Geometry of coadjoint orbits and multiplicity-one branching laws for symmetric pairs

Dedicated to Alexandre Kirillov on the occasion of his 81st birthday with admiration

Toshiyuki Kobayashi
Graduate School of Mathematical Sciences, and Kavli IPMU,
and Salma Nasrin
Department of Mathematics, University of Dhaka

Abstract

Consider the restriction of an irreducible unitary representation π of a Lie group G to its subgroup H. Kirillov's revolutionary idea on the orbit method suggests that the multiplicity of an irreducible H-module ν occurring in the restriction $\pi|_H$ could be read from the coadjoint action of H on $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$, provided π and ν are 'geometric quantizations' of a G-coadjoint orbit $\mathcal{O}^G$ and an H-coadjoint orbit $\mathcal{O}^H$, respectively, where $\mathrm{pr}\colon \sqrt{-1}\mathfrak{g}^* \to \sqrt{-1}\mathfrak{h}^*$ is the projection dual to the inclusion $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras. Such results were previously established by Kirillov, Corwin and Greenleaf for nilpotent Lie groups.

In this article, we highlight specific elliptic orbits $\mathcal{O}^G$ of a semisimple Lie group G corresponding to highest weight modules of scalar type. We prove that the Corwin–Greenleaf number $\sharp(\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H))/H$ is either zero or one for any H-coadjoint orbit $\mathcal{O}^H$, whenever (G, H) is a symmetric pair of holomorphic type. Furthermore, we determine the coadjoint orbits $\mathcal{O}^H$ with nonzero Corwin–Greenleaf number. Our results coincide with the prediction of the orbit philosophy, and can be seen as 'classical limits' of the *multiplicity-free* branching laws of holomorphic discrete series representations (T. Kobayashi [Progr. Math. 2007]).

Mathematics Subject Classifications (2010): Primary 22E46; Secondary 22E60, 32M15, 53C35, 81S10.

Keywords and phrases: orbit method, Corwin–Greenleaf multiplicity function, multiplicity-free representations, highest weight representations, bounded symmetric domains, branching law, coadjoint orbit, geometric quantization.

1 Introduction

The Kirillov–Kostant–Duflo orbit philosophy bridges the unitary dual $\widehat{G}$ of a Lie group G and the set $\sqrt{-1}\mathfrak{g}^*/G$ of coadjoint orbits. The orbit method works perfectly for simply connected nilpotent Lie groups and certain solvable groups G, including the one-to-one correspondence (*Kirillov correspondence*) between $\widehat{G}$ and $\sqrt{-1}\mathfrak{g}^*/G$ and the functorial properties for inductions and restrictions (see [1, 5, 8, 9, 10] and the references therein).

Our interest is in the restriction of representations to subgroups and its counterpart in the geometry of coadjoint orbits. First, we consider the representation-theoretic side. Let π be an irreducible unitary representation of a Lie group G, and H a subgroup of G. Then the restriction $\pi|_H$ is decomposed into the direct integral of irreducible unitary representations of H:

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\nu)\nu d\mu(\nu). \tag{1.1}$$

Here μ is a Borel measure on the unitary dual $\widehat{H}$, and the measurable function $m_\pi\colon \widehat{H} \to \mathbb{N} \cup \{\infty\}$ stands for the multiplicity. The decomposition (1.1) is called *branching law* of the restriction $\pi|_H$. The irreducible decomposition (1.1) is unique up to equivalence if H is of type I, *e.g.*, if H is nilpotent or reductive. We denote by $\mathrm{Supp}_H(\pi|_H)$ the subset of $\widehat{H}$ that is the support of the direct integral (1.1).

Next, we consider the coadjoint orbit side. The *Corwin–Greenleaf multiplicity function*

$$n\colon (\sqrt{-1}\mathfrak{g}^*/G) \times (\sqrt{-1}\mathfrak{h}^*/H) \to \mathbb{N} \cup \{\infty\},$$

counts the number of H-orbits in the intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$, namely,

$$n(\mathcal{O}^G, \mathcal{O}^H) := \sharp\Big(\big(\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)\big)/H\Big), \tag{1.2}$$

where $\mathrm{pr}\colon \sqrt{-1}\mathfrak{g}^* \to \sqrt{-1}\mathfrak{h}^*$ is the natural projection. When G is a simply connected nilpotent Lie group and H is a connected subgroup, Corwin and Greenleaf [1] proved that the multiplicity $m_\pi(\nu)$ coincides with the geometric number $n(\mathcal{O}^G_\pi, \mathcal{O}^H_\nu)$ almost everywhere if $\mathcal{O}^G_\pi \subset \sqrt{-1}\mathfrak{g}^*$ and $\mathcal{O}^H_\nu \subset \sqrt{-1}\mathfrak{h}^*$ are the coadjoint orbits corresponding to $\pi \in \widehat{G}$ and $\nu \in \widehat{H}$, respectively, under the Kirillov correspondence. Thus the result [1] is summarized as

representation theory		geometry of coadjoint orbits
$m_\pi(\nu)$	$=$	$n(\mathcal{O}^G_\pi, \mathcal{O}^H_\nu)$.

In contrast to nilpotent groups, it has been observed by many specialists that the orbit philosophy does not work very well for noncompact semisimple Lie groups G, see *e.g.*, [9, 10, 14]. Indeed, there does not exist a reasonable one-to-one correspondence between $\widehat{G}$ and $\sqrt{-1}\mathfrak{g}^*/G$: 'missing' of coadjoint orbits corresponding to complementary series representations (*cf*. [6, Thm. 2.30]), missing of some 'unipotent representations' that are supposed to be attached to nilpotent coadjoint orbits, and failure of irreducibility or vanishing of nontempered Vogan–Zuckerman $A_\mathfrak{q}(\lambda)$-modules that are supposed to be attached to elliptic coadjoint orbits even for 'positive' λ ([11, 32]) among others, and consequently a rigorous formulation for 'functional properties' in the orbit method is not obvious. Nevertheless, we still expect that Kirillov's orbit philosophy provides useful information and new insights on unitary representation theory and the geometry of coadjoint orbits. In fact, some successful cases about the functorial properties of the orbit method for discretely decomposable restrictions to noncompact reductive subgroups H include Kobayashi–Ørsted [22] for minimal representations attached to minimal coadjoint orbits, Duflo–Vargas [3] for discrete series representations attached to strongly elliptic orbits, and a recent work by Paradan [29] for holomorphic discrete series representations.

In this article, we consider the case where (G, H) is a symmetric pair of holomorphic type (see Definition 2.2 below). A typical example is $(G, H) = (\mathrm{Sp}(n, \mathbb{R}), \mathrm{U}(p, q))$ and $(\mathrm{Sp}(n, \mathbb{R}), \mathrm{Sp}(p, \mathbb{R}) \times \mathrm{Sp}(q, \mathbb{R}))$ with $p+q = n$. We highlight on specific coadjoint orbits $\mathcal{O}^G$ (see (2.1)), and find explicitly the Corwin–Greenleaf function for an arbitrary coadjoint orbit $\mathcal{O}^H$ for any symmetric pair (G, H) of holomorphic type.

