, $I' \equiv S^1 \smallsetminus I$

This immediately implies Haag duality:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I}\;,$$

where $\mathcal{A}(I)'$ is the commutant of $\mathcal{A}(I)$.

We shall say that a Möbius covariant net $\mathcal{A}$ is *irreducible* if $\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff Ω is the unique U-invariant vector (up to scalar multiples), and iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III_1-factors (unless $\mathcal{A}(I) = \mathbb{C}$ identically), see [19].

Every Möbius covariant net $\mathcal{A}$ decomposes uniquely into a direct integral of irreducible Möbius covariant nets (and the analogous is true for the conformal nets below); we shall thus always assume the following.

F. *Irreducibility.* The net $\mathcal{A}$ is irreducible.

Let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving smooth diffeomorphisms of S^1. As is well known $\mathrm{Diff}(S^1)$ is an infinite dimensional Lie group whose Lie algebra is the Virasoro algebra.

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. *Conformal covariance.* There exists a projective unitary representation U of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\mathsf{M\ddot{o}b}$ such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \mathrm{Diff}(S^1), \\ U(g)AU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in \mathrm{Diff}(I'), \end{aligned}$$

where $\mathrm{Diff}(I)$ denotes the group of smooth diffeomorphisms g of S^1 such that $g(t) = t$ for all $t \in I'$.

We shall say that $\mathcal{A}$ satisfies the *split* property if the von Neumann algebra $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ when I_1 and I_2 are intervals with disjoint closures. The split property is entailed by the trace class condition $\mathrm{Tr}(e^{-tL_0}) < \infty$ for all $t > 0$, where L_0 is the conformal Hamiltonian.

Representations. With $\mathcal{A}$ a local conformal net, a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each I a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_{\tilde{I}}\restriction\mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \subset \mathcal{I}\ .$$

π is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation U_π of $\mathsf{M\ddot{o}b}$ (resp. $\mathrm{Diff}^{(\infty)}(S^1)$) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \mathsf{M\ddot{o}b}$ (resp. $g \in \mathrm{Diff}^{(\infty)}(S^1)$). Note that if π is irreducible and diffeomorphism covariant then U is indeed a projective unitary representation of $\mathrm{Diff}(S^1)$.

Following [16], given an interval I and a representation π of $\mathcal{A}$, there is an endomorphism of $\mathcal{A}$ localized in I equivalent to π; namely ρ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to π, such that $\rho_{I'} = \mathrm{id} \restriction_{\mathcal{A}(I')}$. We refer to [19] for basic facts on this structure, in particular for the definition of the dimension $d(\rho)$, that turns out to be equal to the square root of the Jones index [33].

Let h_π be the conformal weight of the representation π, namely the lowest eigenvalue of the conformal Hamiltonian $L_{0,\pi}$ in the representation π. We shall need the following elementary fact.

Lemma 21. *Let $\mathcal{A}$ be a local Möbius covariant conformal net on S^1 and π an irreducible representation with $h_\pi = 0$. Then π is equivalent to the identity representation.*

Proof. Let ξ be a unit vector such that $L_{0,\pi}\xi = 0$. Then $U_\pi(g)\xi = \xi$ for all $g \in \mathsf{M\ddot{o}b}$ (see e.g. [19]). Moreover, as π is irreducible, ξ is cyclic for π.

Given an interval $I \in \mathcal{I}$ and $g_t \equiv \Lambda_I(t)$, $(t \in \mathbb{R})$, we have for every $x \in \mathcal{A}(I)$,

$$(\pi_I(x)\xi, \xi) = (U_\pi(g_t)\pi_I(x)U_\pi(g_t)^{-1}\xi, \xi) = (\pi_I(U(g_t)xU(g_t)^{-1})\xi, \xi) \ .$$

As $t \to \infty$, $U(g_t)xU(g_t)^{-1}$ weakly converges to $(x\Omega, \Omega)$, hence we have

$$(\pi_I(x)\xi, \xi) = (x\Omega, \Omega), \quad x \in \mathcal{A}(I) \ ,$$

yielding the statement by the uniqueness of the GNS representation. $\square$

Nets in a non-vacuum representation. Given a conformal net $\mathcal{A}$ as above and a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$, the map

$$I \in \mathcal{I} \mapsto \mathcal{A}_\pi(I) \subset B(\mathcal{H}_\pi)$$

with $\mathcal{A}_\pi(I) \equiv \pi_I(\mathcal{A}(I))$ satisfies all the above properties **A** to **G** (with $\mathcal{A}_\pi$ and U_π in place of $\mathcal{A}$ and U), except **E**. We can however generalize **E** to **E**$'$ here below.

