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# Stability of Frustration-Free Ground States of Quantum Spin Systems

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# Frustration-Free Quantum Spin Models

A quantum spin system is a collection of quantum systems labeled by $x$ in a finite set $\Lambda$ (with a distance function), each with a finite-dimensional Hilbert space of states $\mathcal{H}_x$. For concreteness, consider $\Lambda \subset \mathbb{Z}^\nu$.

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x.$$

The algebra of observables for the subsystem in $X \subset \Lambda$ is

$$\mathcal{A}_X = \bigotimes_{x\in X} \mathcal{B}(\mathcal{H}_x).$$

The Hamiltonian $H_\Lambda \in \mathcal{A}_\Lambda$ is defined in terms of an interaction $\Phi$: for any finite $X \subset \mathbb{Z}^\nu$, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, and

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X).$$

$\Phi$ is called finite range if there is $R \geq 0$, such that $\Phi(X) = 0$ if $\operatorname{diam} X > R$.

The model defined by a finite-range interaction $\Phi$ is Frustration-Free (FF) if for all finite $\Lambda \subset \mathbb{Z}^\nu$

$$\inf \operatorname{spec} H_\Lambda = \sum_{X \subset \Lambda} \inf \operatorname{spec} \Phi(X).$$

Equivalently, there is a ground state of $H_\Lambda$ that is simultaneously a ground state of all $\Phi(X)$, for $X \subset \Lambda$.

Note that a frustration-free interaction may have infinite volume ground states in which some of the terms $\Phi(X)$ have expectation strictly greater than their minimal eigenvalue. In this situation we distinguish two types of ground states: frustration-free and non-frustration-free ground states.

It is straightforward to extend the notion of frustration freeness to infinite quantum spin systems on a countable set $\Gamma$ with $C^*$-algebra of quasi-local observables given by

$$\mathcal{A}_\Gamma = \overline{\bigcup_{\text{finite } \Lambda \subset \Gamma} \mathcal{A}_\Lambda}^{\|\cdot\|}.$$

# Outline

- A few examples of frustration-free quantum spin models
- Gapped ground state phases
- Stability questions
- The spectral flow and automorphic equivalence
- Stability of the spectral gap
- Stability of gapped ground state phases
- Outlook

# A few examples of frustration-free quantum spin models

1. The first quantum spin model, introduced by Heisenberg (1928), is frustration-free (FF): the ferromagnetic spin-1/2 Heisenberg model.

For each $x \in \mathbb{Z}^\nu$, $\mathcal{H}_x = \mathbb{C}^2$ and

$$H_\Lambda = - \sum_{|x-y|=1} \mathbf{S}_x \cdot \mathbf{S}_y .$$

The ground states are easily found to be the states of maximal spin, which are common eigenvectors of all the terms $-\mathbf{S}_x \cdot \mathbf{S}_y$, with the minimal eigenvalue $-1/4$.

The ground state space is spanned by product states. The continuous symmetry of simultaneous rotations of the spins is broken; hence the there is no gap in the spectrum above the ground state in infinite volume.

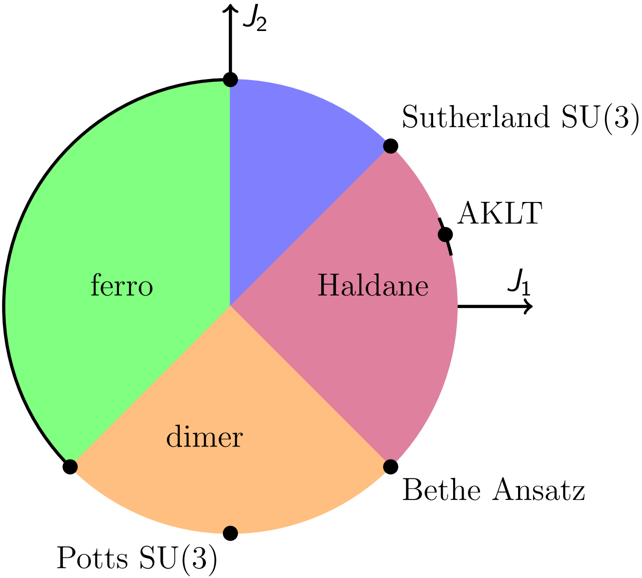
2. The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987-88).
$\Lambda \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$;

$$H_{[1,L]} = \sum_{x=1}^{L-1} \left( \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L-1} P^{(2)}_{x,x+1}$$