Our main results are Theorems A and C which are predicted by the orbit philosophy as the 'classical limit' of multiplicity-free discretely decomposable restrictions of unitary representations that were established earlier (see [13, 17]). Our results can be interpreted also from the viewpoint of symplectic geometry, namely, the momentum map $\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$ for the Hamiltonian action of the subgroup H on $\mathcal{O}^G$ endowed with the Kirillov–Kostant–Souriau symplectic form is a proper map (Corollary 2.5) with explicit description of its image (Theorem D) indicating that the geometric quantization $\mathcal{Q}$ commutes with reduction in this setting, symbolically written as

$$\mathcal{Q}^H \circ \mu = \text{ Restriction} \circ \mathcal{Q}^G.$$

Thus the main features are summarized as follows.

coadjoint orbits		unitary representations	
$\mathcal{O}^G$		π^G	
$\mathcal{O}^G \cap \sqrt{-1}([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp \neq \{0\}$		holomorphic rep. of scalar type	
$n(\mathcal{O}^G, \mathcal{O}^H) \leq 1 \quad (\forall \mathcal{O}^H)$	Thm. A	$\pi^G\|_H$ is multiplicity-free	[13]
$\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$ is proper	Cor. 2.5	$\pi^G\|_H$ is discretely decomposable	[15]
$\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H_\nu) \neq \emptyset$	Thm. C	$\mathrm{Hom}_H(\pi^H_\nu, \pi^G\|_H) \neq \{0\}$	[13]
$\mathrm{Image}(\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*)$	Thm. D	$\mathrm{Supp}_H(\pi^G\|_H)\,(\subset \widehat{H})$	[13]

Theorems A, C and D for $H = K$ (maximal compact subgroups) were proved in the Ph. D. thesis [25] of S. Nasrin at The University of Tokyo in 2003, see also [26, 27]. Alternatively, Theorems A and C for $H = K$ follow from a result of McDuff [24], extended by Deltour [2] that the coadjoint orbit $\mathcal{O}^G$ is symplectomorphic to the vector space $\mathfrak{p}$ and from Paradan [28, Prop. 5.5], too.

The results of this article for noncompact H were delivered at the workshop "Geometric Quantization in the Non-compact Setting" organized by L. Jeffrey, X. Ma and M. Vergne at Oberwolfach, Germany, 13–19 February 2011, and were collected in [19]. Theorem A was announced earlier in [21].

2 Statement of main results

In this section we formulate our main results on the geometry of coadjoint orbits that are predicted by the representation-theoretic results via the orbit method. Theorem A is the counterpart of the multiplicity-freeness property of the restriction $\pi|_H$ (Fact 2.1), and Theorem C is that of its explicit branching law (Fact 2.6).

2.1 Orbit geometry for multiplicity-free representations

Let G be a noncompact, simple Lie group, K a maximal compact subgroup modulo the center of G, θ the corresponding Cartan involution of G, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. We say G is a *Hermitian Lie group* if the associated Riemannian symmetric space G/K is a Hermitian symmetric space, or equivalently, if the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is nonzero. In this case $\mathfrak{c}(\mathfrak{k})$ is actually one-dimensional.

An irreducible representation π of G is said to be a lowest weight module if its underlying $(\mathfrak{g}, K)$-module is $\mathfrak{b}$-finite for some Borel subalgebra $\mathfrak{b}$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Moreover it is said to be *of scalar* type if the minimal K-type of π is one-dimensional. There exists an infinite-dimensional irreducible lowest weight representation of a simple Lie group G if and only if G is a Hermitian Lie group. For any simply-connected Hermitian group G, there exist continuously many lowest weight modules of G of scalar type.

Geometrically, any irreducible lowest weight representation π of scalar type can be realized in the space of holomorphic sections for a G-equivariant holomorphic line bundle over the Hermitian symmetric space G/K. This geometric observation is brought into the orbit method: such π can be seen as a 'geometric quantization' of a coadjoint orbit $\mathcal{O}^G$ satisfying

$$\mathcal{O}^G \cap \sqrt{-1}([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^{\perp} \neq \emptyset. \tag{2.1}$$

We note that $([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^{\perp} \neq \{0\}$ if and only if G is of Hermitian type. In this case the coadjoint orbit $\mathcal{O}^G$ satisfying (2.1) is isomorphic to the Hermitian symmetric space G/K as G-spaces unless $\mathcal{O}^G = \{0\}$.

Let τ be an involutive automorphism of G. We say that (G, H) is a *symmetric pair* if H is an open subgroup of the fixed point group $G^{\tau} := \{g \in G : \tau g = g\}$. In this article, we shall assume H is connected for simplicity.

For the representation theory side, we recall the following multiplicity-free theorem:

Fact 2.1 ([17]). *For any irreducible unitary lowest weight representation π of scalar type of G and for any symmetric pair (G, H), the restriction $\pi|_H$ is multiplicity-free.*

See [13, Thm. A] for the proof based on the theory of 'visible actions' on complex manifolds.

Suppose τ is an involutive automorphism of Hermitian Lie group G such that $\tau\theta = \theta\tau$. Then, τ stabilizes the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$. Since $\dim \mathfrak{c}(\mathfrak{k}) = 1$ for a Hermitian Lie group G, $\tau|_{\mathfrak{c}(\mathfrak{k})}$ is either id or $-$id. On the other hand, since $\tau(K) = K$, τ also acts on G/K as a diffeomorphism. This action is either holomorphic or anti-holomorphic according to $\tau|_{\mathfrak{c}(\mathfrak{k})} = \mathrm{id}$ or $-\mathrm{id}$.

Definition 2.2. The involution τ (or the corresponding symmetric pair (G, H)) is said to be *holomorphic* or *anti-holomorphic*, if $\tau|_{\mathfrak{c}(\mathfrak{k})} = \mathrm{id}$ or $-\mathrm{id}$, respectively.

The Cartan involution θ is always of holomorphic type.

Example 2.3. Let $G = \mathrm{Sp}(n, \mathbb{R})$. Then the pair (G, H) is of holomorphic type if $H = \mathrm{U}(p, q)$ or $\mathrm{Sp}(p, \mathbb{R}) \times \mathrm{Sp}(q, \mathbb{R}) \quad (p + q = n)$, whereas it is of anti-holomorphic type if $H = \mathrm{GL}(n, \mathbb{R})$). See [17, Tables 3.4.1 and 3.4.2] for the list of all the irreducible symmetric pairs (G, H) on the Lie algebra level that are of holomorphic and anti-holomorphic types.

