Image of conformally covariant, symmetry breaking operators for $\mathbb{R}^{p,q}$

Toshiyuki Kobayashi and Alex Leontiev

Abstract We consider the meromorphic continuation of an integral transform that gives rise to a conformally covariant, *symmetry breaking operator* $\mathbb{A}_{\lambda,\nu}$ between the natural family of representations $I(\lambda)$ and $J(\nu)$ of the indefinite orthogonal group $G = O(p+1,q+1)$ and its subgroup $G' = O(p,q+1)$, respectively, realized in function spaces on the conformal compactifications of flat pseudo-Riemannian manifolds $\mathbb{R}^{p,q} \supset \mathbb{R}^{p-1,q}$. In this article, we determine explicitly the image of the renormalized operator $\mathbb{A}_{\lambda,\nu}$ for all $(\lambda,\nu) \in \mathbb{C}^2$. In particular, the complex parameters (λ,ν) for which the image of $\mathbb{A}_{\lambda,\nu}$ coincides with $\{0\}$, $\mathbb{C}$, finite-dimensional representations, the minimal representation, or discrete series representations for pseudo-Riemannian space forms are explicitly classified. A graphic description of the K-types of the image is also provided. Our results extend a part of the prior results of Kobayashi and Speh [Memoirs of Amer. Math. Soc. 2015] in the Riemannian case where $q = 0$.

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1 Introduction

Let $(\pi, \mathscr{H}_\pi)$ be a representation of a group G, and $(\pi', \mathscr{H}_{\pi'})$ the one of a subgroup G'. A *symmetry breaking operator* is a linear map

$$T\colon \mathscr{H}_\pi \to \mathscr{H}_{\pi'}$$

that intertwines the actions of the subgroup G'. Then the image of T is a G'-submodule of π'.

In the last decade, symmetry breaking operators for infinite-dimensional representations of reductive groups $G \supset G'$ have been actively studied as a new line of investigation on branching problems of representation theory [10, 12, 22, 23] and also interacted with some other areas such as automorphic form theory or conformal geometry among others, see [4, 11].

1.1 Conformal representations $I(\lambda)$ and $J(\nu)$ associated with pseudo-Riemannian manifolds $\mathbb{R}^{p,q} \supset \mathbb{R}^{p-1,q}$

In this article we discuss symmetry breaking operators motivated from conformal geometry. Let $G = O(p+1, q+1)$ be the automorphism group of the quadratic form on $\mathbb{R}^{p+q+2}$ of signature $(p+1, q+1)$ defined by

$$Q_{p+1,q+1}(x) = x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2.$$

Let $\mathbb{R}^{p,q}$ be the $(p+q)$-dimensional vector space $\mathbb{R}^{p+q}$ endowed with flat pseudo-Riemannian structure

$$ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

of signature (p, q). Then, the group G acts isometrically on $\mathbb{R}^{p+1,q+1}$, and conformally on the conformal compactification

$$X := (S^p \times S^q)/\{\pm 1\}$$

of $\mathbb{R}^{p,q}$, which is the direct product of p- and q-spheres equipped with pseudo-Riemannian structure $g_{S^p} \oplus (-g_{S^q})$, modulo the direct product of antipodal maps, see Segal [24, Chap. II]. By the general theory of conformal groups [16, Sect. 2], one has a natural family of representations $I(\lambda)$ of G on $C^\infty(X)$ with parameter $\lambda \in \mathbb{C}$. We normalize $I(\lambda)$ such that $I(0)$ is the space of sections, and $I(\dim X)$ is the space of densities. Via the twisted pull-back $\iota_\lambda^*\colon C^\infty(X) \hookrightarrow C^\infty(\mathbb{R}^{p,q})$ of the conformal embedding $\iota\colon \mathbb{R}^{p,q} \hookrightarrow X$, we may realize $I(\lambda)$ on the subspace $\iota_\lambda^*(C^\infty(X))$ of $C^\infty(\mathbb{R}^{p,q})$, see [18, (2.8.6)].

Similarly, another group $G' := O(p, q+1)$ acts on the conformal compactification

$$Y := (S^{p-1} \times S^q)/\{\pm 1\}$$

of the flat pseudo-Riemannian manifold $\mathbb{R}^{p-1,q}$, and one has a natural family of representations $J(\nu)$ of G' on $C^\infty(Y)$ for $\nu \in \mathbb{C}$.

Thus we have a G-module $I(\lambda)$ and a module $J(\nu)$ of the subgroup G' with complex parameters λ and ν. The object of our study is symmetry breaking operators

$$I(\lambda) \to J(\nu)$$

with focus on their images.

1.2 Degenerate principal series representations

The representation $I(\lambda)$ of $G = O(p+1, q+1)$ defined in Section 1.1 by using conformal geometry may be interpreted as a *degenerate principal series representation* of the real reductive Lie group G as follows. Let $P = MAN_+$ be a maximal parabolic subgroup of G with Levi part $MA \simeq O(p,q) \times \{\pm 1\} \times \mathbb{R}$. For a character $\mathbb{C}_\lambda$ of $A \simeq \mathbb{R}$, we regard it as that of P via the quotient map $P \to P/MN_+ \simeq A$, and form a G-equivariant line bundle

$$\mathscr{L}_\lambda = G \times_P \mathbb{C}_\lambda \to G/P.$$

Then the (unnormalized) induced representation $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ is realized in the Fréchet space of smooth sections for the line bundle $\mathscr{L}_\lambda \to G/P$. Our parametrization is chosen in a way that $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ contains a finite-dimensional submodule if $-\lambda \in 2\mathbb{N}$ and a finite-dimensional quotient if $\lambda - (p+q) \in 2\mathbb{N}$. Then we have an isomorphism of G-modules

$$I(\lambda) \simeq \mathrm{Ind}_P^G(\mathbb{C}_\lambda).$$

The realization on $\iota_\lambda^*(C^\infty(X))$ $(\subset C^\infty(\mathbb{R}^{p+q}))$ is referred to as the *N-picture* of $I(\lambda)$.

Similarly to $I(\lambda)$, we have an isomorphism as G'-modules

$$J(\nu) \simeq \mathrm{Ind}_{P'}^{G'}(\mathbb{C}_\nu),$$

where $\mathrm{Ind}_{P'}^{G'}(\mathbb{C}_\nu)$ is the (unnormalized) induced representation of G' from a character $\mathbb{C}_\nu$ of a maximal parabolic subgroup P' with Levi part $O(p-1,q) \times \{\pm 1\} \times \mathbb{R}$.

1.3 Construction of symmetry breaking operators

We realize $\mathbb{R}^{p-1,q}$ as a submanifold of $\mathbb{R}^{p,q}$ by letting $x_p = 0$. This determines the embeddings $Y \hookrightarrow X$ between their conformal compactifications, and $G' = O(p, q+1) \hookrightarrow G = O(p+1, q+1)$ between conformal groups. Applying the general results proven in Kobayashi–Speh [22, Chap. 3] to our specific setting, we see that, for any symmetry breaking operator $T\colon I(\lambda)\to J(\nu)$, there exists a distribution $K_T\in\mathscr{D}'(\mathbb{R}^{p+q})$ such that for all $f\in I(\lambda)$

$$\iota_\nu^*(Tf)(x') = \mathrm{Rest}_{x_p=0}\circ\int_{\mathbb{R}^{p+q}} K_T(x-y)(\iota_\lambda^* f)(y)dy, \tag{1}$$

where $x'=(x_1,\cdots,\widehat{x_p},\cdots,x_{p+q})\in\mathbb{R}^{p-1,q}$ and $x=(x_1,\cdots,x_{p+q})\in\mathbb{R}^{p,q}$.
The distribution kernel K_T satisfies certain covariance properties, which characterize that T is a G'-intertwining operator (*i.e.*, T is a symmetry breaking operator), see [22, Thm. 3.16].

By [21, Lem. 2.22], T is a *differential* symmetry breaking operator if $\mathrm{Supp}(K_T)=\{0\}$. In contrast, T is called a *regular* symmetry breaking operator ([22, Def. 3.3]) if $\mathrm{Supp}(K_T)$ contains an interior point, or equivalently, if $\mathrm{Supp}(K_T)=\mathbb{R}^{p+q}$ in our setting. Differential symmetry breaking operators $I(\lambda)\to J(\nu)$ in our setting were classified in [19], by using the F-method [7, 8], see also [13] for a generalization. On the other hand, there exists a unique holomorphic family of symmetry breaking operators, to be denoted by $\mathbb{A}_{\lambda,\nu}$, up to scalar multiplication, such that $\mathbb{A}_{\lambda,\nu}$ is a regular symmetry breaking operator for an open dense subset of $(\lambda,\nu)\in\mathbb{C}^2$, see Remark 12 below. It is constructed as follows. We set

$$Q_{p,q}(x) := x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2.$$

Theorem 1 (regular symmetry breaking operator). *Suppose that $p,q\geq 1$. We let*

$$(G,G') := (O(p+1,q+1),O(p,q+1))$$

as before. The linear operator $\mathbb{A}_{\lambda,\nu}\colon I(\lambda)\to J(\nu)$, initially defined as the integral operator (1) *with locally integrable kernel function*

$$\mathscr{A}_{\lambda,\nu} := \frac{1}{\Gamma\left(\frac{\lambda-\nu}{2}\right)\Gamma\left(\frac{\lambda+\nu-p-q+1}{2}\right)\Gamma\left(\frac{1-\nu}{2}\right)}\left|x_p\right|^{\lambda+\nu-p-q}|Q_{p,q}(x)|^{-\nu} \tag{2}$$

on $\mathbb{R}^{p,q}$ for $\mathrm{Re}\,\nu\ll 0$ and $\mathrm{Re}(\lambda+\nu)\gg 0$, intertwines the action of the subgroup G', and extends to a family of symmetry breaking operators that depend holomorphically on (λ,ν) in the entire $\mathbb{C}^2$.