In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase). Ground state is frustration free (Valence Bond Solid state (VBS), aka Matrix Product State (MPS), aka Finitely Correlated State (FCS))., and has String Order (den Nijs-Rommelse 1989): ground states are linear combinations of

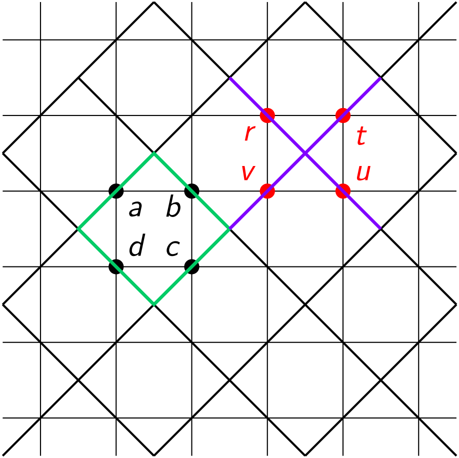
$$\cdots 0\underline{1}0010\underline{1}1000\underline{1}000010\underline{1}\cdots$$

This structure explains the nature of the edges states (Cfr. Ogata's lectures).




$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

3. Toric Code model (Kitaev, 2003). $\Lambda \subset \mathbb{Z}^2$, $\mathcal{H}_x = \mathbb{C}^2$.




$$H = -\sum_p h_p - \sum_s h_s$$

$$h_p = \sigma_a^3 \sigma_b^3 \sigma_c^3 \sigma_d^3$$

$$h_s = \sigma_r^1 \sigma_t^1 \sigma_u^1 \sigma_v^1$$

On a surface of genus $g$, the model has $4^g$ frustration free ground states. This model exhibits topological order and has excitation spectrum described by anyons.

# Gapped ground state phases

The main motivation for the current research on FF models stems from the surge of interest in gapped ground state phases, including topologically ordered phases (Cfr. Ogata's lectures).

The term gapped refers to the existence of a positive lower bound for the energy of excited states with respect to a ground state, uniformly in the size of the system. This implies a gap in the spectrum of the GNS Hamiltonian of the ground state of the infinite system.

The term phase refers to regions in a interaction space where the gap is positive (open). Phase transitions in interaction space can occur when the gap vanishes (closes).

Topological Order and Discrete Symmetry Breaking are often accompanied by a non-vanishing spectral gap.

# The Stability Question(s)

Question 1. Stability of the gapped ground states: Let $\Phi$ and $\Psi$ be short-range interactions.

$$H_\Lambda(\epsilon) = \sum_{X \subset \Lambda} \Phi(X) + \epsilon \Psi(X)$$

Let $\lambda_0(\epsilon)$ and $\lambda_1(\epsilon)$ denote the two smallest eigenvalues of $H_\Lambda(\epsilon)$, and assume there exists $\gamma > 0$ such that $\lambda_1(0) - \lambda_0(0) \geq \gamma$, for arbitrary large $\Lambda$. Assume $\lambda_0(0)$ is simple. Does there exist $\epsilon_0 > 0$, such that for all $|\epsilon| < \epsilon_0$, $\lambda_1(\epsilon) - \lambda_0(\epsilon) \geq \gamma/2$?

Answered positively for a number of special classes of $\Phi$. Best results to date are by Bravyi-Hastings-Michalakis 2010, Bravyi-Hastings 2011, Michalakis-Zwolak (née Pytel) 2013, N-Sims-Young 2016.

# Classification of Gapped Ground State Phases

## Question 2. Invariants within a gapped phase.

If the gap does not close for a range of $\epsilon$, what are the robust features of the system? E.g., under what conditions

(i) is the structure of the set of ground states preserved, which may include 'edges states' for infinite $\Lambda$ with boundary (Cfr. Ogata);

(ii) is the nature of the low-lying excited states, their statistics, are preserved. Of particular interest: in the case of anyons, is the topological S-matrix invariant?

(iii) are there quantities such as Hall conductance that are "quantized"?

It turns out that a tool developed to address (i) (and (ii-iii)) is also an essential ingredient in recent results for Question 1 (stability of the gap).