In this article, we shall treat the case where τ is of holomorphic type. This implies that the branching law (1.1) of the restriction $\pi|_H$ does not contain any continuous spectrum, and is discretely decomposable for any lowest weight module π ([12, 15]).

The first main result of this article is to give the counterpart of Fact 2.1 in terms of the Corwin–Greenleaf function (1.2) in the geometry of coadjoint orbits.

Theorem A. *Let G be a Hermitian Lie group, (G, H) a symmetric pair of holomorphic type, and $\mathcal{O}^G$ a coadjoint orbit in $\sqrt{-1}\mathfrak{g}^*$ of G satisfying* (2.1). *Then*

$$n(\mathcal{O}^G, \mathcal{O}^H) \leq 1$$

for any coadjoint orbit $\mathcal{O}^H$ in $\sqrt{-1}\mathfrak{h}^*$.

It should be noted that the Corwin–Greenleaf function $n(\mathcal{O}^G, \mathcal{O}^H)$ may be infinite in general even for a symmetric pair (G, H) if we drop the assumption (2.1) (see [21] for such an example).

Since H is connected, Theorem A implies the following topological result:

Corollary B. *The intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ is connected for any coadjoint orbit $\mathcal{O}^G$ in $\sqrt{-1}\mathfrak{g}^*$ satisfying* (2.1) *and for any coadjoint orbit $\mathcal{O}^H$ in $\sqrt{-1}\mathfrak{h}^*$.*

In the special case $H = K$, the connectedness of the intersection $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ could be derived also from a general theorem concerning Hamiltonian actions of connected compact group on symplectic manifolds with proper moment maps (see [29, 31]). On the other hand, $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$ can be disconnected when the momentum map $\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$ is not proper (see [21, Fig. 4.6] for such an example).

2.2 Nonvanishing condition for the Corwin–Greenleaf function

The second main result of this article is a necessary and sufficient condition for the Corwin–Greenleaf function $n(\mathcal{O}^G, \mathcal{O}^H)$ to be nonzero. In order to establish it, we need to fix a parametrization of $\mathcal{O}^G$ and $\mathcal{O}^H$.

Suppose G is a simple Lie group of Hermitian type. Then the center $\mathfrak{c}(\mathfrak{k})$ of $\mathfrak{k}$ is one-dimensional, and there exists a characteristic element $Z \in \sqrt{-1}\mathfrak{c}(\mathfrak{k})$ such that

$$\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_- \tag{2.2}$$

is the direct sum decomposition of the eigenspaces of $\mathrm{ad}(Z)$ with eigenvalues 0, $+1$ and -1, respectively. Then G/K carries a G-invariant complex structure with holomorphic tangent bundle $G \times_K \mathfrak{p}_+ \to G/K$.

Suppose τ is an involutive automorphism of G commuting with the Cartan involution θ. We use the same letters τ, θ to denote the complex linear extensions of their differentials. We take a maximal abelian subspace of $\mathfrak{k}^\tau = \mathfrak{h} \cap \mathfrak{k}$, and extend it to a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{k}$. Then $\mathfrak{t}^\tau = \mathfrak{h} \cap \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}^\tau$. Let $\Delta(\mathfrak{k}, \mathfrak{t})$ ($\subset \sqrt{-1}\mathfrak{t}^*$) and $\Delta(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$ ($\subset \sqrt{-1}(\mathfrak{t}^\tau)^*$) be the root systems of the pair $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $(\mathfrak{k}^\tau_{\mathbb{C}}, \mathfrak{t}^\tau_{\mathbb{C}})$, respectively. Then we can choose compatible positive systems $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$ in the sense that

$$\alpha|_{\mathfrak{t}^\tau} \in \Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau) \quad \text{for any } \alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t}). \tag{2.3}$$

We write $\sqrt{-1}(\mathfrak{t}^*)_+$ for the dominant Weyl chamber with respect to the positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and $\sqrt{-1}(\mathfrak{t}^\tau)^*_+$ to $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau)$.

Hereafter we assume that τ is of holomorphic type (Definition 2.2). Since $\tau Z = Z$, the direct sum decomposition (2.2) is stable under τ. Thus we have a direct sum decomposition

$$\mathfrak{p}_+ = \mathfrak{p}_+^\tau + \mathfrak{p}_+^{-\tau},$$

where we set

$$\mathfrak{p}_+^{\pm\tau} := \{X \in \mathfrak{p}_+ : \tau X = \pm X\}.$$

For a $\mathfrak{t}^\tau$-stable subspace F in $\mathfrak{p}_+$, let $\Delta(F)$ denote the set of weights of F with respect to $\mathfrak{t}^\tau$. It is a finite set in $\sqrt{-1}(\mathfrak{t}^\tau)^*$.

The subgroup $G^{\tau\theta}$ is locally isomorphic to the direct product of a compact normal subgroup $G^{(0)}$ and noncompact simple Lie subgroups $G^{(i)}$ $(1 \le i \le L)$. Correspondingly, the Lie algebra $\mathfrak{g}^{\tau\theta}$ is decomposed into the direct sum:

$$\mathfrak{g}^{\tau\theta} \simeq \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(L)}. \tag{2.4}$$

Remark 2.4. By the classification [17, Table 3.4.1], we see

$$L = 1 \quad \text{or} \quad 2.$$

For example, when $G = \mathrm{Sp}(p+q, \mathbb{R})$, $L = 1$ if $H = \mathrm{U}(p, q)$ and $L = 2$ if $H = \mathrm{Sp}(p, \mathbb{R}) \times \mathrm{Sp}(q, \mathbb{R})$.

Two roots α and β are called *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. For each i $(1 \le i \le L)$, we denote by r_i (≥ 1) the real rank of $\mathfrak{g}^{(i)}$, and take a maximal set of strongly orthogonal roots $\{\nu_1^{(i)}, \cdots, \nu_{r_i}^{(i)}\}$ in $\Delta(\mathfrak{p}_+ \cap \mathfrak{g}^{(i)})$ $(\subset \sqrt{-1}(\mathfrak{t}^\tau)^*)$ such that

1) $\nu_1^{(i)}$ is the highest in $\Delta(\mathfrak{p}_+ \cap \mathfrak{g}^{(i)})$.
2) $\nu_k^{(i)}$ is the highest in the set of all ν in $\Delta(\mathfrak{p}_+ \cap \mathfrak{g}^{(i)})$ such that ν is strongly orthogonal to $\nu_1^{(i)}, \cdots, \nu_{k-1}^{(i)}$ $(2 \le k \le r_i)$.

We note that the split rank of the semisimple symmetric space G/H equals the real rank of $G^{\tau\theta}$, which is given by $r := r_1 + \cdots + r_L$.