Remark 2 *The Gamma factor in* (2) *is chosen in an optimal way in the sense that*

- *$\mathbb{A}_{\lambda,\nu}$ depends holomorphically on $(\lambda,\nu)\in\mathbb{C}^2$;*
- *the set of the zeros of $\mathbb{A}_{\lambda,\nu}$ is a discrete subset in $\mathbb{C}^2$ (see Theorem 4) below.*

Remark 3 *Theorem 1 gives a generalization of Kobayashi–Speh [22, Thm. 1.5] which treated the $q=0$ case. We note that the normalizing Gamma factor is different in the $q=0$ case.*

1.4 Image of the symmetry breaking operators $\mathbb{A}_{\lambda,\nu}$

The goal of this article is to determine the image of the holomorphic continuation of the regular symmetry breaking operator

$$\mathbb{A}_{\lambda,\nu} : I(\lambda) \to J(\nu)$$

for all $(\lambda,\nu) \in \mathbb{C}^2$ given in Theorem 1. We note that $I(\lambda)$ is a G-module and $J(\nu)$ is a G'-module and that $G \supsetneqq G'$. This sort of problem was first studied in Kobayashi–Speh [22, Chap. 13], and a complete solution was given in the Riemannian case where $q = 0$. In this article, we shall consider a more general case where $q > 0$. This means that $\mathbb{R}^{p,q}$ is of indefinite metric and the conformal group $G = O(p+1,q+1)$ has real rank greater than one. For simplicity of the exposition, we confine ourselves to the case $p > 1$ in this article.

Our main theorem (Theorem 15) will be formulated in Section 3 after preparing some combinatorial notation. As an introduction, we avoid complicated definitions in the general case, and focus on specific features of Theorem 15 instead, giving explicit criteria for the parameter (λ,ν) to fulfill the following conditions:

(1) $\mathrm{Image}(\mathbb{A}_{\lambda,\nu}) = \{0\}$, *i.e.*, $\mathbb{A}_{\lambda,\nu}$ vanishes (Theorem 4);
(2) $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ is finite-dimensional (Theorem 5);
(3) $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ is the trivial one-dimensional representation (Corollary 7);
(4) $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ is the minimal representation (Theorem 8);
(5) $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ is a discrete series for the pseudo-Riemannian space form (Theorem 9).

(1) **Vanishing condition of $\mathbb{A}_{\lambda,\nu}$.**

In the theory of symmetry breaking operators, it is an important question to determine the zeros and poles of the meromorphic continuation of the regular symmetry breaking operators. Once it is normalized as in Remark 2, of particular importance is to find precisely the parameters for which the holomorphic continuation of the normalized regular symmetry breaking operator vanishes. In those places, we expect that the representations are reducible and that the dimension of symmetry breaking operators jumps up, see [22, Thm. 11.4] for instance.

In the case $q = 0$, it is proved in [22, Thm. 8.1] that the zeros of the normalized regular symmetry breaking operator are given by the following discrete set:

$$L_{\mathrm{even}} := \{(\lambda,\nu) \in \mathbb{Z}^2 : \lambda \le \nu \le 0,\ \lambda \equiv \nu \mod 2\}. \tag{3}$$

In the case $q \ge 1$, the zeros of $\mathbb{A}_{\lambda,\nu}$ are given as follows. For simplicity, we assume $p \ne 1$.

Theorem 4. *Suppose $p \ge 2$ and $q \ge 1$. Then the following two conditions on $(\lambda,\nu) \in \mathbb{C}^2$ are equivalent:*

(i) $\mathbb{A}_{\lambda,\nu} = 0$.
(ii) $(\lambda,\nu) \in /\!/ \cap |||$.

The definition of the subsets $//$ and $|||$ will be given $\mathbb{C}^2$ in Section 3.1. For here we present a more concrete formula of the intersection $// \cap |||$ by comparing it with L_{even}. For this, we consider the following discrete set in $\mathbb{C}^2$.

$$\Gamma := \{(\lambda, \nu) \in \mathbb{Z}^2 : 0 < \nu \text{ and } \lambda \leq \nu\}.$$

Then by (12) and (14) below, one sees

$$// \cap ||| = \begin{cases} (L_{\text{even}} \cap (2\mathbb{Z})^2) \sqcup (\Gamma \cap (2\mathbb{Z})^2) & \text{if } q \text{ is odd,} \\ L_{\text{even}} \sqcup (\Gamma \cap (2\mathbb{Z}+1)^2) & \text{if } q \text{ is even.} \end{cases} \tag{4}$$

(2) **When is** $\text{Image}(\mathbb{A}_{\lambda,\nu})$ **finite-dimensional?**

Whereas the vanishing condition of the symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ depends on the parity of q in the previous theorem, see (4), it turns out that the condition on $(\lambda, \nu) \in \mathbb{C}^2$ for $\text{Image}(\mathbb{A}_{\lambda,\nu})$ to be a (nonzero) finite-dimensional vector space is independent of the parities of p and q as below.

Theorem 5 (finite-dimensional image). *Suppose $p \geq 2$ and $q \geq 1$. Then the following two conditions on $(\lambda, \nu) \in \mathbb{C}^2$ are equivalent.*

(i) $\text{Image}(\mathbb{A}_{\lambda,\nu} : I(\lambda) \to J(\nu))$ *is nonzero and finite-dimensional.*
(ii) $\nu \in 2\mathbb{Z}$, $\nu \leq 0$ *and λ satisfies one of the following:*

- $\lambda \in 2\mathbb{Z}$ *and* $\nu < \lambda$;
- $\lambda \in \mathbb{C} - 2\mathbb{Z}$.

Graphically, Theorem 5 corresponds to the colored red left corner bounded by the "barrier A^{++}" in Case A of Theorems 20, 26, 31 and 35 in later sections. We note that if one of (therefore both of) the equivalent conditions (i) and (ii) in Theorem 5 are fulfilled, then $\text{Image}(\mathbb{A}_{\lambda,\nu})$ is an irreducible representation of the subgroup G'.

Remark 6 *See [22, Thm. 13.1] for an analogous theorem in the case $q = 0$ and [22, Thm. 14.9] for some application.*

(3) **When is** $\text{Image}(\mathbb{A}_{\lambda,\nu})$ **isomorphic to the trivial one-dimensional representation?**

The trivial one-dimensional representation of G' occurs as a subrepresentation of the degenerate principal series representation $J(\nu)$ with $\nu = 0$. Then the equivalence (i) $\Leftrightarrow$ (iii) in the following corollary is an immediate consequence of Theorem 5 with $\nu = 0$. The equivalence (i) $\Leftrightarrow$ (ii) follows from a trick in Kobayashi–Speh [22, Chap. 14].

Corollary 7 *Suppose $p \geq 2$ and $q \geq 1$. Then the following three conditions on $\lambda \in \mathbb{C}$ are equivalent:*

(i) $\text{Image}(\mathbb{A}_{\lambda,0} : I(\lambda) \to J(0))$ *is the trivial one-dimensional representation of G'.*
(ii) *The regular symmetry breaking operator $\mathbb{A}_{\lambda,0}$ induces a nonzero G-intertwining operator*

$$I(\lambda) \to C^\infty(G/G'). \tag{5}$$

(iii) $\lambda \in \mathbb{C} - \{-2, -4, -6, \cdots\}$.

In Corollary 7, the resulting G-intertwining operator (5) is nothing but a (generalized) Poisson transform to the semisimple symmetric space G/G', which is the pseudo-Riemannian space form $M^+_{p,q+1}$ of signature $(p, q+1)$ with positive constant curvature.

(4) **When is** $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ **isomorphic to the minimal representation?**

For $p+q$ odd (≥ 7), there exists an irreducible unitary representation ϖ of $G' = O(p, q+1)$, referred to as the *minimal representation*. Here by minimal representation we mean that the annihilator of the smooth representation ϖ^∞ in the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ is the Joseph ideal [3]. This is the unique irreducible unitary representation of G' whose Gelfand–Kirillov dimension is equal to $p+q-2$, which is smaller than that of any other infinite-dimensional unitary representation of G'. It is remarkable that our symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ constructs the minimal representation as its image when $p+q \equiv 1 \mod 4$ by a specific choice of (λ, ν) as follows.

Theorem 8 (minimal representation). *Suppose that $p+q = 4k+1$ for some $k \in \mathbb{Z}$ with $k \geq 2$. If we take*

$$(\lambda, \nu) := (2k+1, 2k-1),$$

then the underlying $(\mathfrak{g}', K')$-module of $\mathrm{Image}(\mathbb{A}_{\lambda,\nu} \colon I(\lambda) \to J(\nu))$ *is isomorphic to that of the minimal representation ϖ of the subgroup $G' = O(p, q+1)$. In particular, $\mathbb{A}_{\lambda,\nu} f$ is a Yamabe harmonic on the pseudo-Riemannian manifold $Y = (S^p \times S^q)/\{\pm 1\}$ for any $f \in I(\lambda)$.*

The last statement of Theorem 8 follows from the geometric construction of the minimal representation proved in [16, Thms. 3.4.2 and 3.6.1]. See Case E in Theorem 26 for a graphic interpretation of Theorem 8.

(5) **When is** $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ **isomorphic to a discrete series representation for a generalized hyperboloid?**

Let $M^\varepsilon_{p,q}$ be the $(p+q)$-dimensional pseudo-Riemannian space form of signature (p,q) of constant sectional curvature $+1$ ($\varepsilon = +$) and -1 ($\varepsilon = -$), referred also to as a generalized hyperboloid from its realization as hypersurfaces in $\mathbb{R}^{p+q+1}$:

$$M^+_{p,q} = \{x \in \mathbb{R}^{p+1,q} : Q_{p+1,q}(x) = 1\},$$
$$M^-_{p,q} = \{x \in \mathbb{R}^{p,q+1} : Q_{p,q+1}(x) = -1\}.$$

For simplicity, we treat only the case $\varepsilon = -$ here. Then the group $G' = O(p, q+1)$ acts isometrically and transitively on $M^-_{p,q}$. As a homogeneous space, we have the following diffeomorphism

$$M^-_{p,q} \simeq O(p, q+1)/O(p,q).$$

Let $\square_{p,q}$ be the Laplacian of the pseudo-Riemannian manifold $M^-_{p,q}$. For $p > 0$ and $q > 0$, the Laplacian $\square_{p,q}$ is not an elliptic operator. In this case, there exist countably many L^2-eigenvalues of $\square_{p,q}$ on $M^-_{p,q}$, given explicitly by

$$\{\nu(\nu-2\rho) : \nu \in \mathbb{Z}, \rho < \nu\}, \tag{6}$$

see Faraut [1] or Strichartz [25]. Here we set

$$\rho := \frac{1}{2}(p+q-1). \tag{7}$$

The isometry group G' acts irreducibly on the Hilbert space of L^2-eigenfunctions of $\square_{p,q}$ for each eigenvalue $\nu(\nu-2\rho)$. Following the same notation as in [17, Sect. 5], we write $\pi^{p,q+1}_{-,\nu-\rho}$ for the resulting irreducible unitary representation. The representation $\pi^{p,q+1}_{-,\nu-\rho}$ is referred to as a *discrete series representation* for the generalized hyperboloid $M^-_{p,q}$.