A locally normal state ω on $\mathcal{A}_\pi$ is, by definition, a family $\{\omega_I,\ I \in \mathcal{I}\}$, where ω_I is a normal state on $\mathcal{A}_\pi(I)$, such that

$$\omega_{\tilde{I}} \restriction \mathcal{A}_\pi(I) = \omega_I \quad \text{if} \quad I \subset \tilde{I} \ .$$

- **E$'$.** *Existence of the vacuum state.* There exists a locally normal state ω on $\mathcal{A}_\pi$ that is $\mathsf{M\ddot{o}b}$ covariant:

$$\omega_I = \omega_{gI} \cdot \mathrm{Ad}U_\pi(g), \quad I \in \mathcal{I},\ g \in \mathsf{M\ddot{o}b} \ .$$

The state ω is defined by $\omega_I \equiv (\pi_I^{-1}(\cdot)\Omega, \Omega)$ once we start with a the vacuum representation, but **E**$'$ can be taken as an axiom if we start directly in the representation π. In this case, in order to obtain the vacuum representation, one can perform the GNS procedure associated with ω. One needs however to supplement **E**$'$ to the positivity of the energy in the vacuum state, namely ω must be a ground state. Equivalently one can require the local KMS property, that follows immediately from the above discussed Bisognano-Wichmann property if we had started from the vacuum sector.

E″. *Local KMS property.* The modular group associated with $(\mathcal{A}_\pi(I), \omega_I)$, $I \in \mathcal{I}$, is $\mathrm{Ad}U_\pi(\Lambda_I(-2\pi t))$.

By definition a local Möb covariant net $\mathcal{A}_\pi$ (in an representation) is a map $I \in \mathcal{I} \mapsto \mathcal{A}_\pi(I)$ that satisfies the properties **A,B,C,D** and **E′,E″**. We shall say that $\mathcal{A}_\pi$ is conformal if it satisfies **G** and the vacuum representation is diffeomorphism covariant.

Proposition 22. *Let $\mathcal{A}_\pi$ be a local* Möb *covariant net in a representation. There exists a local* Möb *covariant net $\mathcal{A}$ in the vacuum representation and a DHR representation π of $\mathcal{A}$ such that $\mathcal{A}_\pi(I) = \pi_I(\mathcal{A}(I))$.*

Proof. Let $\{\mathcal{H}_I, \sigma_I, \Omega_I\}$ be the GNS triple associated with ω_I and $\mathcal{A}(I) \equiv \sigma_I(\mathcal{A}_\pi)$. Clearly, if $I \subset \tilde{I}$, we can identify H_I with a Hilbert subspace of $\mathcal{H}_{\tilde{I}}$ and Ω_I with $\Omega_{\tilde{I}}$. The usual Reeh-Schlieder analyticity argument with the KMS property **E″** then shows that indeed $\mathcal{H} \equiv H_I = \mathcal{H}_{\tilde{I}}$, thus $\mathcal{H}$ is independent of I. The rest is now clear (cf. [20]). $\square$

B Appendix. Trace and determinants

This appendix contains elementary known facts. Its purpose is to make explicit formula (40), as it helps to understand our definitions.

Let $\mathcal{H}$ be an Hilbert space and $\Gamma_\pm(\mathcal{H})$ the Bose/Fermi Fock Hilbert space over $\mathcal{H}$. If $a \in B(\mathcal{H})$ and $||a|| \leq 1$ the second quantization of $A_\pm \equiv \Gamma_\pm(a)$ is the linear contraction on $\Gamma_\pm(\mathcal{H})$ defined by

$$A_\pm \equiv 1 \oplus a \oplus (a \otimes a) \oplus (a \otimes a \otimes a) \oplus \cdots$$

where the $a \otimes \cdots \otimes a$ acts on the symmetric/anti-symmetric part $\mathcal{H}_\pm^{\otimes^n}$ of $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ depending on the Bose/Fermi alternative. The following is well known, see e.g. [8].