We set, for $1 \le i \le L$,

$$\begin{aligned} C_+^{(i)} &:= \{(t_j^{(i)})_{1 \le j \le r_i} \in \mathbb{R}^{r_i} : t_1^{(i)} \ge \cdots \ge t_{r_i}^{(i)} \ge 0\}, \\ \Lambda^{(i)} &:= C_+^{(i)} \cap \mathbb{Z}^{r_i}, \end{aligned}$$

and define a closed convex cone in $\sqrt{-1}(\mathfrak{t}^\tau)^*$ by

$$\mathrm{Cone}(\mathfrak{p}_+^{-\tau}) := \{\sum_{i=1}^{L} \sum_{j=1}^{r_i} t_j^{(i)} \nu_j^{(i)} : (t_j^{(i)})_{1 \le j \le r_i} \in C_+^{(i)} \quad \text{for all } i \ (1 \le i \le L)\}. \tag{2.5}$$

By using the Killing form, we identify $\sqrt{-1}\mathfrak{g}$ with $\sqrt{-1}\mathfrak{g}^*$, and regard

$$\sqrt{-1}\mathfrak{c}(\mathfrak{k})^* \subset \sqrt{-1}(\mathfrak{t}^\tau)^* \subset \sqrt{-1}\mathfrak{t}^* \subset \sqrt{-1}\mathfrak{k}^* \subset \sqrt{-1}\mathfrak{g}^*$$

corresponding to the inclusion $\mathfrak{c}(\mathfrak{k}) \subset \mathfrak{t}^\tau \subset \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$. Via the identification $\sqrt{-1}\mathfrak{g}^* \simeq \sqrt{-1}\mathfrak{g}$, the condition (2.1) for a coadjoint orbit $\mathcal{O}^G$ is equivalent to the condition

$$\mathcal{O}^G \cap \sqrt{-1}\mathfrak{c}(\mathfrak{k}) \neq \emptyset$$

for an adjoint orbit $\mathcal{O}^G$ (by abuse of notation) because $\sqrt{-1}([\mathfrak{k},\mathfrak{k}]+\mathfrak{p})^\perp = \sqrt{-1}\mathfrak{c}(\mathfrak{k})^*$.

A coadjoint orbit $\mathcal{O}^G$ is said to be an *elliptic orbit* if $\mathcal{O}^G \cap \sqrt{-1}\mathfrak{k}^* \neq \emptyset$. In particular, $\mathcal{O}^G$ is elliptic if (2.1) is satisfied. If $\mathcal{O}^G$ is an elliptic coadjoint orbit, then $\mathcal{O}^G$ meets at a single point, say μ, in the dominant Weyl chamber $\sqrt{-1}(\mathfrak{t}^*)_+$ with respect to $\Delta^+(\mathfrak{k}, \mathfrak{t})$. We shall write $\mathcal{O}^G_\mu$ for $\mathcal{O}^G$ if $\mathcal{O}^G \cap \sqrt{-1}(\mathfrak{t}^*)_+ = \{\mu\}$. Likewise, an elliptic coadjoint orbit $\mathcal{O}^H$ is written as $\mathcal{O}^H_\mu = \mathrm{Ad}^*(H)\mu$ for some dominant element $\mu \in \sqrt{-1}(\mathfrak{t}^\tau)^*_+$.

The coadjoint orbit $\mathcal{O}^G$ satisfying (2.1) is a special case of elliptic orbits. In this case, $\mathcal{O}^G$ is of the form $\mathcal{O}^G_\lambda$ for some $\lambda \in \sqrt{-1}\mathfrak{c}(\mathfrak{k})^*$. If $\lambda \neq 0$ then we have either

$$\langle \lambda, \beta \rangle > 0 \text{ for any } \beta \in \Delta(\mathfrak{p}_+) \tag{2.6}$$

or

$$\langle \lambda, \beta \rangle < 0 \text{ for any } \beta \in \Delta(\mathfrak{p}_+).$$

Without loss of generality, we may and do assume that the condition (2.6) is satisfied.

We shall see in Proposition 3.4 below that if $\mathcal{O}^G$ satisfies (2.1) and $\mathcal{O}^H$ is a coadjoint orbit in $\sqrt{-1}\mathfrak{h}^*$ such that $n(\mathcal{O}^G_\lambda, \mathcal{O}^H) \neq 0$ then $\mathcal{O}^H$ must be an elliptic orbit, equivalently, $\mathcal{O}^H$ is of the form $\mathcal{O}^H_\mu = \mathrm{Ad}^*(H)\mu$ for some $\mu \in \sqrt{-1}(\mathfrak{t}^\tau)^*_+$. Then we determine elliptic coadjoint orbits $\mathcal{O}^H$ with $n(\mathcal{O}^G_\lambda, \mathcal{O}^H) \neq 0$ as follows:

Theorem C. *Let G be a Hermitian Lie group, and (G, H) a symmetric pair of holomorphic type. Suppose $\mathcal{O}^G_\lambda = \mathrm{Ad}^*(G)\lambda$ with λ satisfying* (2.6)*. Then the following three conditions on $\mu \in \sqrt{-1}(\mathfrak{t}^\tau)^*_+$ are equivalent:*

(i) $n(\mathcal{O}^G_\lambda, \mathcal{O}^H_\mu) \neq 0$;

(ii) $n(\mathcal{O}^G_\lambda, \mathcal{O}^H_\mu) = 1$;

(iii) $\mu \in \lambda + \mathrm{Cone}(\mathfrak{p}_+^{-\tau})$.

The restriction of the projection $\mathrm{pr}\colon \sqrt{-1}\mathfrak{g}^* \to \sqrt{-1}\mathfrak{h}^*$ to a coadjoint orbit $\mathcal{O}^G$ is identified with the momentum map $\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$ for the Hamiltonian action on the symplectic manifold $\mathcal{O}^G$. Then the following corollary is deduced readily from Theorem C.

Corollary 2.5. *Let G be a Hermitian Lie group, and (G, H) a symmetric pair of holomorphic type. Suppose $\mathcal{O}^G$ is a coadjoint orbit satisfying* (2.6)*. Then the momentum map*

$$\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$$

is proper.

The representation-theoretic counterpart for Theorem C is branching laws of scalar holomorphic discrete series representations $\pi^G(\lambda)$ with respect to symmetric pairs (G, H) of holomorphic type, and that for Corollary 2.5 is discrete decomposability of the restriction $\pi^G(\lambda)|_H$. To describe the branching law explicitly, we fix some notation. A holomorphic discrete series representation of G is parametrized by its minimal K-type. We denote by $\pi^G(\lambda)$ if its minimal K-type has highest weight λ with respect to $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, see also [17, Sect. 8.1] for the convention when G is not simple. Similar notation is applied to holomorphic discrete series representations $\pi^H(\mu)$ of H.