Theorem 9 (discrete series for pseudo-Riemannian space form). *Suppose $p \geq 2$ and $q \geq 1$. Let $\nu \in \mathbb{Z}$ satisfy $\rho < \nu$. Then the underlying $(\mathfrak{g}, K)$-module of* $\mathrm{Image}(\mathbb{A}_{\lambda,\nu} : I(\lambda) \to J(\nu))$ *is isomorphic to that of a discrete series representation for the pseudo-Riemannian space form $M^-_{p,q}$ if and only if $\nu \equiv q+1 \mod 2$ and λ satisfies the following conditions.*
Case 1. *q is even.*
$(\lambda \in 2\mathbb{Z}$ *and* $\nu < \lambda)$ *or* $(\lambda \in \mathbb{C} - 2\mathbb{Z})$.
Case 2. *q is odd.*
$(\lambda \in 2\mathbb{Z}+1$ *and* $\nu < \lambda)$ *or* $(\lambda \in \mathbb{C} - (2\mathbb{Z}+1))$.

To see Theorem 9, we use a realization of $\pi^{p,q+1}_{-,\nu-\rho}$ as a subrepresentation of $J(\nu)$ with K-types E^{+-}_{ν} given by the barrier $A^{+-}_{p,q+1,-\nu}$, see Example 14. Then Theorem 9 follows from a graphic description of $\mathrm{Image}(\mathbb{A}_{\lambda,\nu})$ in

Cases E, E′, and E$_{bis}$	in Theorem 20;
Cases G and G$_{bis}$	in Theorem 26 with $\nu > \rho$;
Cases C and C$_{bis}$	in Theorem 31 with $\nu > \rho$;
Cases B, B′, and B$_{bis}$	in Theorem 35.

The paper is organized as follows. In Section 2 we give brief comments on our problem from a perspective on the general problem of restrictions of representations, in particular, for pairs of reductive groups $G \supset G'$. In Section 3, we determine $\mathrm{Image}(\mathbb{A}_{\lambda,\nu} : I(\lambda) \to J(\nu))$ for all $(\lambda, \nu) \in \mathbb{C}^2$ in Theorem 15 when $(G, G') = (O(p+1, q+1), O(p, q+1))$ with $p \geq 2$ and $q \geq 1$. A graphic description of Theorem 15 is given in Sections 4–7 depending on the parities of p and q, from which theorems in Introduction follow. A detailed proof of Theorem 15 will appear elsewhere.

Notation. $\mathbb{N} := \{0, 1, 2, \cdots\}$. For two subsets A and B of a set, we write $A - B := \{a \in A : a \notin B\}$ rather than the usual notation $A \setminus B$.

2 Branching program ABC for restriction of representations

In this section, we provide some brief comments on the topic treated here from a perspective on the general problem of restriction of (infinite-dimensional) representations of real reductive Lie groups. See [10, 11] for more details.

2.1 *Finiteness criterion for multiplicities in branching of representations*

Suppose $G \supset G'$ are a pair of reductive groups and π is an irreducible representation of G. The restriction of π to the subgroup G' is no more irreducible in general as a representation of G'. If G is compact, then any irreducible π is finite-dimensional and splits into a finite direct sum

$$\pi|_{G'} = \bigoplus_{\pi' \in \widehat{G'}} m(\pi, \pi')\pi'$$

of irreducibles π' of G' with multiplicities $m(\pi, \pi')$. In this case, the multiplicity $m(\pi, \pi')$ is given by

$$\dim \mathrm{Hom}_{G'}(\pi', \pi|_{G'}) = \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \pi'). \tag{8}$$

However, for noncompact G' and for infinite-dimensional π, the restriction $\pi|_{G'}$ is not always a direct sum of irreducible representations, even if π is a unitary representation of G. In general, we need the notion of direct integral of Hilbert spaces to give an irreducible decomposition of the restriction $\pi|_{G'}$. Sometimes there is no continuous spectrum in the irreducible decomposition of the restriction $\pi|_{G'}$, even when G' is noncompact. See [5, 6] for the condition that the restriction $\pi|_{G'}$ is discretely decomposable.

For the more general case where π is nonunitary, the equality (8) does not hold: both of the spaces $\mathrm{Hom}_{G'}(\cdot,\cdot)$ depend on the underlying topologies on the representation spaces of π and π'.

To clarify our formulation, we recall that, associated to a continuous representation π of a Lie group on a Banach space $\mathscr{H}$, a continuous representation π^∞ is defined on the Fréchet space $\mathscr{H}^\infty$ of C^∞-vectors of the Banach representation on $\mathscr{H}$. Given another representation π' of the subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}\left(\pi^\infty|_{G'}, (\pi')^\infty\right). \tag{9}$$

If both π and π' are admissible representations of finite length of reductive Lie groups G and G', respectively, then the dimension of the space (9) is determined by the underlying $(\mathfrak{g}, K)$-module π_K of π and the $(\mathfrak{g}', K')$-module $\pi'_{K'}$ of π', and is independent of the choice of Banach globalizations because π^∞ and $(\pi')^\infty$ are determined uniquely by π_K and $\pi'_{K'}$, respectively, by the Casselman–Wallach theory [27, Chap. 11]. We denote by $m(\pi,\pi')$ the dimension of (9), and call it the *multiplicity* of π' in the restriction $\pi|_{G'}$.

In general, the multiplicity $m(\pi,\pi')$ may be infinite, even when G' is a maximal reductive subgroup of G and π is irreducible. This happens even when (G,G') is a symmetric pair. By using the theory of real spherical spaces initiated in Kobayashi–Oshima [20], the criterion for finite-multiplicities is discovered in [9, 20] as follows.

Fact 10 *Let (G,G') be a pair of real reductive Lie groups, and $(G_{\mathbb{C}},G'_{\mathbb{C}})$ its complexification.*

(1) *The multiplicity $m(\pi,\pi')$ is finite for all irreducible representations π of G and all irreducible representations π' of G' if and only if a minimal parabolic subgroup of G' has an open orbit on the real flag variety of G.*
(2) *The multiplicity $m(\pi,\pi')$ is uniformly bounded if and only if a Borel subgroup of $G'_{\mathbb{C}}$ has an open orbit on the complex flag variety of $G_{\mathbb{C}}$.*

The complete classification of symmetric pairs (G,G') satisfying the above geometric criteria was accomplished in Kobayashi–Matsuki [15]. The $(G,G)=(O(p+1,q+1),O(p,q+1))$ satisfies the criterion in (2) (and in particular, the criterion in (1), too), and therefore, $m(\pi,\pi')$ is uniformly bounded. Furthermore, Sun–Zhu [26] proved that $m(\pi,\pi')\leq 1$.

In the theory of symmetry breaking operators, we consider "quotient map" from a representation of a group G to that of the subgroup G'. On the other hand, one may reverse arrows and consider "embedding map" from a representation of a subgroup G' to that of G, *e.g.,* consider the following spaces:

$$\mathrm{Hom}_{G'}\left((\pi')^\infty,\pi^\infty|_{G'}\right)\ \text{ or }\ \mathrm{Hom}_{\mathfrak{g}',K'}\left(\pi'_{K'},\pi_K|_{\mathfrak{g}',K'}\right).$$

We observe that there are canonical injective maps:

$$\mathrm{Hom}_{G'}\left(\left((\pi')^\vee\right)^\infty,(\pi^\vee)^\infty|_{G'}\right)\subset \mathrm{Hom}_{G'}\left(\pi^\infty|_{G'},\pi'^\infty\right),$$
$$\mathrm{Hom}_{\mathfrak{g}',K'}\left((\pi')^\vee_{K'},(\pi^\vee)_K|_{\mathfrak{g}',K'}\right)\subset \mathrm{Hom}_{\mathfrak{g}',K'}\left(\pi_K|_{\mathfrak{g}',K'},\pi'_{K'}\right),$$

where the symbol $^\vee$ stands for the contragredient representation. The study of these objects in the left-hand sides is closely related to the theory of discretely decomposable restrictions, [5, 6], which we do not discuss here. Concerning the right-hand sides for symmetry breaking operators in the category of $(\mathfrak{g},K)$-modules and in the category of admissible smooth representations of moderate growth, we raised a question in [9, Sect. 10] about automatic continuity property for symmetry breaking operators as a generalization of the theory of Casselman–Wallach ($G=G'$ case): it is plausible that if (G,G') satisfies one of (therefore any of) the equivalent conditions in Fact 10 (1), then the natural injection map below is surjective

$$\mathrm{Hom}_{G'}\left(\pi^\infty|_{G'},\pi'^\infty\right)\hookrightarrow \mathrm{Hom}_{\mathfrak{g}',K'}\left(\pi_K|_{\mathfrak{g}',K'},\pi'_{K'}\right),$$

see [9, Rem. 10.2 (4)].

2.2 Program ABC for branching

The study of restriction of representations (branching problem) is an important but involves different types of difficult problems even in very special cases. The first author analysed various (in fact, *wild*) features and phenomena about restrictions for reductive Lie groups, and proposed in [10] a program for studying the restriction of representations of reductive groups, which may be summarized as follows:

Stage A. Abstract features of the restriction;
Stage B. Branching law of $\pi|_{G'}$;
Stage C. Construction of symmetry breaking operators.

Fact 10 is an example for Stage A in branching problems. Stage A aims for developing the general theory of the restrictions $\pi|_{G'}$ (*e.g.*, spectrum, multiplicity), which would single out the *good* triples (G, G', π). In turn, we could expect concrete and detailed study of those restrictions $\pi|_{G'}$ in Stages B and C.

For instance, Fact 10 assures the following *a priori* estimate:

$$m(\pi, \pi') \text{ is uniformly bounded}$$

if the pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ is a real form of $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ or $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$, in particular, if (G, G') is of the form

$$(G, G') = (O(p+1, q+1), O(p, q+1)). \tag{10}$$

"Stage B" is a traditional question, however, it is often very difficult to compute explicitly branching laws of infinite-dimensional representations of (noncompact) reductive groups. The first systematic study of "Stage C" is given by a monograph by Kobayashi–Speh [22], which corresponds to the case $q = 0$, more precisely, the case

π: spherical principal series representations of $G = O(n+1, 1)$,
π': spherical principal series representations of $G' = O(n, 1)$,

Stage C includes the following subproblems.