# Spectral Flow and Automorphic Equivalence

Let $\Phi_s, 0, \leq s \leq 1$, be a differentiable family of short-range interactions, i.e., assume that for some $a, M > 0$, the interactions $\Phi_s$ satisfy

$$\sup_{x,y\in\mathbb{Z}} e^{ad(x,y)} \sum_{\substack{X\subset\mathbb{Z}^d \\ x,y\in X}} \|\Phi_s(X)\| + |X|\|\partial_s\Phi_s(X)\| \leq M.$$

E.g,

$$\Phi_s = \Phi_0 + s\Psi$$

with both $\Phi_0$ and $\Psi$ finite-range and uniformly bounded. Let $\Lambda_n \subset \mathbb{Z}^\nu$, be a sequence of finite volumes, satisfying suitable regularity conditions and suppose that the spectral gap above the ground state (or a low-energy interval) of

$$H_{\Lambda_n}(s) = \sum_{X\subset\Lambda_n} \Phi_s(X)$$

is uniformly bounded below by $\gamma > 0$.

Let $\mathcal{S}(s)$ be the set of thermodynamic limits of ground states of $H_{\Lambda_n}(s)$. E.g., if there is only one ground state, this set contains the state obtained by taking the limit of the infinite lattice: for each observable $A$,

$$\omega(A) = \lim_{\Lambda_n \to \mathbb{Z}^d} \langle \psi_{\Lambda_n} \mid A \psi_{\Lambda_n} \rangle$$

## Theorem (Bachmann, Michalakis, N, Sims, 2012)

*Under the assumptions of above, there exist automorphisms $\alpha_s$ of the algebra of observables such that $\mathcal{S}(s) = \mathcal{S}_0 \circ \alpha_s$, for $s \in [0, 1]$.*

*The automorphisms $\alpha_s$ can be constructed as the thermodynamic limit of the s-dependent "time" evolution for an interaction $\Omega(X, s)$, which decays almost exponentially.*

Concretely, the action of the quasi-local automophisms $\alpha_s$ on observables is given by

$$\alpha_s(A) = \lim_{n \to \infty} V_n^*(s) A V_n(s)$$

where $V_n(s) \in \mathcal{A}_{\Lambda_n}$ is unitary solution of a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = -iD_n(s) V_n(s), \quad V_n(0) = \mathbb{1},$$

with $D_n(s) = \sum_{X \subset \Lambda_n} \Omega(X, s)$.

The $\alpha_s$ satisfy a Lieb-Robinson bound of the form

$$\|[\alpha_s(A), B]\| \leq \|A\|\|B\| \min(|X|, |Y|)(e^{\tilde{v}s} - 1)F(d(X, Y)),$$

where $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $0 < d(X, Y)$ is the distance between $X$ and $Y$. $F(r)$ can be chosen of the form

$$F(r) = Ce^{-\frac{2}{7}\frac{br}{(\log br)^2}}.$$

with $b \sim \gamma/v$, where $\gamma$ and $v$ are bounds for the gap and the Lieb-Robinson velocity of the interactions $\Phi_s$, i.e., $b \sim a\gamma M^{-1}$.

$$D_\Lambda(s) = \int_{-\infty}^{\infty} w_\gamma(t) \int_0^t e^{iuH_\Lambda(s)} \left[\frac{d}{ds}H_\Lambda(s)\right] e^{-iuH_\Lambda(s)} du\, dt$$

Proofs of a gap: Affleck-Kennedy-Lieb-Tasaki (1988), Fannes-N-Werner (1992), N (1996), Kitaev (2006), Bachmann-Hamza-N-Young (2014), Bravyi-Gosset (2015), Gosset-Mozgunov (2015), Bishop-N-Young (2016).

Proofs of stability of the gap (Question 1):
'classical' results by Kennedy-Tasaki, Datta-Fernandez-Fröhlich, Borgs-Kotecky-Uetlschi, Matsui, and others(1980-90s),

More recently: Yarotsky (2004), Bravyi-Hastings-Michalakis (2010), Michalakis-Zwolak (2013), Cirac-Michalakis-PerezGarcia-Schuch (2013), Szehr-Wolf (2015), N-Sims-Young (in prep)

# Structure of Ground State Spaces: Topological vs Landau Order

Consider a quantum spin Hamiltonian on a finite set $\Lambda$, $H_\Lambda \in \mathcal{A}_\Lambda$, defined in terms of a finite range interaction $\Phi$:

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

In a ‘gapped phase’ (and with suitable boundary conditions), we often expect the spectrum of $H_\Lambda$ to have the following structure:

$$\mathrm{spec}(H_\Lambda) \subset [E_\Lambda(0), E_\Lambda(0) + \delta_\Lambda] \cup [E_\Lambda(0) + \delta_\Lambda + \gamma_\Lambda, \infty)$$

for some $\delta_\Lambda \geq 0$ and $\gamma_\Lambda > 0$. The simplest situation is when $\delta_\Lambda \to 0$ as $\Lambda \to \mathbb{Z}^\nu$, and $\gamma_\Lambda \geq \gamma > 0$, for all $\Lambda$.
Let $\mathcal{G}_\Lambda$ denote the spectral subspace associated with the spectrum in $[E_\Lambda(0), E_\Lambda(0) + \delta_\Lambda]$.

For concreteness, suppose $\Phi$ has a local symmetry described by a finite group $G$: for every $x \in \Lambda$, there is a unitary representation $u_x(g)$, $g \in G$, acting on $\mathcal{H}_x$, such that

$$[\Phi(X), U_X(g)] = 0, U_X(g) = \bigotimes_{x \in X} u_x(g), \text{ for all } g \in G.$$

If this symmetry is fully broken in the (infinite-volume) ground states, we expect a decomposition of $\mathcal{G}_\Lambda$ labeled by $g \in G$:

$$\mathcal{G}_\Lambda = \bigoplus_{g \in G} \mathcal{G}_\Lambda^g.$$

Example: the $\mathbb{Z}_2$-symmetry of the Ising model. In general, direct sum is not necessarily orthogonal.

Let $P_\Lambda$ denote the $\perp$ projection onto $\mathcal{G}_\Lambda$ and $P^g_\Lambda$ the $\perp$ projection onto $\mathcal{G}^g_\Lambda$. For a suitable sequence of finite volumes $\Lambda_n$ we can obtain the symmetry broken ground states in the thermodynamic limit:

$$\omega^g(A) = \lim_{n\to\infty} \frac{\mathrm{Tr} P^g_{\Lambda_n} A}{\mathrm{Tr} P^g_{\Lambda_n}},$$

for any local observable $A$. Symmetry breaking means that there is a local order parameter that distinguishes the states:

$$\omega^g(m) = m^g.$$

If there is translation invariance it follows that any two states giving different values to $m$ must become orthogonal in the infinite volume limit.

In this situation, we expect that for all local observable $A \in \mathcal{A}_X$, and unit vectors $\psi^i_{\Lambda_n} \in \mathcal{G}^{g_i}_{\Lambda_n}$, $i = 1, 2$, we have

$$\lim_n \langle \psi^1_{\Lambda_n}, A\psi^2_{\Lambda_n} \rangle = 0, \text{ if } g_1 \neq g_2.$$

For different $\psi_{\Lambda_n} \in \mathcal{G}^g_{\Lambda_n}$, with the same $g$, we often have the Local Topological Quantum Order (LTQO) property first introduced by Bravyi, Hastings, and Michalakis. Generalized to the situation with a symmetry $G$, asserts the following: there is a $q > 2(d + 1)$, and $\alpha \in (0, 1)$, such that for all $r \leq (\mathrm{diam}\,\Lambda)^\alpha$, and all $A \in \mathcal{A}_{B_x(r)}$, such that $[A, U_\Lambda(g)] = 0$, $g \in G$,

$$\|P_{B_x(r+\ell)} A P_{B_x(r+\ell)} - \omega_\Lambda(A) P_{B_x(r+\ell)}\| \leq C\|A\|\ell^{-q},$$

with

$$\omega_\Lambda(A) = \mathrm{Tr}(P_\Lambda A)/\mathrm{Tr}(P_\Lambda).$$

If spontaneous symmetry breaking occurs, then for all $A \in \mathcal{A}_{B_x(r)}$

$$\|P^g_{B_x(r+\ell)} A P^h_{B_x(r+\ell)} - \delta_{g,h}\omega^g_\Lambda(A) P^g_{B_x(r+\ell)}\| \leq C\|A\|\ell^{-q}.$$

with

$$\omega^g_\Lambda(A) = \mathrm{Tr}(P^g_\Lambda A)/\mathrm{Tr}(P^g_\Lambda).$$

We call this LTQO with symmetry breaking.