Fact 2.6 (Hua–Kostant–Schimd–Kobayashi, [13])**.** *Suppose (G, H) is a symmetric pair of holomorphic type. Assume $\lambda \in \sqrt{-1}\mathfrak{c}(\mathfrak{k})^*$ satisfies* (2.6)*. Then the restriction of $\pi^G(\lambda)$ to H is decomposed into a multiplicity-free direct sum of irreducible representation of H:*

$$\pi^G(\lambda)|_H \simeq \sum\nolimits^{\oplus} \pi^H(\lambda|_{\mathfrak{t}^\tau} + \sum_{i=1}^{L} \sum_{j=1}^{r_i} a_j^{(i)} \nu_j^{(i)}), \tag{2.7}$$

where the sum is taken over the following countable set:

$$(a_j^{(i)})_{1 \le j \le r_i} \in \Lambda^{(i)} \quad (1 \le i \le L). \tag{2.8}$$

When H is a maximal compact subgroup K, each summand in (2.7) is finite-dimensional, and the formula (2.7) was known by Hua [7] (classical groups), Kostant (unpublished), and Schmid [30]. The general case for noncompact H was given in Kobayashi [13] with detailed proof in [17, Thm. 8.3]. See also [20, Cor. 3.12] for a formulation in the category $\mathcal{O}$.

In comparison to Fact 2.6, Theorem C may be restated as follows:

Theorem D. *Suppose λ satisfies* (2.6)*. Then $n(\mathcal{O}_\lambda^G, \mathcal{O}_\mu^H) \neq 0$ if and only if*

$$\mu \in \operatorname{Conv}\bigl(\operatorname{Supp}_H\bigl(\pi^G(\lambda)|_H\bigr)\bigr),$$

where $\operatorname{Conv}(S)$ *denotes the convex hull of a set S, and $\mathcal{O}_\lambda^G := \operatorname{Ad}^*(G) \cdot \lambda$ and $\mathcal{O}_\mu^H :=$ $\operatorname{Ad}^*(H) \cdot \mu$.*

In Theorem D, we have regarded $\operatorname{Supp}_H(\pi^G(\lambda)|_H)$ as a subset of dominant integral weights with respect to the positive system $\Delta^+(\mathfrak{k}^\tau, \mathfrak{t}^\tau) = \Delta^+(\mathfrak{h} \cap \mathfrak{k}, \mathfrak{t}^\tau)$, namely,

$$\operatorname{Supp}_H(\pi^G(\lambda)|_H) = \bigcup \{\lambda|_{\mathfrak{t}^\tau} + \sum_{i=1}^{L} \sum_{j=1}^{r_i} a_j^{(i)} \nu_j^{(i)}\}, \tag{2.9}$$

where the union is taken over the countable set (2.8).

3 Proof of the main theorems

This section gives the proof of Theorems A and C.

3.1 (G, G^τ) and its associated symmetric pair $(G, G^{\tau\theta})$

In general, it is not easy to describe H-coadjoint orbits on the intersection

$$\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H)$$

for a pair (G, H) of reductive Lie groups. When (G, H) is a symmetric pair, our key idea is to use another symmetric pair (G, H^a), referred to as the *associated symmetric pair*, defined as follows.

Let τ be an involutive automorphism of G commuting with the Cartan involution θ. Then, the composition $\tau\theta$ is also an involutive automorphism. We set

$$H := (G^\tau)_0, \quad H^a := (G^{\tau\theta})_0$$

the identity components of the fixed point groups G^τ and $G^{\tau\theta}$, respectively. Then the reductive groups H and H^a have the following Cartan decompositions

$$H = (H \cap K)\exp(\mathfrak{p}^\tau), \quad H^a = (H \cap K)\exp(\mathfrak{p}^{-\tau}),$$

respectively. We observe that both H and H^a have the same maximal compact subgroups $H \cap K$, and that the 'noncompact part' is complementary to each other, namely,

$$\mathfrak{p} = \mathfrak{p}^\tau + \mathfrak{p}^{-\tau} \qquad \text{(direct sum decomposition).}$$

This observation will be crucial in the proof of Theorems A and C below.

3.2 Hermitian symmetric space $H^a/H^a \cap K$

We return to our previous setting where τ is of holomorphic type, equivalently, $\tau Z = Z$. Since $\theta Z = Z$, the involution $\tau\theta$ is also of holomorphic type, and consequently, $\tau\theta$ preserves the decomposition (2.2). Therefore, we have a compatible direct sum decomposition of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}^{\tau\theta}$ of H^a:

$$\mathfrak{g}_{\mathbb{C}}^{\tau\theta} = (\mathfrak{k}_{\mathbb{C}}^\tau) + (\mathfrak{p}_+^{-\tau}) + (\mathfrak{p}_-^{-\tau}).$$

This decomposition makes $H^a/(H^a \cap K)$ a Hermitian symmetric space, which is naturally embedded into the Hermitian symmetric space G/K.

We first prepare notation when $H^a = (G^{\tau\theta})_0$ contains only one noncompact simple factor, namely, $L = 1$ by applying the structural results of Hermitian symmetric spaces [23] to $H^a/(H^a \cap K)$. In this case we shall write $\{\nu_1, \cdots, \nu_r\}$ for the maximal set of strongly orthogonal roots in $\Delta(\mathfrak{p}_+^{-\tau})$ instead of $\{\nu_1^{(i)}, \cdots, \nu_{r_i}^{(i)}\}$ as in Section 2.2. For each j, we define an $\mathfrak{sl}_2$-triple $\{H_j, E_j, E_{-j}\}$ in $\mathfrak{g}_{\mathbb{C}}^{\tau\theta}$ as follows:

$$E_j \in (\mathfrak{g}_{\mathbb{C}}^{\tau\theta})_{\nu_j},\ E_{-j} \in (\mathfrak{g}_{\mathbb{C}}^{\tau\theta})_{-\nu_j}, \text{ and } H_j \in \sqrt{-1}\mathfrak{t}^{\tau\theta}.$$

Here $(\mathfrak{g}_{\mathbb{C}}^{\tau\theta})_{\nu_j}$ denotes the root space in $\mathfrak{g}_{\mathbb{C}}^{\tau\theta}$ corresponding to $\nu_j \in \sqrt{-1}(\mathfrak{t}^\tau)^*$, and $H_j := \frac{2\nu_j}{\langle \nu_j, \nu_j \rangle}$ if we identify $\sqrt{-1}\mathfrak{t}^*$ with $\sqrt{-1}\mathfrak{t}$ by the Killing form. Furthermore, we may and do choose E_j and E_{-j} such that the following elements X_j and Y_j belong to the real Lie algebra $\mathfrak{g}$:

$$X_j := E_j + E_{-j}, \quad Y_j := -\sqrt{-1}(E_j - E_{-j}).$$

Then X_j, $Y_j \in \mathfrak{p}^{\tau\theta} = \mathfrak{p}^{-\tau}$. Next, let us define the following two subspaces:

$$\begin{aligned} \mathfrak{a} &:= \bigoplus_{j=1}^{r} \mathbb{R}X_j \qquad\quad \subset \mathfrak{p}^{\tau\theta}(= \mathfrak{p}^{-\tau}), \qquad (3.1)\\ \mathfrak{t}^- &:= \sqrt{-1}\bigoplus_{j=1}^{r} \mathbb{R}H_j \quad \subset \mathfrak{t}^{\tau\theta}(= \mathfrak{t}^{\tau}). \end{aligned}$$

Let $\mathfrak{t}^+$ be the orthogonal complement of $\mathfrak{t}^-$ in $\mathfrak{t}^\tau$ with respect to the Killing form. Then $\mathfrak{t}^+ + \mathfrak{a}$ is a maximally split Cartan subalgebra of $\mathfrak{g}^{\tau\theta}$.