(C1) construct symmetry breaking operators explicitly;
(C2) classify all symmetry breaking operators;
(C3) find residue formulæ for symmetry breaking operators;
(C4) study functional equations among symmetry breaking operators;
(C5) determine the images of subquotients by symmetry breaking operators.

The subprogram (C1)–(C5) was considered by Kobayashi–Speh [22] with a complete answer for the pair $(G,G') = (O(n+1,1), O(n,1))$ of real rank one groups (10).

In [14], we discussed the subprograms (C1)–(C4) for degenerate spherical principal series representations $\pi = I(\lambda)$ of G and $\pi' = J(\nu)$ of G' for the pair of higher real rank groups. The (C5) is the main issue of this article.

3 Main theorem

In this section we determine the image of the meromorphic continuation of the regular symmetry breaking operator

$$\mathbb{A}_{\lambda,\nu} : I(\lambda) \to J(\nu)$$

for all $(\lambda,\nu) \in \mathbb{C}^2$. The statement of the main results uses the following notation:

(1) subsets $//$, $\backslash\backslash$, $|||$, $\mathbb{X}$ and $\mathbb{X}_+$ (see Section 3.1),
(2) description of G'-submodules of the target space $J(\nu)$ (see Section 3.2).

Theorems 4–9 are special cases of the main theorem of this section (Theorem 15).

3.1 Subsets $//$, $|||$, $\backslash\backslash$ and $\mathbb{X}$ in $\mathbb{C}^2$

We introduce some subsets of $\mathbb{C}^2$. It should be noted that the symbols $//$, $\backslash\backslash$, $||$, and $|||$ below are defined as subsets of $\mathbb{C}^2$, and are not as binary relations.

Definition 11 *We let*

$$\backslash\backslash := \{(\lambda,\nu) \in \mathbb{C}^2 : p+q-1-\lambda-\nu \in 2\mathbb{N}\}, \tag{11}$$
$$// := \{(\lambda,\nu) \in \mathbb{C}^2 : \nu-\lambda \in 2\mathbb{N}\}, \tag{12}$$
$$\mathbb{X} := // \cap \backslash\backslash, \tag{13}$$
$$||| := \{(\lambda,\nu) \in \mathbb{C}^2 : \nu \in -2\mathbb{N} \cup (q+1+2\mathbb{Z})\}. \tag{14}$$

For the sets $//$, $\backslash\backslash$, and $\mathbb{X}$, we have adopted the same notation with the one introduced in [22] which dealt with the $q=0$ case. It is easy to see

$$\mathbb{X} \cap \{(\lambda,\nu) \in \mathbb{C}^2 : \nu \in \mathbb{Z}\} = \emptyset \quad \text{if and only if } p+q \text{ is even.} \tag{15}$$

As in [22], we define $\ell \in \mathbb{N}$ and and $k \in \mathbb{N}$ by

$$2\ell = \nu - \lambda \text{ for } (\lambda,\nu) \in //, \tag{16}$$
$$2k = p+q-1-\lambda-\nu \text{ for } (\lambda,\nu) \in \backslash\backslash. \tag{17}$$

We define two subsets of $\mathbb{X} = \backslash\backslash \cap //$ by

$$\mathbb{X}_+ := \{(\lambda,\nu) \in \mathbb{X} : \rho \le \nu\} = \{(\rho-\ell-k, \rho+\ell-k) : \ell, k \in \mathbb{N}, \ell \ge k\},$$
$$\mathbb{X}_- := \{(\lambda,\nu) \in \mathbb{X} : \rho > \nu\} = \{(\rho-\ell-k, \rho+\ell-k) : \ell, k \in \mathbb{N}, \ell < k\}.$$

Here we recall from (7) that $\rho = \frac{1}{2}(p+q-1)$.

We decompose the set $// \cup \backslash\backslash$ into a disjoint union

$$// \cup \backslash\backslash = // \sqcup (\backslash\backslash - \mathbb{X}),$$

and further decompose the set $//$ into three subsets

$$// = (// - (\mathbb{X}_+ \cup |||)) \sqcup (// \cap |||) \sqcup (\mathbb{X}_+ - |||),$$

where we have used $\mathbb{X}_+ \subset //$. Combining these decompositions together, we have a decomposition of the parameter set $\mathbb{C}^2$ of (λ,ν) into a disjoint union of five subsets as follows

$$\mathbb{C}^2 = (// \cup \backslash\backslash)^c \sqcup (// - (\mathbb{X}_+ \cup |||)) \sqcup (// \cap |||) \sqcup (\mathbb{X}_+ - |||) \sqcup (\backslash\backslash - \mathbb{X}). \quad (18)$$

Here we set $(// \cup \backslash\backslash)^c := \mathbb{C}^2 - (// \cup \backslash\backslash)$. Then the image of the regular symmetry breaking operator is described according to which subset the parameter (λ,ν) belongs to.

Remark 12 *The support of the distribution kernel $\mathscr{A}_{\lambda,\nu}$ of the symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ does not contain an interior point, if $(\lambda,\nu) \in //, \backslash\backslash$, or if $\nu \in 1+2\mathbb{N}$, respectively, when $p,q \ge 1$ ([14, Thm. 6.3]).*

3.2 Description of submodules of the principal series $J(\nu)$

The degenerate spherical principal series representation $J(\nu)$ of the group $G' = O(p,q+1)$ has at most four irreducible subquotients. The number of irreducible subquotients depends on $\nu \in \mathbb{C}$ and on the parities of p and q. In this section, we give a quick review of the socle filtration of $J(\nu)$ from Howe and Tan [2]. We note that our group is $G' = O(p,q+1)$ whereas their group in [2] is $O(p,q)$.

Let $K' = O(p) \times O(q+1)$. Then K' is a maximal compact subgroup $G' = O(p,q+1)$, and the G'-module $J(\nu)$ is multiplicity-free as K'-modules for any $\nu \in \mathbb{C}$. To describe its K'-types, it is convenient to use the notion of spherical harmonics, which we recall now. The space of spherical harmonics of degree $a \in \mathbb{N}$ is defined by

$$\begin{aligned}\mathscr{H}^a(\mathbb{R}^p) :=& \{F \in \mathrm{Pol}[x_1, \cdots, x_p] : \sum_{j=1}^{p} \frac{\partial^2 F}{\partial x_j^2} = 0,\ \sum_{j=1}^{p} x_j \frac{\partial F}{\partial x_j} = aF\} \\ \simeq& \{f \in C^\infty(S^{p-1}) : \Delta_{S^{p-1}} f = -a(a+p-2) f\},\end{aligned}$$

where the second isomorphism is induced by the restriction map $F \mapsto f = F|_{S^{p-1}}$. Then $\mathscr{H}^a(\mathbb{R}^p) \neq \{0\}$ if $p \geq 2$ or if $p = 1$ and $a \in \{0,1\}$, and the orthogonal group $O(p)$ acts irreducibly on $\mathscr{H}^a(\mathbb{R}^p)$.

With this notation, the space $J(\nu)_{K'}$ of K'-finite vectors is decomposed into the multiplicity-free direct sum of irreducible K'-modules as follows.

$$J(\nu)_{K'} \simeq \bigoplus_{(a,b)\in\mathbb{N}^2_{\text{even}}} \mathscr{H}^a(\mathbb{R}^p) \boxtimes \mathscr{H}^b(\mathbb{R}^{q+1}), \tag{19}$$

where we set

$$\mathbb{N}^2_{\text{even}} := \{(a,b) \in \mathbb{N}^2 : a \equiv b \mod 2\}.$$

Therefore, any G'-submodule is characterized by its K'-types, which, in turn, is parametrized as a subset of $\mathbb{N}^2_{\text{even}}$ via (19). We introduce the following notation.

Definition 13 *We set*

$$E^{++}_\nu := \begin{cases} \{(a,b) \in \mathbb{N}^2_{\text{even}} : a+b \leq -\nu\} & \textit{if } \nu \in -2\mathbb{N}, \\ \mathbb{N}^2_{\text{even}} & \textit{if } \nu \notin -2\mathbb{N}, \end{cases}$$
$$E^{+-}_\nu := \begin{cases} \{(a,b) \in \mathbb{N}^2_{\text{even}} : a-b \leq -\nu+q-1\} & \textit{if } 1-\nu+q \in 2\mathbb{Z}, \\ \mathbb{N}^2_{\text{even}} & \textit{if } 1-\nu+q \notin 2\mathbb{Z}, \end{cases}$$
$$E^{-+}_\nu := \begin{cases} \{(a,b) \in \mathbb{N}^2_{\text{even}} : a-b \geq \nu-p+2\} & \textit{if } \nu-p \in 2\mathbb{Z}, \\ \mathbb{N}^2_{\text{even}} & \textit{if } \nu-p \notin 2\mathbb{Z}, \end{cases}$$
$$E^{--}_\nu := \begin{cases} \{(a,b) \in \mathbb{N}^2_{\text{even}} : a+b \geq \nu+3-p-q\} & \textit{if } p+q-1-\nu \in -2\mathbb{N}, \\ \mathbb{N}^2_{\text{even}} & \textit{if } p+q-1-\nu \notin -2\mathbb{N}. \end{cases}$$

Then the K'-types of any nonzero G'-submodules of $J(\nu)$ are given by the intersection of some $E^{\delta,\varepsilon}_\nu$ $(\delta,\varepsilon = \pm)$.