Lemma 23. *If a is selfadjoint, $0 \leq a < 1$, then*

$$\mathrm{Tr}\, A_\pm = \det(1 \mp a)^{\mp 1}, \tag{38}$$

$$\log \mathrm{Tr}\, A_\pm = \pm\, \mathrm{Tr} \log(1 \pm a) \tag{39}$$

Proof. Assume first that $\mathcal{H}$ is one-dimensional, thus $a = \lambda$ is a scalar $0 \leq \lambda < 1$. In the Bose case, $\mathcal{H}_+^{\otimes^n}$ is also one-dimensional for all n, thus we have $A_+ = \oplus_{n=0}^\infty \lambda^n$, so $\mathrm{Tr}\, A_+ = \sum_{n=0}^\infty \lambda^n = (1 - \lambda)^{-1}$.

For a general a, we may decompose $\mathcal{H} = \oplus_i \mathcal{H}_i$ so that $\dim\mathcal{H}_i = 1$ and $a = \oplus_i \lambda_i$. Then $\Gamma_+(\mathcal{H}) = \otimes_i^{\{\Omega_i\}} \Gamma_+(\mathcal{H}_i)$, where Ω_i is the vacuum vector of $\Gamma_+(\mathcal{H}_i)$, and $A_+ = \otimes_i \Gamma_+(a_i)$. It follows that

$$\mathrm{Tr}\, A_+ = \prod_i \mathrm{Tr}\, \Gamma_+(a_i) = \prod_i (1 - \lambda_i)^{-1} = \det(1 - a)^{-1}.$$

as for the Fermi case, if $\mathcal{H}$ is one-dimensional then $\mathcal{H}_-^{\otimes^n} = \{0\}$ if $n \geq 2$ and is one-dimensional if $n = 0, 1$; if $a = \lambda$ we then have $A_- = 1 \oplus \lambda$ so $\operatorname{Tr} A_- = 1 + \lambda$. Since, also in the Fermi case, there is a canonical equivalence between $\Gamma_-(a \oplus b)$ and $\Gamma_-(a) \otimes \Gamma_-(b)$, we have

$$\operatorname{Tr} A_- = \prod_i \operatorname{Tr} \Gamma_-(\lambda_i) = \prod_i (1 + \lambda_i) = \det(1 + a) \ ,$$

where $a = \oplus_i \lambda_i$.

Concerning the second formula, notice that

$$\det a = e^{\operatorname{Tr} \log a},$$

hence

$$\log \operatorname{Tr} A_\pm = \mp \log \det(1 \mp a) = \mp \operatorname{Tr} \log(1 \mp a).$$

$\square$

Lemma 24. *Let h be a positive selfadjoint operator on $\mathcal{H}$ and H the Fermi second quantization of h, namely $H = \Gamma_-(h)$. Then*

$$\frac{\log \operatorname{Tr}(e^{-tH})}{\operatorname{Tr}(e^{-th})} = O(t) \qquad t \to 0^+ \ . \tag{40}$$

Proof. We shall show that

$$\log 2 \leq \liminf_{t \to 0^+} \frac{\log \operatorname{Tr}(e^{-tH})}{\operatorname{Tr}(e^{-th})} \leq \limsup_{t \to 0^+} \frac{\log \operatorname{Tr}(e^{-tH})}{\operatorname{Tr}(e^{-th})} = 1.$$

By Lemma 23 it suffices to show that

$$\log 2 \leq \liminf_{t \to 0^+} \frac{\operatorname{Tr} \log(1 + e^{-th})}{\operatorname{Tr}(e^{-th})} \leq \limsup_{t \to 0^+} \frac{\operatorname{Tr} \log(1 + e^{-th})}{\operatorname{Tr}(e^{-th})} = 1.$$

We have

$$\log 2 \cdot e^{-th} \leq \log(1 + e^{-th}) \leq e^{-th}$$

because of the corresponding function inequalities, that obviously implies the previous inequality. $\square$

The Bose version of the above lemma is omitted (the $U(1)$-current algebra local conformal net is not rational).

C Appendix. Index and entropy

In this appendix we develop abstract mathematical results, concerning Jones index and Connes-Haagerup noncommutative measure theory, that are necessary for our work. We refer to Takesaki's book [51] for the basic theory.