For the general case where L may be greater than 1, we write $X_j^{(i)}$ instead of X_j $(1 \le j \le r_i, 1 \le i \le L)$. We take a positive system $\Sigma^+(\mathfrak{g}^{\tau\theta}, \mathfrak{a})$ such that the corresponding dominant Weyl chamber $\mathfrak{a}_+$ is given by

$$\begin{aligned} \mathfrak{a}_+^{(i)} &:= \{\sum_{j=1}^{r_i} t_j^{(i)} X_j^{(i)} : (t_j^{(i)})_{1\le j\le r_i} \in C_+^{(i)}\} \quad (1 \le i \le L),\\ \mathfrak{a}_+ &:= \{\sum_{i=1}^{L}\sum_{j=1}^{r_i} t_j^{(i)} X_j^{(i)} : (t_j^{(i)})_{1\le j\le r_i} \in C_+^{(i)} \quad \text{for } 1 \le i \le L\}. \qquad (3.2) \end{aligned}$$

Correspondingly, we define a subset of the connected abelian group $A = \exp(\mathfrak{a})$ by

$$A_+ = \exp(\mathfrak{a}_+) = \exp(\mathfrak{a}_+^{(1)}) \cdots \exp(\mathfrak{a}_+^{(L)}). \qquad (3.3)$$

Via the Killing form, the projection $\mathrm{pr}\colon \sqrt{-1}\mathfrak{g}^* \to \sqrt{-1}\mathfrak{k}^*$ is identified with the map

$$\mathrm{pr}^\theta : \sqrt{-1}\mathfrak{g} \to \sqrt{-1}\mathfrak{k}, \quad X \mapsto \frac{1}{2}(X + \theta X) \qquad (3.4)$$

We recall from [27, Prop. 2.4 and Lem. 2.5] an explicit formula for $\mathrm{pr}^\theta\big(\mathrm{Ad}(a)Z\big)$ applied to each noncompact simple factor $G^{(i)}$ of H^a.

Lemma 3.1. *Suppose $1 \le i \le L$. Let $Z^{(i)} \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}^{(i)})$ be the characteristic element of the simple Hermitian Lie algebra $\mathfrak{g}^{(i)} = \mathfrak{k}^{(i)} + \mathfrak{p}^{(i)}$. For $t_1^{(i)}, \cdots, t_{r_i}^{(i)} \in \mathbb{R}$, we define an element $a^{(i)}$ of A by*

$$a^{(i)} := \exp(\sum_{j=1}^{r_i} t_j^{(i)} X_j^{(i)}).$$

Then we have

1. $\mathrm{pr}^\theta(\mathrm{Ad}(a^{(i)})Z^{(i)}) = Z^{(i)} + \sum_{j=1}^{r_i}(\sinh t_j^{(i)})^2 H_j^{(i)}$*;*
2. $\mathrm{pr}^\theta(\mathrm{Ad}(a^{(i)})Z^{(i)}) \in \sqrt{-1}(\mathfrak{t}^\tau)_+^*$*;*
3. $a^{(i)} \mapsto \mathrm{pr}^\theta(\mathrm{Ad}(a^{(i)})Z_i)$ *is injective when restricted to* $A_+^{(i)} := \exp(\mathfrak{a}_+^{(i)})$.

In (2) we have identified $\sqrt{-1}\mathfrak{g}$ with $\sqrt{-1}\mathfrak{g}^*$ via the Killing form.

3.3 The formula for $\mathrm{pr}\big(\mathrm{Ad}(a)\lambda\big)$

Suppose $\lambda \in \sqrt{-1}([\mathfrak{k},\mathfrak{k}]+\mathfrak{p})^{\perp}$. By identifying $\sqrt{-1}\mathfrak{g}^*$ with $\sqrt{-1}\mathfrak{g}$ we see that λ is of the form $\lambda = cZ$ for some $c \in \mathbb{R}$.

Similarly to the map $\mathrm{pr}^{\theta} \colon \mathfrak{g} \to \mathfrak{k}$ (see (3.4)), we define a linear map

$$\mathrm{pr}^{\tau} \colon \sqrt{-1}\mathfrak{g} \to \sqrt{-1}\mathfrak{h}, \quad X \mapsto \frac{1}{2}(X+\tau X),$$

which is identified with the projection $\mathrm{pr} \colon \sqrt{-1}\mathfrak{g}^* \to \sqrt{-1}\mathfrak{h}^*$. Then we have:

Proposition 3.2. *Suppose $\lambda = cZ$ with $c > 0$. Recall from* (2.5) *the definition of the closed cone* $\mathrm{Cone}(\mathfrak{p}_+^{-\tau})$ *in* $\sqrt{-1}(\mathfrak{t}^{\tau})^*$*. Then,*

$$\mathrm{pr}^{\tau}\big(\mathrm{Ad}(A_+)\lambda\big) = \mathrm{pr}^{\theta}\big(\mathrm{Ad}(A_+)\lambda\big) = \lambda + \mathrm{Cone}(\mathfrak{p}_+^{-\tau}).$$

Proof. We recall that $\mathfrak{a}_+ \subset \mathfrak{g}^{-\tau} \cap \mathfrak{g}^{-\theta} \subset \mathfrak{g}^{\tau\theta}$. Since τ is of holomorphic type, we have $\tau\lambda = \lambda$, and therefore, $\mathrm{Ad}(A_+)\lambda \subset \sqrt{-1}\mathfrak{g}^{\tau\theta}$. Since $\tau = \theta$ on $\mathfrak{g}_{\mathbb{C}}^{\tau\theta}$, we have

$$\mathrm{pr}^{\tau} = \frac{1}{2}(\mathrm{id}+\tau) = \frac{1}{2}(\mathrm{id}+\theta) = \mathrm{pr}^{\theta} \quad \text{on } \mathfrak{g}_{\mathbb{C}}^{\tau\theta}. \tag{3.5}$$

In particular, we have

$$\mathrm{pr}^{\tau}\big(\mathrm{Ad}(A_+)\lambda\big) = \mathrm{pr}^{\theta}\big(\mathrm{Ad}(A_+)\lambda\big).$$