# Stability under uniformly small perturbations

The Michalakis-Zwolak stability result (CMP, 2013) applies to models with frustration-free finite-range interactions on periodic boxes in $\mathbb{Z}^\nu$. We (N-Sims-Young) recently obtained a generalization which includes situations with discrete symmetry breaking and more general lattices and boundary conditions. Let $B_x(R)$ denote the ball of radius $R$ centered at $x \in \mathbb{Z}^\nu$, and $\Lambda$ is a finite subset of $\Gamma$. Then,

$$H_\Lambda(0) = \sum_{\substack{x\in\Lambda \\ B_x(R)\subset\Lambda}} Q_x,$$

where each term $Q_x \in \mathcal{A}_{B_x(R)}$, satisfies $0 \leq Q_x \leq M\mathbb{1}$, and $[Q_x, U_{B_x(R)}(g)] = 0$, for all $g \in G$.

We consider perturbations of the following form:

$$H_\Lambda(\epsilon) = H_\Lambda(0) + \epsilon \sum_{X \subset \Lambda} \Phi(X).$$

and we will assume that there exists $a > 0$ such that

$$\|\Phi\|_a = \sup_{x,y \in \Gamma} e^{ad(x,y)} \sum_{\substack{X \subset \Gamma \\ x,y \in X}} \|\Phi(X)\| < \infty,$$

and

$$[\Phi(X), U_X(g)] = 0,$$

for all $g \in G$, $X \subset \mathbb{Z}^\nu$.

The assumptions on the unperturbed model are:
- It is Frustration Free: $\ker H_\Lambda(0) \neq \{0\}$; Let $P_\Lambda(\epsilon)$ denote the orthogonal projection onto $\ker H_\Lambda(\epsilon)$. Assume convergence

$$\omega(A) = \lim_n \frac{1}{\dim \ker H_{\Lambda_n}} \mathrm{Tr} P_{\Lambda_n}(0) A, \quad A \in \mathcal{A}_{\mathrm{loc}},$$

for a suitable sequence $\Lambda_n \nearrow \Gamma$.
- Local Gap: there is $\gamma > 0$ such that the gap above the ground state of $H_{B_x(r)} \geq \gamma$ for all $x$ and $r$;
- Local Topological Quantum Order (LTQO): there is a $q > 2(d+1)$, and $\alpha \in (0,1)$, for all $r \leq (\mathrm{diam}\,\Lambda_n)^\alpha$, and all $A \in \mathcal{A}_{B_x(r)}$, $[A, U_{B_x(r)}(g)] = 0$, for all $g \in G$.

$$\|P_{B_x(r+\ell)} A P_{B_x(r+\ell)} - \omega(A) P_{B_x(r+\ell)}\| \leq C \|A\| \ell^{-q}.$$

# Stability of the Spectral Gap

Let $E_\Lambda(\epsilon) = \inf \mathrm{spec}(H_\Lambda(\epsilon))$. The gap of $H_\Lambda(\epsilon)$ is defined taking into account that the perturbation may produce a splitting up to an amount $\delta_\Lambda$ of the zero eigenvalue of $H_\Lambda(0)$, which is in general degenerate:

$$\gamma_\delta(H_\Lambda(\epsilon)) = \sup\{\eta > 0 \mid (\delta, \delta+\eta) \cap \mathrm{spec}(H_\Lambda(\epsilon) - E_\Lambda(\epsilon)\mathbb{1}) = \emptyset\}$$

## Theorem (Bravyi-Hastings (2011), Michalakis-Zwolak (2013), N-Sims-Young (in prep))

*Let $H_\Lambda(0)$ be a finite-range $G$-symmetric Hamiltonian satisfying the assumptions of above and $\Phi$ an exponentially decaying $G$-symmetric perturbation. Then, for any $0 < \gamma_0 < \gamma(H_\Lambda(0))$ there is an $\epsilon_0 > 0$ such that for sufficiently large $\Lambda$,*

$$\gamma_{\delta_\Lambda}(H_\Lambda(\epsilon)) \geq \gamma_0, \ \text{if } |\epsilon| \leq \epsilon_0,$$

*where $\delta_\Lambda \leq C(\mathrm{diam}\,\Lambda)^{-q}$, for some $q > 0$.*

# Stability of the Ground State Phases

Next, consider the situation where in the unperturbed model we have spontaneous breaking of the symmetry $G$ in the frustration-free ground states. Concretely, we will assume the following:

The unperturbed model is defined on finite volume $\Lambda$ as before:

$$H_\Lambda(0) = \sum_{\substack{x \in \Lambda \\ B_x(R) \subset \Lambda}} Q_x,$$

where each term $Q_x \in \mathcal{A}_{B_x(R)}$, satisfies $0 \leq Q_x \leq M\mathbb{1}$, and $[Q_x, U_{B_x(R)}(g)] = 0$, for all $g \in G$.