Let R be a von Neumann algebra on a Hilbert space $\mathcal{H}$, $S = R'$ its commutant. Given a n.f.s. (normal, faithful, semifinite) weight φ on R and a n.f.s. weight ψ on S the Connes spatial derivative $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ is a canonical positive non-singular selfadjoint operator on $\mathcal{H}$ such that $\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\big)^{it}$ implements σ_t^φ on R (the modular group of (R,φ)) and $\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\big)^{-it}$ implements σ_t^ψ on S. One has $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi} = \big(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\big)^{-1}$.

If ψ_0 is another n.f.s. weight on S there holds

$$\Big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\Big)^{it} = \Big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\Big)^{it}(D\psi : D\psi_0)_t \ , \tag{41}$$

where $(D\psi : D\psi_0)$ is the unitary Connes Radon-Nikodym cocycle in S w.r.t. ψ_0 and ψ. The following proposition is known.

Proposition 25. *Let R and $S = R'$ be von Neumann algebras on a Hilbert space $\mathcal{H}$, and and V a one-parameter unitary group on $\mathcal{H}$ such that $\mathrm{Ad}V(t)R = R$, $t \in \mathbb{R}$. Given a n.f.s. weight φ on R such that $\mathrm{Ad}V(t)\restriction_R = \sigma_t^\varphi$, there is a unique n.f.s. ψ weight on S such that $\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\big)^{it} = V(t)$.*

If ψ_0 is an arbitrary n.f.s. weight on S one has

$$(D\psi : D\psi_0)_t = u_t \tag{42}$$

where $u_t \equiv V(-t)\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\big)^{it}$.

Proof. With ψ_0 an arbitrary n.f.s. weight on S, both $\big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\big)^{it}$ and $V(t)$ implements σ_t^φ on R, thus u_t belongs to S and is a unitary σ^{ψ_0}-cocycle. By Connes theorem, there exists a n.f.s. weight ψ on S such that $u_t = (D\psi : D\psi_0)_t$. The rest follows by formula (41). $\square$

Corollary 26. *Suppose that, in Prop. 25, φ is the state on R given by a cyclic and separating vector ξ. If K is the infinitesimal generator of V we have*

$$\psi(1) = (e^{-K}\xi, \xi),$$

in particular ψ is a bounded functional iff ξ belongs to the domain of $e^{-\frac{1}{2}K}$.

Proof. Let ψ_0 be the vector state on S implemented by ξ. Then

$$\begin{aligned}\psi(1) &= \underset{t\to -i}{\text{anal.cont.}}\, \psi_0\big((D\psi : D\psi_0)_t\big) \\ &= \underset{t\to -i}{\text{anal.cont.}}\, (V(-t)\Big(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\Big)^{it}\xi, \xi) = \underset{t\to -i}{\text{anal.cont.}}\, (V(-t)\xi, \xi) = (e^{-K}\xi, \xi)\end{aligned} \tag{43}$$

where we have made use that $\frac{d\varphi}{d\psi_0}$ is the modular operator of (R,ξ), thus $\frac{d\varphi}{d\psi_0}\xi = \xi$. $\square$

Let N_1, N_2 be commuting factors on a Hilbert space $\mathcal{H}$ with $N_1 \vee N_2 = B(\mathcal{H})$. Set $M_1 \equiv N_2'$, $M_2 \equiv N_1'$, thus $N_i \subset M_i$ are irreducible inclusion of factors $(i = 1, 2)$. Let φ_i be a normal faithful state on N_i and V a one-parameter unitary group on $\mathcal{H}$ such that

$$\mathrm{Ad}V(t)\restriction_{N_1} = \sigma_t^{\varphi_1}, \quad \mathrm{Ad}V(-t)\restriction_{N_2} = \sigma_t^{\varphi_2}, \quad t \in \mathbb{R},$$

where σ^{φ_i} is the modular group of (N_i, φ_i). Let ψ_1 be the n.f.s. weight on M_1 associated with V and φ_2 by Prop. 25, namely ψ_1 is characterized by

$$K = \log\big(\frac{d\varphi_2}{d\psi_1}\big)\ ,$$

and analogously let ψ_2 be the n.f.s. weight on M_2 associated with V and φ_1.

There exists a unique n.f.s. operator valued weight $\mathcal{E}_i : M_i \to N_i$ such that $\varphi_i \cdot \mathcal{E}_i = \psi_i$. The existence of $\mathcal{E}_i$ follows by Haagerup theorem because $\sigma^{\psi_i}\restriction_{N_i} = \sigma^{\varphi_i}$. Then $\mathcal{E}_i$ is faithful and unique up to a positive scalar multiple because $N_i' \cap M_i = \mathbb{C}$.