Therefore, we shall focus on $\mathrm{pr}^{\theta}(\mathrm{Ad}(A_+)\lambda)$ from now. According to the direct sum decomposition (2.4), the characteristic element $Z \in \mathfrak{c}(\mathfrak{k})$ is decomposed as

$$Z = Z^{(0)} + Z^{(1)} + \cdots + Z^{(L)},$$

where $Z^{(i)} \in \sqrt{-1}\mathfrak{c}(\mathfrak{k}^{(i)})$ for $0 \le i \le L$, and $Z^{(i)}$ $(1 \le i \le L)$ are the characteristic elements for the Hermitian Lie algebra $\mathfrak{g}^{(i)}$. We apply Lemma 3.1 (1) for the computation of $\mathrm{pr}^{\theta}\big(\mathrm{Ad}(a)\lambda\big)$ with $a = a^{(1)} \cdots a^{(L)} \in A_+$, and get

$$\begin{aligned}\mathrm{pr}^{\theta}\big(\mathrm{Ad}(A_+)\lambda\big) =& c\,\{Z^{(0)} + \sum_{i=1}^{L}(Z^{(i)} + \sum_{j=1}^{r_i}(\sinh t_j^{(i)})^2 H_j^{(i)}) : (t_j^{(i)})_{1\le j\le r_i} \in C_+^{(i)}\ (1 \le i \le L)\}\\ =& c\,\{Z + \sum_{i=1}^{L}\sum_{j=1}^{r_i}(\sinh t_j^{(i)})^2 H_j^{(i)} : (t_j^{(i)})_{1\le j\le r_i} \in C_+^{(i)}\ (1 \le i \le L)\}.\end{aligned}$$

Hence we have proved Proposition 3.2 □

Next we prove the following proposition.

Proposition 3.3. *Fix $\lambda = cZ$ with $c > 0$. Then the following three conditions on $a, a' \in A_+$ are equivalent:*

(i) $\mathrm{pr}^{\tau}(\mathrm{Ad}(a)\lambda)$ *and* $\mathrm{pr}^{\tau}(\mathrm{Ad}(a')\lambda)$ *are conjugate by the adjoint action of H;*

(ii) $\mathrm{pr}^\tau(\mathrm{Ad}(a)\lambda)$ *and* $\mathrm{pr}^\tau(\mathrm{Ad}(a')\lambda)$ *are conjugate by the adjoint action of* $H \cap K$*;*

(iii) $a = a'$.

Proof. Since $\mathrm{pr}^\tau\big(\mathrm{Ad}(a)\lambda\big) = \mathrm{pr}^\theta\big(\mathrm{Ad}(a)\lambda\big)$ for any $a \in A$, we see that $\mathrm{pr}^\tau(\mathrm{Ad}(a)\lambda) \in \sqrt{-1}(\mathfrak{t}^\tau)^*_+$ from Lemma 3.1 (2). Since two elements in $\sqrt{-1}(\mathfrak{t}^\tau)^*_+$ is conjugate under $H = (G^\tau)_0$ if and only if they coincide, we get the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) by Lemma 3.1 (3). The implication (iii) $\Rightarrow$ (i) is obvious. Thus Proposition 3.3 is proved. $\square$

3.4 Proof of Theorems A and C

Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{-\tau} \cap \mathfrak{g}^{-\theta}$, we have the generalized Cartan decomposition [4, Thm. 4.1] for the semisimple symmetric pair (G, H):

$$G = HA_+K. \tag{3.6}$$

Suppose $\lambda \in \sqrt{-1}\mathfrak{c}(\mathfrak{k})^*$. Since K stabilizes λ, the decomposition (3.6) implies

$$\mathcal{O}^G_\lambda = \mathrm{Ad}(G)\lambda = \mathrm{Ad}(H)\,\mathrm{Ad}(A_+)\lambda. \tag{3.7}$$

Proof of Theorem A. We take any two elements x, $x' \in \mathcal{O}^G \cap (\mathrm{pr}^\tau)^{-1}(\mathcal{O}^H)$. We shall prove that $x' \in \mathrm{Ad}(H)x$. It follows from the generalized Cartan decomposition (3.6) that there exist a, $a' \in A_+$ and h, $h' \in H$ such that

$$x = \mathrm{Ad}(h)\,\mathrm{Ad}(a)\lambda,\ x' = \mathrm{Ad}(h')\,\mathrm{Ad}(a')\lambda. \tag{3.8}$$

Since the projection $\mathrm{pr}^\tau\colon \mathfrak{g} \to \mathfrak{h}$ respects H-action, we have

$$\mathrm{pr}^\tau(x) = \mathrm{Ad}(h)\,\mathrm{pr}^\tau(\mathrm{Ad}(a)\lambda),\ \mathrm{pr}^\tau(x') = \mathrm{Ad}(h')\,\mathrm{pr}^\tau(\mathrm{Ad}(a')\lambda).$$

By our assumption, both $\mathrm{pr}^\tau(x)$ and $\mathrm{pr}^\tau(x')$ are contained in the same H-orbit $\mathcal{O}^H$. Therefore, $\mathrm{pr}^\tau(\mathrm{Ad}(a)\lambda)$ and $\mathrm{pr}^\tau(\mathrm{Ad}(a')\lambda)$ are conjugate by an element of H. By Proposition 3.3, we conclude $a = a'$. Using (3.8) again, we see that x is conjugate to x' under H. This is what we wanted to prove. $\square$

Finally, we shall determine H-coadjoint orbits $\mathcal{O}^H$ such that $n(\mathcal{O}^G, \mathcal{O}^H) \neq 0$. The first step is to show that $\mathcal{O}^H$ must be an elliptic orbit.

Proposition 3.4. *Let* $\mathcal{O}^G$ *and* $\mathcal{O}^H$ *be coadjoint orbits in* $\sqrt{-1}\mathfrak{g}^*$ *and* $\sqrt{-1}\mathfrak{h}^*$*, respectively. Suppose* $\mathcal{O}^G$ *satisfies* (2.1)*. If* $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H) \neq \emptyset$*, then* $\mathcal{O}^H$ *is an elliptic orbit.*

Proof. If $\mathcal{O}^G = \{0\}$, then the condition $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H) \neq \emptyset$ obviously implies $\mathcal{O}^H = \{0\}$. Hence $\mathcal{O}^H$ is an elliptic orbit.