We now assume that there are $N = |G|$ pure infinite-volume frustration-free ground states, $\omega^1, \ldots, \omega^N$, and the symmetries, $g \in G$, act transitively as permutations on this set.

For sufficiently large $m$, there are $N$ non-zero orthogonal projections $P^1_{b_x(m)}, \ldots, P^N_{b_x(m)}$, onto subspaces of $\ker H_{b_x(m)}$ such that the following properties hold:

- It is Frustration Free: $\ker H_\Lambda(0) \neq \{0\}$; Let $P_\Lambda(\epsilon)$ denote the orthogonal projection onto $\ker H_\Lambda(\epsilon)$. Assume convergence

$$\omega(A) = \lim_n \frac{1}{\dim \ker H_{\Lambda_n}} \mathrm{Tr} P_{\Lambda_n}(0) A, \quad A \in \mathcal{A}_{\mathrm{loc}},$$

for a suitable sequence $\Lambda_n \nearrow \Gamma$.

- Local Gap: there is $\gamma > 0$ such that the gap above the ground state of $H_{B_x(r)} \geq \gamma$ for all $x$ and $r$;

- LTQO with symmetry breaking: there exist $q > 2(d+1)$, and $\alpha \in (0,1)$, such that for all $r \leq (\mathrm{diam}\, \Lambda_n)^\alpha$, and all $A \in \mathcal{A}_{B_x(r)}$,

$$\left\| P^i_{b_x(r+\ell)} A P^j_{b_x(r+\ell)} - \delta_{ij} \omega^i(A) P^i_{b_x(r+\ell)} \right\| \leq C \|A\| \ell^{-q}. \qquad (1)$$

It is natural to assume that

$$\omega_0 = \frac{1}{N} \sum_{i=1}^{N} \omega^i \tag{2}$$

is the unique $G$-invariant frustration-free ground state.
The perturbations are of the form

$$H_\Lambda(\epsilon) = H_\Lambda(0) + \epsilon \sum_{X \subset \Lambda} \Phi(X).$$

such that $[\Phi(X), U_X(g)] = 0$, for all $g \in G$, and $\|\Phi\|_a < \infty$ for some $a > 0$.
Let $\mathcal{S}_\epsilon$ denote the set of all thermodynamic limits of ground states of $H_\Lambda(\epsilon)$

## Theorem (N-Sims-Young, in prep)

*There exists* $\epsilon_0 > 0$ *such that if* $|\epsilon| \leq \epsilon_0$*, then the set* $\mathcal{S}_\epsilon$ *is an N-dimensional simplex. Each of the extreme points (pure states) satisfies LTQO and has a non-vanishing spectral gap in the spectrum of its GNS Hamiltonian.*

The main tool in the proof is the spectral flow, which is constructed using Lieb-Robinson bounds for the dynamics and related transformations (Hastings (2005), Bachmann et al (2012)).

We also prove that the thermodynamic limit of the spectral flow yields quasi-local automorphisms $\alpha_\epsilon$ such that

$$\mathcal{S}_\epsilon = \{\omega \circ \alpha_\epsilon \mid \omega \in \mathcal{S}_0\}.$$

Therefore, the entire phase structure is preserved under the perturbations.

# Outlook

- Frustration free (FF) models turned out to be an essential tool to help us understand gapped ground state phases and their classification.
- Progress in estimating the spectral gap above the ground state has come from studying FF models.
  Well-understood in one dimension. Next step: higher dimensions (so far only special classes of examples in $d$ dimensions (Bishop-N-Young, JSP 2016).
- Stability results of ground states is based on FF models.
  Next steps: 1) 'stability' of non-FF ground states of FF models (in prep. and ongoing work Cha-Naaijkens-N); 2) relax the FF condition.
- Progress in stability results of superselection sectors is based on FF models with commuting terms: next step: treat more physically realistic interactions.