Proposition 27. *The following are equivalent:*

- (a) *There exists a normal expectation $\varepsilon_i : M_i \to N_i$;*
- (b) *ψ_i is bounded.*

If the above hold, then $\mathcal{E}_i = \psi_i(1)\varepsilon_i$ and

$$K = -\log\frac{d\varphi_1 \cdot \varepsilon_1}{d\varphi_2} + \log\psi_1(1) \tag{44}$$

$$= \log\frac{d\varphi_2 \cdot \varepsilon_2}{d\varphi_1} + \log\psi_2(1)\ . \tag{45}$$

Proof. If (a) holds, say with $i = 1$, then $\mathcal{E}_1 = \lambda\varepsilon_1$ for some $\lambda > 0$, thus $\psi_1 = \varphi_1 \cdot \mathcal{E}_1 = \psi_1 = \lambda\varphi_1 \cdot \varepsilon_1$ is bounded. Conversely if (b) holds then ψ_1 is a normal, faithful, positive linear functional on M_1 whose modular group $\sigma_t^{\psi_1} = \mathrm{Ad}V(t)$ leaves N_1 globally invariant, so there is a normal expectation $\varepsilon : M_1 \to N_1$ by Takesaki theorem.

Clearly, if the above hold, then $\mathcal{E}_1(1) = \lambda$, thus $\psi_1(1) = \varphi_1 \cdot \mathcal{E}_1(1) = \lambda$, and the rest of the statement statement follows. $\square$

Assume there exists a faithful normal expectation $\varepsilon_1 : M_1 \to N_1$. Denote by $\varepsilon^{-1} : M_2 \to N_2$ the dual operator valued weight. This is the unique n.f.s. operator valued weight $M_2 \to N_2$ such that

$$\frac{d\omega_1 \cdot \varepsilon_1}{d\omega_2} = \big(\frac{d\omega_2 \cdot \varepsilon^{-1}}{d\omega_1}\big)^{-1}$$

for all n.f.s. weight ω_1 on N_1 and ω_2 on N_2.

According to Kosaki definition [32], the inclusion $N_1 \subset M_1$ has finite index iff ε^{-1} is bounded and the index is defined to be $\varepsilon^{-1}(1)$, namely

$$\varepsilon^{-1} = [M_1 : N_1]\varepsilon_2 \ ,$$

where ε_2 is the unique normal expectation from M_2 onto N_2.

Proposition 28. *We have*

$$[M_1 : N_1] = \psi_1(1) \cdot \psi_2(1) \ .$$

Proof. By definition

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}\varphi_2} = e^K, \quad \frac{\mathrm{d}\psi_2}{\mathrm{d}\varphi_1} = e^{-K} \ .$$

Thus

$$\frac{\mathrm{d}\varphi_1 \cdot \mathcal{E}_1}{\mathrm{d}\varphi_2} = \Big(\frac{\mathrm{d}\varphi_2 \cdot \mathcal{E}_2}{\mathrm{d}\varphi_1}\Big)^{-1} \ ;$$

setting $\lambda_i \equiv \psi_i(1)$, since $\mathcal{E}_i = \lambda_i \varepsilon_i$ we then have

$$\lambda_1\lambda_2 \frac{\mathrm{d}\varphi_1 \cdot \varepsilon_1}{\mathrm{d}\varphi_2} = \Big(\frac{\mathrm{d}\varphi_2 \cdot \varepsilon_2}{\mathrm{d}\varphi_1}\Big)^{-1} \ .$$

On the other hand we have

$$\frac{\mathrm{d}\varphi_1 \cdot \varepsilon_1}{\mathrm{d}\varphi_2} = [M_1 : N_1]^{-1} \Big(\frac{\mathrm{d}\varphi_2 \cdot \varepsilon_2}{\mathrm{d}\varphi_1}\Big)^{-1}$$

showing that $[M_1 : N_1] = \lambda_1\lambda_2$. $\square$

Corollary 29. *If ξ_i is a cyclic and separating vector for N_i such that $\varphi_i(x) = (x\xi_i, \xi_i)$, $x \in N_i$, we have*