Example 14 *In Section 1.4, we discussed discrete series representations $\pi^{p,q+1}_{-,\nu-\rho}$ ($\nu \in \mathbb{Z}$ and $\nu > \rho$) for the pseudo-Riemannian space form $M^-_{p,q} = O(p,q+1)/O(p,q)$. Then, if $\nu \equiv q+1 \mod 2$, then the smooth representation $(\pi^{p,q+1}_{-,\nu-\rho})^\infty$ of $\pi^{p,q+1}_{-,\nu-\rho}$ is isomorphic to the subrepresentation of $J(\nu)$ with K-types given by E^{+-}_ν, see [17, Sect. 5].*

As in [2], we define functions of $\mathbb{R}^2$ by

$$\begin{aligned} A^{++}_{p,q,c}(a,b) &:= c-a-b, \\ A^{+-}_{p,q,c}(a,b) &:= c-a+b+q-2, \\ A^{-+}_{p,q,c}(a,b) &:= c+a-b+p-2, \\ A^{--}_{p,q,c}(a,b) &:= c+a+b+p+q-4. \end{aligned}$$

Then we may characterize $E^{\pm\pm}_\nu$ by the "barriers" as follows:

- when $\nu \in 2\mathbb{Z}$,

$$E^{++}_\nu = \{(a,b) \in \mathbb{N}^2_{\text{even}} : A^{++}_{p,q+1,-\nu}(a,b) \geqslant 0\};$$

- when $1-\nu+q\in 2\mathbb{Z}$,

$$E_\nu^{+-}=\{(a,b)\in\mathbb{N}^2_{\text{even}}:A^{+-}_{p,q+1,-\nu}(a,b)\geqslant 0\};$$

- when $\nu-p\in 2\mathbb{Z}$,

$$E_\nu^{-+}=\{(a,b)\in\mathbb{N}^2_{\text{even}}:A^{-+}_{p,q+1,-\nu}(a,b)\geqslant 0\};$$

- when $p+q-1-\nu\in 2\mathbb{Z}$,

$$E_\nu^{--}=\{(a,b)\in\mathbb{N}^2_{\text{even}}:A^{--}_{p,q+1,-\nu}(a,b)\geqslant 0\}.$$

In later sections, we use the symbol $A^{\pm\pm}$ to refer to the line (or "barrier") defined by the zero locus of the functions $A^{\pm\pm}(x,y)$, indicating that the submodules graphically given by the barrier.

3.3 Main theorem: image of $\mathbb{A}_{\lambda,\nu}$

As we mentioned, since $J(\nu)$ is multiplicity-free as a K'-module, any G'-submodule of $J(\nu)$ is characterized by its K'-types, or equivalently, the corresponding subset of $\mathbb{N}^2_{\text{even}}$ via (19).

Theorem 15. *Suppose $p\geq 2$ and $q\geq 1$. Then* $\operatorname{Image}\left(\mathbb{A}_{\lambda,\nu}:I(\lambda)\to J(\nu)\right)$ *is a G'-submodule of $J(\nu)$ which is characterized by its K'-types according to the decomposition* (18) *of the parameter space as follows:*

$$\begin{cases} E_\nu^{++}\cap E_\nu^{+-} & \textit{if } (\lambda,\nu)\in\mathbb{C}^2-(//\cup\backslash\backslash),\\ E_\nu^{++}\cap E_\nu^{+-} & \textit{if } (\lambda,\nu)\in //-(\mathbb{X}_+\cup |||),\\ \emptyset & \textit{if } (\lambda,\nu)\in //\cap |||,\\ E_\nu^{++}\cap E_\nu^{+-}\cap E_\nu^{-+}\cap E_\nu^{--} & \textit{if } (\lambda,\nu)\in \mathbb{X}_+-|||,\\ E_\nu^{++}\cap E_\nu^{+-}\cap E_\nu^{-+}\cap E_\nu^{--} & \textit{if } (\lambda,\nu)\in \backslash\backslash-\mathbb{X}. \end{cases}$$

3.4 Restatement of Theorem 15

The conditions on the parameter (λ,ν) in Theorem 15 may look somewhat complicated, however, the subsets of $\mathbb{C}^2$ given in (18) are of simpler forms when we specify the parities of p,q and ν as follows.

Proposition 16 *Let $\rho=\frac{1}{2}(p+q-1)$ as in* (7). *Suppose $\nu\in\mathbb{Z}$. Then the subsets in* (18) *reduce to the following sets in Table 1 when we impose conditions on p,q and ν in the left two columns in the table.*

To be precise about Table 1, we use the following convention. Under the conditions on p,q and ν described in the left two columns, the symbol in each box gives the

		$//-(\mathbb{X}_+\cup \vert\vert\vert)$	$//\cap \vert\vert\vert$	$\mathbb{X}_+ - \vert\vert\vert$	$\backslash\backslash - \mathbb{X}$
p even q even	ν even $\nu \leq 0$	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash$
	ν even $\nu > 0$	$//$	$\emptyset$	$\emptyset$	$\backslash\backslash$
	ν odd	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash$
p odd q even	ν even $\nu \leq 0$	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash - \mathbb{X}$
	ν even $0 < \nu < \rho$	$//$	$\emptyset$	$\mathbb{X}_+$	$\backslash\backslash - \mathbb{X}$
	ν even $\rho \leq \nu$	$//-\mathbb{X}_+$	$\emptyset$	$\mathbb{X}_+$	$\emptyset$
	ν odd	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash - \mathbb{X}$
p even q odd	ν even $\nu < \rho$	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash - \mathbb{X}$
	ν even $\rho \leq \nu$	$\emptyset$	$//$	$\emptyset$	$\emptyset$
p odd q odd	ν odd	$//$	$\emptyset$	$\emptyset$	$\backslash\backslash$
	ν even	$\emptyset$	$//$	$\emptyset$	$\backslash\backslash$

Table 1: Decomposition of $\mathbb{C}^2 - (//\cup\backslash\backslash)$ in (18)

same subset of (λ,ν) with that of the 0-th row above the box. For example, $//-\mathbb{X}_+$ in the sixth row in the left column means that if p is odd and q is even, then

$$(//-\mathbb{X}_+)\cap V = (//-(\mathbb{X}\cup|||))\cap V,$$

where $V := \{(\lambda,\nu)\in\mathbb{C}\times 2\mathbb{Z} : \rho\leq\nu\}$. Proposition 16 is a set-theoretic assertion, and is easy to be verified. For the sake of completeness, we provide a quick proof for Proposition 16 in Sections 4–7, depending on the parities of p and q. We shall give graphic description of Theorem 15 in Sections 4–7, from which we can easily derive Theorems 4–9 in Section 1. The proof of Theorem 15 will be given elsewhere.

4 Graphic description of the image of the regular symmetry breaking operators: Case p even and q even

In Sections 4–7, we give a graphic description of Theorem 15 according to the parities of p and q. In this section, we treat the case where both p and q are even.

4.1 Socle filtration of the target space $J(\nu)$

The image of the symmetry breaking operator $\mathbb{A}_{\lambda,\nu} : I(\lambda)\to J(\nu)$ is a G'-submodule of the (degenerate) principal series representation $J(\nu)$ of $G' = O(p,q+1)$. Since the G'-module $J(\nu)$ is of finite length, there are at most finitely many candidates for Image $(\mathbb{A}_{\lambda,\nu})$. In our setting, the structure of G'-submodules (*socle filtration*) of $J(\nu)$ is known for all $\nu \in \mathbb{C}$, see [2]. Although we do not need the results of [2] for the proof of Theorem 15, it is helpful to use the socle filtration of the G'-module $J(\nu)$ when we "visualize" Theorem 15.

Since $J(\nu)$ is K'-multiplicity free, any G'-submodule of $J(\nu)$ is characterized by its K'-types. By abuse of notation, we use the symbols $E_\nu^{\delta\varepsilon}$ $(\delta, \varepsilon = \pm)$, see Definition 13, to denote the G'-submodule of the principal series representation $J(\nu)$ of $G' = O(p,q+1)$ having K'-types parametrized by the subset $E_\nu^{\delta\varepsilon}$ of $\mathbb{N}^2_{\mathrm{even}}$. However, we keep the notation $\{0\}$ and $J(\nu)$ instead of $\emptyset$ and $\mathbb{N}^2_{\mathrm{even}}$, respectively.

We review from [2] the socle filtration of the principal series representations $J(\nu)$ of G' with p even and q even.

Fact 17 *Suppose p and q are even. Let $\rho = \frac{1}{2}(p+q-1)$ as in* (7).

(1) *The G'-module $J(\nu)$ is irreducible if and only if $\nu \in \mathbb{C} - \mathbb{Z}$.*
(2) *Suppose $\nu \in \mathbb{Z}$. Then G'-submodules of $J(\nu)$ are classified by their K'-types as follows.*

- *For ν even,*

$$\begin{aligned} &\{0\} \subsetneqq E_\nu^{++} \subsetneqq E_\nu^{-+} \subsetneqq J(\nu) && \text{if } \nu \leq 0, \\ &\{0\} \subsetneqq E_\nu^{-+} \subsetneqq J(\nu) && \text{if } \nu > 0. \end{aligned}$$

- *For ν odd,*

$$\begin{aligned} &\{0\} \subsetneqq E_\nu^{+-} \subsetneqq J(\nu) && \text{if } \nu < 2\rho, \\ &\{0\} \subsetneqq E_\nu^{+-} \subsetneqq E_\nu^{--} \subsetneqq J(\nu) && \text{if } \nu \geq 2\rho. \end{aligned}$$

The following lemma is readily seen from Definition 13 set theoretically, and fits well with Fact 17.

Lemma 18 *Suppose that p and q are even. For $\nu \in \mathbb{Z}$, the G'-modules with the K'-types $\bigcap_{\varepsilon \in \{\pm\}} E_\nu^{+\varepsilon}$ or $\bigcap_{\delta,\varepsilon \in \{\pm\}} E_\nu^{\delta\varepsilon}$ are given as follows. By abuse of notation, we identify G'-modules of $J(\nu)$ with their K'-types parametrized by subsets of $\mathbb{N}^2_{\mathrm{even}}$.*

		$E_\nu^{++} \cap E_\nu^{+-}$	$E_\nu^{++} \cap E_\nu^{+-} \cap E_\nu^{-+} \cap E_\nu^{--}$
ν even	$\nu \leq 0$	E_ν^{++}	E_ν^{++}
	$\nu > 0$	$J(\nu)$	E_ν^{-+}
ν odd		E_ν^{+-}	E_ν^{+-}

4.2 Reduction of the parameter set for (λ, ν)

In this section we discuss the parameter space $\mathbb{C}^2$ of (λ, ν). We recall the disjoint union (18) of $\mathbb{C}^2$ and analyze its main part, namely, the following disjoint decomposition.

$$//\cup\backslash\backslash = (//-(\mathbb{X}_+\cup |||))\sqcup(//\cap |||)\sqcup(\mathbb{X}_+ - |||)\sqcup(\backslash\backslash - \mathbb{X})\,. \tag{20}$$

Lemma 19 *For p and q both even, Proposition 16 holds.*

Proof. By (15), $\mathbb{X}\cap(\mathbb{C}\times\mathbb{Z})=\emptyset$ because $p+q$ is even. Hence, under the condition that $\nu\in\mathbb{Z}$, the four sets in the right-hand side of (20) amount to the sets in the second row of the table (21) below. Here we have used the same convention as in Proposition 16.

$//-(\mathbb{X}_+\cup \|\|\|)$	$//\cap \|\|\|$	$\mathbb{X}_+ - \|\|\|$	$\backslash\backslash - \mathbb{X}$
$//-\|\|\|$	$//\cap \|\|\|$	$\emptyset$	$\backslash\backslash$

(21)

Since q is even, the definition of $|||$ (see (14)) shows:

$(\lambda,\nu)\in|||$ if and only if $\nu\le 0$ when ν is even;
$(\lambda,\nu)\in|||$ for any odd integer ν.

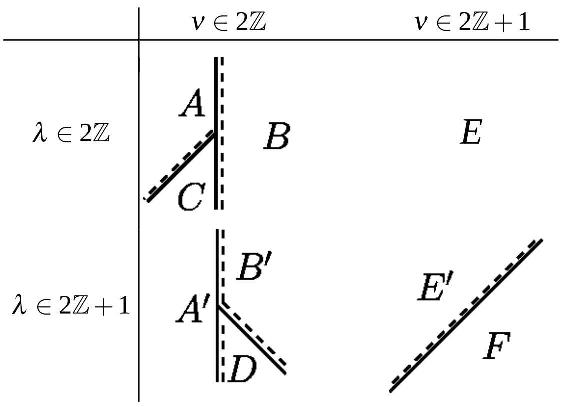
Therefore the first two sets $//-|||$ and $//\cap|||$ in the second row of (21) reduce to the sets in the following table according to conditions on $\nu\in\mathbb{Z}$.

		$//-\|\|\|$	$//\cap \|\|\|$
ν even	$\nu\le 0$	$\emptyset$	$//$
	$\nu>0$	$//$	$\emptyset$
ν odd		$\emptyset$	$//$

Thus Lemma 19 is proved. □

4.3 *Description of the image of symmetry breaking operators (p even, q even)*

For p and q both even, the critical cases are when $(\lambda,\nu)\in\mathbb{Z}^2$. We divide the parameter space $\mathbb{Z}^2$ into the following regions (see Theorem 20 below for the precise definition). Here, we follow the convention of Kobayashi–Speh [22] that ν is for the x-axis and λ is for the y-axis.



We are ready to describe graphically the image of the regular symmetry breaking operators for p and q both even as follows.

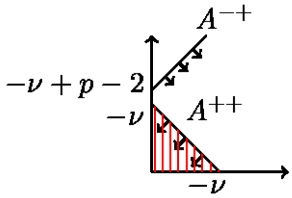
Theorem 20. *Let p and q be both even.*

(1) *Suppose $\nu \notin \mathbb{Z}$. Then the regular symmetry breaking operator $\mathbb{A}_{\lambda,\nu}\colon I(\lambda) \to J(\nu)$ is surjective for any $\lambda \in \mathbb{C}$.*

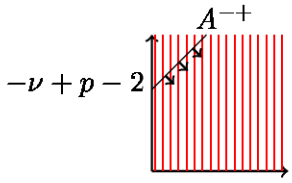
(2) *Suppose $\nu \in 2\mathbb{Z}$.*

(2-a) *For $\lambda \in 2\mathbb{Z}$, the K'-types of the image of $\mathbb{A}_{\lambda,\nu}$ are given by the subsets of $\mathbb{N}^2_{\mathrm{even}}$ in the following colored red regions via* (19).

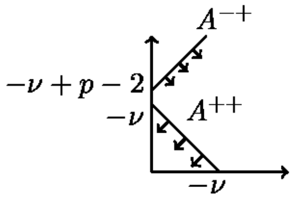
Case A:
$$\begin{cases} \nu \le 0, \\ \nu < \lambda. \end{cases}$$



Case B:
$0 < \nu.$



Case C:
$$\begin{cases} \nu \le 0, \\ \lambda \le \nu. \end{cases}$$

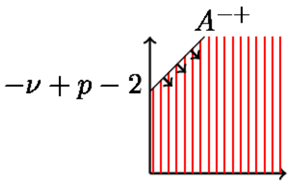


(2-b) *For $\lambda \in 2\mathbb{Z}+1$, we divide the parameter set $(\lambda,\nu) \in (2\mathbb{Z}+1)\times(2\mathbb{Z})$ into the following three cases.*

Case A$'$: $\nu \le 0$,
Case B$'$: $0 < \nu, \lambda+\nu > p+q-1$,
Case D: $0 < \nu, \lambda+\nu \le p+q-1$.

Then the image of $\mathbb{A}_{\lambda,\nu}$ in Cases A$'$ or B$'$ is described graphically by the same diagram with the one in Cases A or B, respectively, whereas the one in Case D is given as follows.

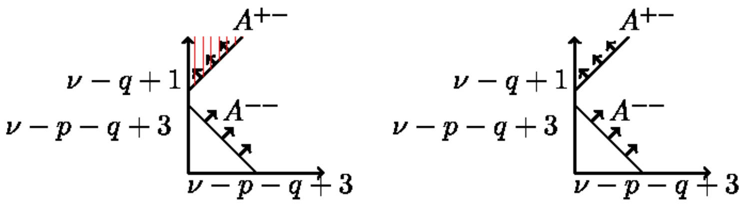
Case D:



(2-c) *For $\lambda \notin \mathbb{Z}$, we divide the parameter space $(\mathbb{C}-\mathbb{Z})\times 2\mathbb{Z}$ into two cases.*

Case A_{bis}: $\nu \leq 0$,
Case B_{bis}: $0 < \nu$.
Then the image of the regular symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ in Cases A_{bis} or B_{bis} is described graphically by the same diagram with the one in Cases A or B, respectively.

(3) *Suppose $\nu \in 2\mathbb{Z}+1$. We divide the parameter space $\mathbb{C} \times (2\mathbb{Z}+1)$ into the following four cases:*
Case E: $\lambda \in 2\mathbb{Z}$,
Case E′: $\lambda \in 2\mathbb{Z}+1$, $\nu < \lambda$,
Case F: $\lambda \in 2\mathbb{Z}+1$, $\nu \geq \lambda$,
Case E_{bis}: $\lambda \in \mathbb{C} - \mathbb{Z}$.
Then the image of $\mathbb{A}_{\lambda,\nu}$ is described graphically by the following diagram.



Cases E, E′ and E_{bis} Case F

Remark 21 *In each case, the arrangement of the barriers $A^{\pm\pm}$ may vary.*

Assuming Theorem 15, we complete the proof of Theorem 20.

Proof of Theorem 20. By Lemmas 18 and 19, Theorem 20 follows readily from Theorem 15. □

5 Graphic description of the image of the regular symmetry breaking operator: Case p odd (≥ 3) and q even

In this section we give a graphic description of Theorem 15 in the case where p is odd ($\geq$ 3) and q is even.

5.1 Socle filtration of the target space $J(\nu)$

In this section we give a graphic description of Theorem 15 in the case where p is odd ($\geq$ 3) and q is even. We review from [2] the socle filtration of the principal series representations $J(\nu)$ of $G' = O(p, q+1)$ with p odd (≥ 3) and q even as in Fact 17. We keep the notation from (7) that $\rho = \frac{1}{2}(p+q-1)$.

Fact 22 *Suppose p is odd (≥ 3) and q is even.*

(1) *The G'-module $J(\nu)$ is irreducible if and only if $\nu \in \mathbb{C} - \mathbb{Z}$ or ν is an even integer satisfying* $0 < \nu < 2\rho$.
(2) *For $\nu \in \mathbb{Z}$, G'-submodules of $J(\nu)$ are classified by their K'-types as follows:*

- *For ν even,*

$$\begin{aligned}
&\{0\} \subsetneqq E_\nu^{++} \subsetneqq J(\nu) && \textit{if } \nu \le 0,\\
&\{0\} \subsetneqq J(\nu) && \textit{if } 0 < \nu < 2\rho,\\
&\{0\} \subsetneqq E_\nu^{--} \subsetneqq J(\nu) && \textit{if } 2\rho \le \nu.
\end{aligned}$$

- *For ν odd,*

$$\begin{aligned}
&\{0\} \subsetneqq E_\nu^{+-} \cap E_\nu^{-+} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq J(\nu) && \textit{if } \nu < \rho,\\
&\{0\} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq J(\nu) && \textit{if } \nu = \rho,\\
&\{0\} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq E_\nu^{+-} \oplus E_\nu^{-+} \subsetneqq J(\nu) && \textit{if } \rho < \nu.
\end{aligned}$$

The following lemma formulated with the same convention as in Lemma 18 is readily seen from Definition 13 set theoretically, and fits well with Fact 22.

Lemma 23 *Suppose that p is odd (≥ 3) and q is even. We retain the notation that $\rho = \frac{1}{2}(p+q-1)$. For $\nu \in \mathbb{Z}$, the G'-modules with the K'-types $\bigcap_{\varepsilon \in \{\pm\}} E_\nu^{+\varepsilon}$ or $\bigcap_{\delta,\varepsilon \in \{\pm\}} E_\nu^{\delta\varepsilon}$ are given in the following table.*

		$E_\nu^{++} \cap E_\nu^{+-}$	$E_\nu^{++} \cap E_\nu^{+-} \cap E_\nu^{-+} \cap E_\nu^{--}$
ν even	$\nu \le 0$	E_ν^{++}	E_ν^{++}
	$0 < \nu < 2\rho$	$J(\nu)$	$J(\nu)$
	$2\rho \le \nu$	$J(\nu)$	E_ν^{--}
ν odd	$\nu < \rho$	E_ν^{+-}	$E_\nu^{+-} \cap E_\nu^{-+}$
	$\nu = \rho$	E_ν^{+-}	$\{0\}$
	$\rho < \nu$	E_ν^{+-}	$\{0\}$

5.2 Reduction of the parameter set for (λ, ν)

For $p+q$ odd, we use the following observation:

Lemma 24 *Suppose $p+q$ is odd.*

(1) *If $2\nu \ge p+q-1$, then $(\lambda,\nu) \in \backslash\backslash$ implies $(\lambda,\nu) \in //$.*
(2) *If $2\nu \le p+q-1$, then $(\lambda,\nu) \in //$ implies $(\lambda,\nu) \in \backslash\backslash$.*

Proof. Clear from the definition. $\square$

Then the four sets in the decomposition (20) have a simpler form as follows.