$$[M_1 : N_1] = (e^K\xi_1, \xi_1)(e^{-K}\xi_2, \xi_2) \ .$$

Suppose further that there exists a unitary U such that $UM_1U^* = M_2$, $UN_1U^* = N_2$ and $\varphi_2 = \varphi_1 \cdot \mathrm{Ad}U$. Then $\psi_1(1) = \psi_2(1) = [M_1 : N_1]^{\frac{1}{2}}$ and

$$(e^K\xi_1, \xi_1) = (e^{-K}\xi_2, \xi_2) = [M_1 : N_1]^{\frac{1}{2}} \ ,$$

thus

$$K = -\log \frac{\mathrm{d}\varphi_1 \cdot \varepsilon_1}{\mathrm{d}\varphi_2} + \frac{1}{2}\log[M_1 : N_1] \ . \tag{46}$$

Proof. The first equality follows by Cor. 26 and Prop. 28.

The second equality then follows because U interchanges the triples of (M_1, N_1, φ_1) and (M_2, N_2, φ_2), thus the canonical quantities $(e^K\xi_1, \xi_1)$ and $(e^{-K}\xi_2, \xi_2)$ must coincide.

The last identity (46) now follows by equation (45). $\square$

Araki relative entropy. Before concluding this appendix we recall the definition of Araki relative entropy between two faithful normal states φ_1 and φ_2 of von Neumann algebra M:

$$S(\varphi_1|\varphi_2) \equiv -(\log \Delta_{\xi_2,\xi_1}\xi_1, \xi_1)\ .$$

Here M is in a standard form with respect to a cyclic and separating vector Ω, the vector ξ_i is the canonical representative of φ_i in the natural positive cone $L^2(M,\Omega)_+$ and Δ_{ξ_2,ξ_1} is the relative modular operator, namely the polar decomposition of S_{ξ_2,ξ_1} is

$$S_{\xi_2,\xi_1} = J\Delta^{1/2}_{\xi_2,\xi_1}$$

where S_{ξ_2,ξ_1} is the closure of the anti-linear operator on $M\xi_1$ defined by $S_{\xi_2,\xi_1}x\xi_1 = x^*\xi_2$.

It easy to check that $S_{\xi_2,\xi_1} = S_{\eta_2,\eta_1}$ if η_1 implements the same state of ξ_1 on M and η_2 implements the same state of ξ_2 on M' , namely $\varphi_1 = (\cdot\eta_1, \eta_1)\restriction_M$ and $\psi_2 \equiv \varphi_2 \cdot \mathrm{Ad}J = (\cdot\eta_2, \eta_2)\restriction_{M'}$. Thus $S(\varphi_1|\varphi_2)$ depends only on the states φ_1 and ψ_2 and we have

$$S(\varphi_1|\varphi_2) = S(\varphi_1|\psi_2) \equiv -(\log(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi_2})\xi_1, \xi_1)\ .$$

We finally note, that, by taking expectation values, equation (46) gives

$$\begin{aligned}(K\xi_2, \xi_2) &= -(\log\frac{\mathrm{d}\varphi_1\cdot\varepsilon_1}{\mathrm{d}\varphi_2}\xi_2, \xi_2) + \frac{1}{2}\log[M_1 : N_1]\\ &= S(\varphi_2|\varphi_1\cdot\varepsilon_1) + \tfrac{1}{2}H(M_1|N_1)\ ,\end{aligned}$$

where $H(M_1|N_1) = \log[M_1 : N_1]$ is the Pimsner-Popa entropy [41].

D Final comments

Adding a massive charge to a black hole should increase the total mass of the black hole, hence make a change of the spacetime itself and of the entropy. In a theory of quantum gravity, the spacetime itself should be noncommutative [15] from the start. In the setting of QFT on a curved spacetime the backreaction from the gravitational field is ignored and the spacetime is classical. In the previous work [35] one considered the addition of a single charge: the increment of entropy is there an "higher order effect" and become visible in the associated noncommutative geometry, while the classical spacetime remains fixed [37]. The entropy in the present work also has a noncommutative geometrical nature, but rather reflects the global noncommutative geometrical complexity of the system.

It would be interesting to relate our setting with Connes' Noncommutative Geometry [10]. A link should be possible in a supersymmetric context, where cyclic cohomology appears. In this respect model analysis with our point of view, in particular in the supersymmetric frame, may be of interest. Note also that Connes' spectral action concerns the Hamiltonian spectral density behavior, see [28].

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