Lemma 25 *For p odd (≥ 3) and q even, Proposition 16 holds.*

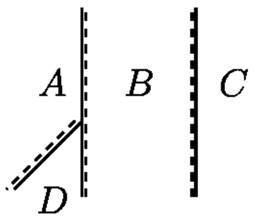
Proof. As in the proof of Lemma 19, we have

$(\lambda,\nu)\in|||$ if and only if $\nu\leq 0$ when ν is even;
$(\lambda,\nu)\in|||$ for any odd integer ν,

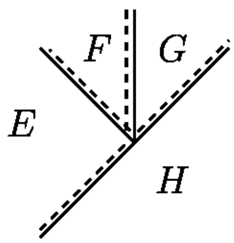
because q is even. We also note that $\mathbb{X}_+\cap\{(\lambda,\nu):\nu<\rho\}=\emptyset$ by definition. Now Lemma 25 is clear from Lemma 24. ◻

5.3 *Description of the image of symmetry breaking operators (p odd ≥ 3, q even)*

For p odd and q even, the critical cases are when $(\lambda,\nu)\in(2\mathbb{Z})^2$ or $(2\mathbb{Z}+1)^2$. In each case, we divide the parameter space into the following four regions (see Theorem 26 below for the precise definition). We remind again that ν is for the x-axis, and λ is for the y-axis, as in Section 4.3.



$(\lambda,\nu)\in(2\mathbb{Z})^2$



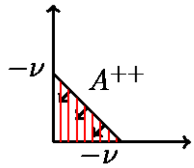
$(\lambda,\nu)\in(2\mathbb{Z}+1)^2$

We are ready to describe the image of the regular symmetry breaking operators for p odd (≥ 3) and q even.

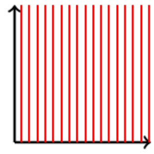
Theorem 26. *Let p be odd (≥ 3) and q be even.*

(1) *Suppose $\nu\notin\mathbb{Z}$. Then the regular symmetry breaking operator $\mathbb{A}_{\lambda,\nu}\colon I(\lambda)\to J(\nu)$ is surjective for any $\lambda\in\mathbb{C}$.*
(2) *Suppose $\nu\in 2\mathbb{Z}$. For $\lambda\in 2\mathbb{Z}$, the K'-types of the image of $\mathbb{A}_{\lambda,\nu}$ are given by the subsets of $\mathbb{N}^2_{\mathrm{even}}$ in the following colored red regions via* (19).

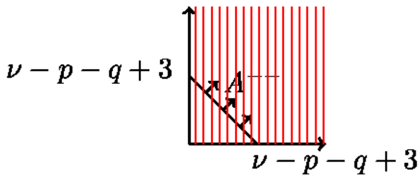
Case A:
$\begin{cases} \nu \le 0, \\ \nu < \lambda. \end{cases}$



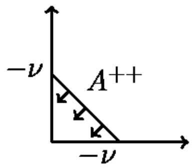
Case B:
$0 < \nu < p+q-1.$



Case C:
$p+q-1 \le \nu.$



Case D:
$\begin{cases} \nu \le 0, \\ \lambda \le \nu. \end{cases}$

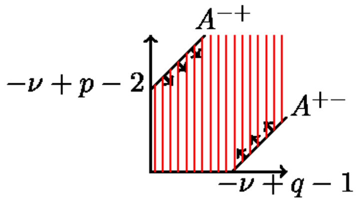


For $\lambda \notin 2\mathbb{Z}$, we divide of the parameter set $(\mathbb{C}-2\mathbb{Z}) \times 2\mathbb{Z}$ into three cases.
Case A_{bis}: $\nu \le 0$,
Case B_{bis}: $0 < \nu < \frac{1}{2}(p+q-1)$,
Case C_{bis}: $\frac{1}{2}(p+q-1) \le \nu$.
Then the image of the regular symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ in Case T_{bis} (T=A, B, or C) is described graphically by the same diagram with the one in Case T ($T = A$, B, or C, respectively).

(3) *Suppose $\nu \in 2\mathbb{Z}+1$. For $\lambda \in 2\mathbb{Z}+1$, the K'-types of the image of $\mathbb{A}_{\lambda,\nu}$ are described graphically as follows.*

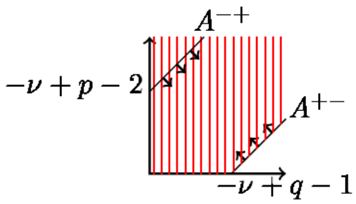
Case E:

$$\begin{cases} \lambda + \nu \leq p + q - 1, \\ \nu < \lambda. \end{cases}$$



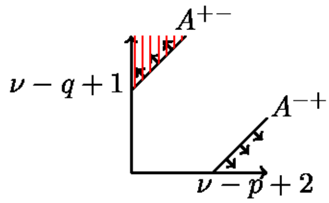
Case F:

$$\begin{cases} \nu < \frac{1}{2}(p + q - 1), \\ \lambda + \nu > p + q - 1. \end{cases}$$



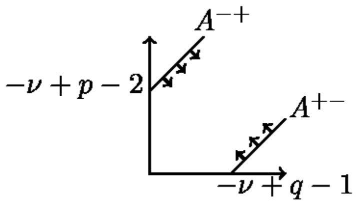
Case G:

$$\begin{cases} \frac{1}{2}(p + q - 1) \leq \nu, \\ \nu < \lambda. \end{cases}$$



Case H:

$\lambda \leq \nu$.



For $\lambda \notin 2\mathbb{Z} + 1$, *we divide the parameter set* $(\mathbb{C} - (2\mathbb{Z} + 1)) \times (2\mathbb{Z} + 1)$ *into the two cases.*

Case F_{bis}: $\nu < \frac{1}{2}(p + q - 1)$,

Case G_{bis}: $\frac{1}{2}(p + q - 1) \leq \nu$.

Then the image of the regular symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ *in Cases* F_{bis} *or* G_{bis} *is described graphically by the same diagram with the one in Cases F or G, respectively.*

Remark 27 *The arrangement of the barriers* A^{+-} *and* A^{-+} *may vary in Case H.*

Proof of Theorem 26. By Lemmas 23 and 25, Theorem 26 follows readily from Theorem 15. □

6 Graphic description of the image of regular symmetry breaking operators: Case p even and q odd

In this section we give a graphic description of Theorem 15 in the case where p is even and q is odd.

6.1 Socle filtration of the target space $J(\nu)$

We review from [2] the socle filtration of the (degenerate) principal series representations $J(\nu)$ of $G' = O(p, q+1)$ with p even and q odd as in Facts 17 and 22.

Fact 28 *Suppose p is even and q is odd. We recall $\rho = \frac{1}{2}(p+q-1)$.*

(1) *The G'-module $J(\nu)$ is irreducible if and only if $\nu \in \mathbb{C} - 2\mathbb{Z}$.*
(2) *For $\nu \in 2\mathbb{Z}$, G'-submodules of $J(\nu)$ are classified by their K'-types as follows:*

$$\begin{array}{ll}
\{0\} \subsetneqq E_\nu^{++} \subsetneqq E_\nu^{+-} \cap E_\nu^{-+} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq J(\nu) & \text{if } \nu \le 0,\\
\{0\} \subsetneqq E_\nu^{+-} \cap E_\nu^{-+} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq J(\nu) & \text{if } 0 < \nu < \rho,\\
\{0\} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq J(\nu) & \text{if } \nu = \rho,\\
\{0\} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq E_\nu^{+-} \oplus E_\nu^{-+} \subsetneqq J(\nu) & \text{if } \rho < \nu < 2\rho,\\
\{0\} \subsetneqq E_\nu^{+-}, E_\nu^{-+} \subsetneqq E_\nu^{+-} \oplus E_\nu^{-+} \subsetneqq E_\nu^{--} \subsetneqq J(\nu) & \text{if } 2\rho \le \nu.
\end{array}$$

The following lemma is readily seen from Definition 13 set theoretically, and fits well with Fact 28.

Lemma 29 *Suppose that p is even and q is odd. For $\nu \in 2\mathbb{Z}$, the G'-modules with the K'-types $\bigcap_{\varepsilon \in \{\pm\}} E_\nu^{+\varepsilon}$ or $\bigcap_{\delta,\varepsilon \in \{\pm\}} E_\nu^{\delta\varepsilon}$ are given in the following table. Here we identify, as before, G'-submodules of $J(\nu)$ with their K-types parametrized by subsets of $\mathbb{N}^2_{\text{even}}$.*

	$E_\nu^{++} \cap E_\nu^{+-}$	$E_\nu^{++} \cap E_\nu^{+-} \cap E_\nu^{-+} \cap E_\nu^{--}$
$\nu \le 0$	E_ν^{++}	E_ν^{++}
$0 < \nu < \rho$	E_ν^{+-}	$E_\nu^{+-} \cap E_\nu^{-+}$
$\rho \le \nu$	E_ν^{+-}	$\{0\}$

6.2 Reduction of the parameter set for (λ, ν)

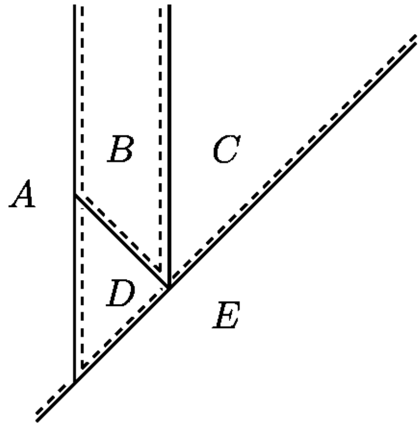
The four sets in the decomposition (20) have a simpler form as follows.

Lemma 30 *For p even and q odd, Proposition 16 holds.*

Proof. Since we are dealing with $\nu \in 2\mathbb{Z}$, (λ, ν) belongs automatically to $|||$ by definition (11) for every $\lambda \in \mathbb{C}$. Now Lemma 30 is clear from Lemma 24. $\square$

6.3 Description of the image of symmetry breaking operators (p even, q odd)

For p even and q odd, the critical case is when $(\lambda, \nu) \in (2\mathbb{Z})^2$. We divide the parameter space $(2\mathbb{Z})^2$ into the following five regions (see Theorem 31 below for the precise definition). We remind that ν is for the x-axis and λ is for the y-axis as in [22].



$(\lambda, \nu) \in (2\mathbb{Z})^2$

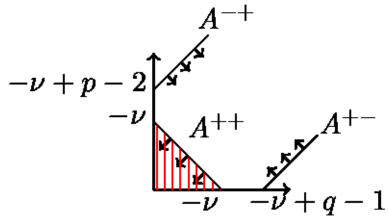
We are ready to describe the image of the regular symmetry breaking operators for p even and q odd.

Theorem 31. *Let p be even and q odd.*

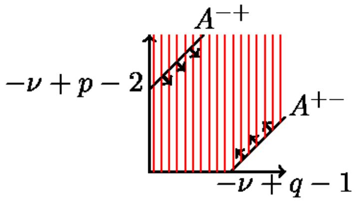
(1) *Suppose $\nu \notin 2\mathbb{Z}$. Then the symmetry breaking operator $\mathbb{A}_{\lambda,\nu} \colon I(\lambda) \to J(\nu)$ is surjective for any $\lambda \in \mathbb{C}$.*

(2) *Suppose* $(\lambda,\nu)\in(2\mathbb{Z})^2$. *Then the* K'*-types of the image of* $\mathbb{A}_{\lambda,\nu}$ *are given by the subsets of* $\mathbb{N}^2_{\mathrm{even}}$ *in the following colored regions via* (19).

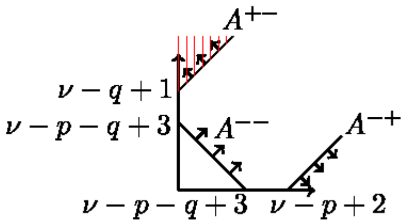
Case A:
$$\begin{cases}\nu\le 0,\\ \nu<\lambda.\end{cases}$$



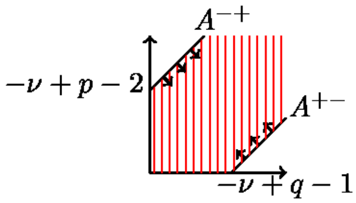
Case B:
$$\begin{cases}0<\nu<\frac{1}{2}(p+q-1),\\ \lambda+\nu>p+q-1.\end{cases}$$



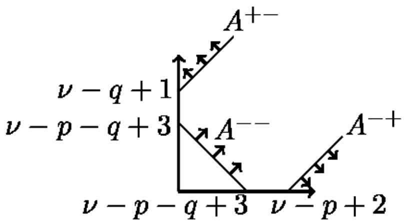
Case C:
$$\begin{cases}\frac{1}{2}(p+q-1)\le\nu,\\ \nu<\lambda.\end{cases}$$



Case D:
$$\begin{cases}0<\nu,\\ \nu<\lambda,\\ \lambda+\nu\le p+q-1.\end{cases}$$



Case E:
$\lambda\le\nu$.



(3) *Suppose* $\nu\in 2\mathbb{Z}$ *and* $\lambda\in\mathbb{C}-2\mathbb{Z}$.

Case A_{bis}: $\nu\le 0$;
Case B_{bis}: $0<\nu<\frac{1}{2}(p+q-1)$;
Case C_{bis}: $\frac{1}{2}(p+q-1)\le\nu$.

The image of the regular symmetry breaking operator for Case T_{bis} (T=A, B or C) is described graphically by the same diagram for Case T (T=A, B, or C, respectively).

Remark 32 *The arrangement of the barrier A^{--} in Case C and also that of the barriers A^{+-}, A^{--}, and A^{-+} may vary in case E according to the value of ν.*

Proof of Theorem 31. By Lemmas 29 and 30, Theorem 31 follows readily from Theorem 15. □

7 Graphic description of the image of regular symmetry breaking operators: Case p odd (≥ 3) and q odd

In this section we give a graphic description of Theorem 15 in the case where p is odd (≥ 3) and q is odd.

7.1 Socle filtration of the target space $J(\nu)$

We review from [2] the socle filtration of the principal series representations $J(\nu)$ of $G' = O(p,q+1)$ with p odd (≥ 3) and q odd as in Facts 17, 22, and 28.

Fact 33 *Suppose p is odd (≥ 3) and q is odd. We recall $\rho = \frac{1}{2}(p+q-1)$.*

(1) *The G'-module $J(\nu)$ is irreducible if and only if $\nu \in \mathbb{C} - \mathbb{Z}$.*
(2) *For $\nu \in \mathbb{Z}$, G'-submodules of $J(\nu)$ are classified by their K'-types as follows:*

- *For ν even,*

$$\begin{aligned} &\{0\} \subsetneqq E_\nu^{++} \subsetneqq E_\nu^{+-} \subsetneqq J(\nu) && \text{if } \nu \leq 0, \\ &\{0\} \subsetneqq E_\nu^{+-} \subsetneqq J(\nu) && \text{if } 0 < \nu. \end{aligned}$$

- *For ν odd,*

$$\begin{aligned} &\{0\} \subsetneqq E_\nu^{-+} \subsetneqq J(\nu) && \text{if } \nu < 2\rho, \\ &\{0\} \subsetneqq E_\nu^{-+} \subsetneqq E_\nu^{--} \subsetneqq J(\nu) && \text{if } 2\rho \leq \nu. \end{aligned}$$

7.2 Reduction of the parameter set for (λ, ν)

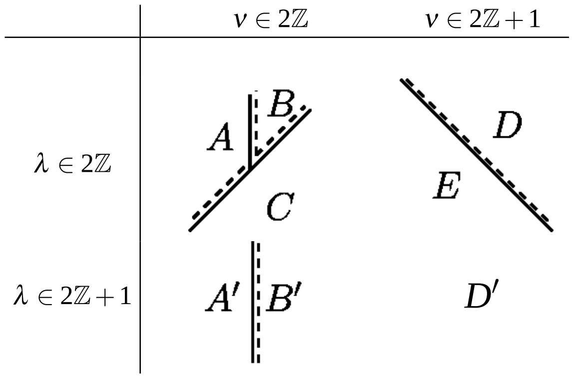
The four sets in the decomposition (20) have a simpler form as follows.

Lemma 34 *For p odd and q odd, Proposition 16 holds.*

Proof. By (15), $\mathbb{X} \cap (\mathbb{C} \times \mathbb{Z}) = \emptyset$ because $p+q$ is even. Hence, we have the same table with (21) in this case. Since q is odd, it follows from the definition (14) of $|||$ that $(\lambda, \nu) \in ||| \iff \nu \in 2\mathbb{Z}$. Now Lemma 34 is clear from (21). $\square$

7.3 Description of the image of the symmetry breaking operators $\mathbb{A}_{\lambda,\nu}$ (*p* odd ≥ 3, *q* odd)

For p and q both odd, the interesting cases are when $(\lambda, \nu) \in \mathbb{Z}^2$. We divide the parameter space $\mathbb{Z}^2$ into the following regions (see Theorem 35 below for the precise definition).



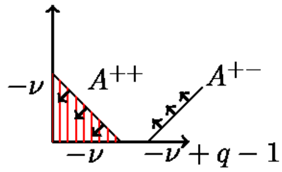
Theorem 35. *Let p be odd* (≥ 3) *and q odd.*

(1) *Suppose* $\nu \notin \mathbb{Z}$. *Then the regular symmetry breaking operator* $\mathbb{A}_{\lambda,\nu} \colon I(\lambda) \to J(\nu)$ *is surjective for any* $\lambda \in \mathbb{C}$.

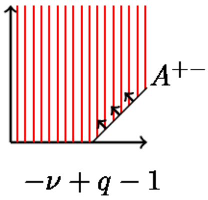
(2) *Suppose* $\nu \in 2\mathbb{Z}$.

(2-a) *For* $\lambda \in 2\mathbb{Z}$, *the* K'*-types of the image of* $\mathbb{A}_{\lambda,\nu}$ *are given by the subsets of* $\mathbb{N}^2_{\mathrm{even}}$ *in the following red regions via* (19).

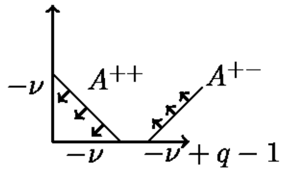
Case A:
$$\begin{cases} \nu \le 0, \\ \nu < \lambda. \end{cases}$$



Case B:
$$\begin{cases} 0 < \nu, \\ \nu < \lambda. \end{cases}$$



Case C:
$\nu \ge \lambda$.



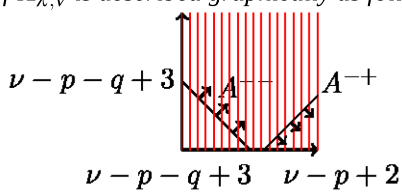
In Case C, the barrier A^{++} does not appear when $\nu > 0$.

(2-b) *For $\lambda \in (2\mathbb{Z}+1) \cup (\mathbb{C}-\mathbb{Z}) = \mathbb{C}-2\mathbb{Z}$, we use the following decomposition of the parameter set $(\lambda,\nu) \in (\mathbb{C}-2\mathbb{Z}) \times (2\mathbb{Z})$.*
Case $\mathrm{A}' \cup \mathrm{A}_{bis}$: $\nu \le 0$,
Case $\mathrm{B}' \cup \mathrm{B}_{bis}$: $0 < \nu$.
Then the image of $\mathbb{A}_{\lambda,\nu}$ in Case $T' \cup T_{bis}$ (T=A or B) is described graphically by the same diagram with the one in Case A or Case B, respectively.

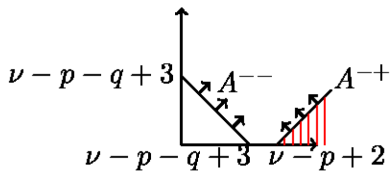
(3) *Suppose $\nu \in 2\mathbb{Z}+1$.*

(3-a) *For $\lambda \in 2\mathbb{Z}$, the image of $\mathbb{A}_{\lambda,\nu}$ is described graphically as follows.*

Case D:
$\lambda+\nu > p+q-1$.



Case E:
$\lambda+\nu \le p+q-1$.



In each case the barrier A^{--} does not appear when $\nu < p+q-1$.

(3-b) *The remaining case for $\nu \in 2\mathbb{Z}+1$ is the following.*
Case D': $\lambda \in 2\mathbb{Z}+1$.
Case D_{bis}: $\lambda \in \mathbb{C}-\mathbb{Z}$.

In both cases, the image of $\mathbb{A}_{\lambda,\nu}$ *is the same as in Case D, that is,* $\mathbb{A}_{\lambda,\nu}: I(\lambda) \to J(\nu)$ *is surjective. Again the barrier* A^{--} *does not appear when* $\nu < p+q-1$.

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