From now, we assume that $\mathcal{O}^G \neq \{0\}$. Without loss of generality, we may assume $\mathcal{O}^G = \mathrm{Ad}(G)\lambda$ where $\lambda = cZ$ $(c > 0)$ via the identification of $\sqrt{-1}\mathfrak{g}$ with $\sqrt{-1}\mathfrak{g}^*$ as before. If $\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H) \neq \emptyset$, we find $g \in G$ such that

$$\mathrm{pr}\big(\mathrm{Ad}(g)\lambda\big) \in \mathcal{O}^H. \tag{3.9}$$

We write

$$g = hak \in G \quad (h \in H,\, a \in A_+,\, k \in K)$$

according to (3.6). Then it follows from (3.9) that $\mathrm{pr}\big(\mathrm{Ad}(a)\lambda\big) \in \mathcal{O}^H$ because $\mathrm{Ad}(k)\lambda = \lambda$. By Proposition 3.2, we have $\mathrm{pr}\big(\mathrm{Ad}(a)\lambda\big) \in \sqrt{-1}(\mathfrak{t}^\tau)^*_+$. Hence $\sqrt{-1}(\mathfrak{t}^\tau)^*_+ \cap \mathcal{O}^H \neq \emptyset$. Therefore, $\mathcal{O}^H$ must be an elliptic orbit. □

Proof of Theorem C. Suppose $\mathcal{O}^G = \mathcal{O}^G_\lambda$ with $\lambda = cZ \ \ (c > 0)$. Then the proof of Proposition 3.4 asserts that if $\mathcal{O}^G_\lambda \cap \mathrm{pr}^{-1}(\mathcal{O}^H) \neq \emptyset$, then $\mathrm{pr}\big(\mathrm{Ad}(a)\lambda\big) \in \mathcal{O}^H$ for some $a \in A_+$. Clearly, the opposite implication also holds. Thus we have shown that $n(\mathcal{O}^G_\lambda, \mathcal{O}^H) \neq 0$ if and only if $\mathcal{O}^H \cap \big(\lambda + \mathrm{Cone}(\mathfrak{p}_+^{-\tau})\big) \neq \emptyset$ because

$$\mathrm{pr}\big(\mathrm{Ad}(A_+)\lambda\big) = \lambda + \mathrm{Cone}(\mathfrak{p}_+^{-\tau})$$

by Proposition 3.4. Hence we have the equivalence (i) ⇔ (iii) in Theorem C. The equivalence (i) ⇔ (ii) follows from Theorem A. □.

4 Visible actions on coadjoint orbits

We end this article with discussion about another aspect on the geometry of the coadjoint orbits.

A holomorphic action of a Lie group H on a connected complex manifold M is said to be *strongly visible* if there exist a totally real submanifold S, referred to as a *slice*, and an anti-holomorphic diffeomorphism σ of M which preserves every H-orbit in M such that generic H-orbits meet S and $\sigma|_S = \mathrm{id}$, see [16, Def. 3.3.1]. The proof of the multiplicity-free theorem (Fact 2.1) is based on the following fact:

Fact 4.1 ([18])**.** *Let G be a Hermitian Lie group. For any symmetric pair (G, H), the H-action on G/K is strongly visible.*

Any nonzero coadjoint orbit $\mathcal{O}^G$ satisfying the condition (2.1) is isomorphic to the Hermitian symmetric space G/K. Hence Fact 4.1 may be seen as a result on the geometry of coadjoint orbits:

Fact 4.2. *Let $\mathcal{O}^G$ be a coadjoint orbit satisfying* (2.1)*. For any symmetric pair (G, H), the H-action on the coadjoint orbit $\mathcal{O}^G$ is strongly visible.*

In this case, the slice S can be taken to be $\mathring{A}_+ \cdot o$, where $\mathring{A}_+$ denotes the set of interior points of A_+ and o is the fixed point of K with notation for the generalized Cartan decomposition (3.6).

In turn, the Cayley transform of A_+ for the subgroup $H^a = (G^{\tau\theta})_0$ (not for $H = (G^\tau)_0$) followed by the shift of $\lambda|_{\mathfrak{t}^\tau}$ gives the support $\mathrm{Supp}_H(\pi^G|_H)$, as is seen in (2.9), where π^G is the irreducible unitary representation of G attached to the coadjoint orbit $\mathcal{O}^G$ and λ is determined by $\mathcal{O}^G$ by the condition $\mathcal{O}^G \cap \sqrt{-1}\mathfrak{c}(\mathfrak{k})^* = \{\lambda\}$. This viewpoint from visible actions provides yet another perspective of Theorem D on the Kirillov correspondence between branching laws of unitary representations and coadjoint orbits with momentum map $\mu\colon \mathcal{O}^G \to \sqrt{-1}\mathfrak{h}^*$ for the Hamiltonian action of the subgroup H on $\mathcal{O}^G$.

Acknowledgements: The author would like to express his gratitude to the organizers, A. Borodin, A. Kirillov, S. Morier-Genoud, A. Okounkov, V. Ovsienko, M. Pevzner, N. Rozhkovskaya, M. Schlichenmaier, and R. Yu, of the conference 'Representation Theory at the Crossroads of Modern Mathematics' in honor of Alexandre A. Kirillov on his 81st birthday at Reims, May 29–June 2, 2017 for their warm hospitality during the stimulating conference.

This work was partially supported by Grant-in-Aid for Scientific Research (A) (18H03669), Japan Society for the Promotion of Science.

References

[1] Corwin, L. and Greenleaf, F., *Spectrum and multiplicities for restrictions of unitary representations in nilpotent Lie groups*, Pacific J. Math. **135** (1988), 233–267.

[2] Deltour, G., *On a generalization of a theorem of McDuff*, J. Differential Geom. **93** (2013), 379–400.

[3] Duflo, M. and Vargas, J. A., *Branching laws for square integrable representations*, Proc. Japan Acad. Ser. A, Math. Sci., **86**, (2010), 49–54.

[4] Flensted-Jensen, M., *Spherical functions of a real semisimple Lie group. A method of reduction to the complex case*, J. Funct. Anal. **30** (1978), 106–146.

[5] Fujiwara, H., *Représentation monomiales des groups de Lie résolubles exponentiels*, The Orbit Method in Representation Theory, Progr. Math. Birkhäuser, 1990, 61–84.

[6] Hilgert, J. Kobayashi, T. and Möllers, J., *Minimal representations via Bessel operators*, J. Math. Soc. Japan **66** (2014), 349–414.

[7] Hua, L.-K., *Harmonic analysis of functions of several complex variables in the classical domains*, Translated from the Russian by Leo Ebner and Adam Korányi, Amer. Math. Soc., Providence, R. I., 1963.

[8] Kirillov, A. A., *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** 1962, 57–110.

[9] Kirillov, A. A., *Merits and demerits of the orbit method*. Bull. Amer. Math. Soc. (N.S.) **36** (1999), 433–488.

[10] Kirillov, A. A., *Lectures on the Orbit Method*, Graduate Studies in Mathematics, **64**, Amer. Math. Soc. 2004.

[11] Kobayashi, T., *Singular Unitary Representations and Discrete Series for Indefinite Stiefel Manifolds* $U(p,q;\mathbb{F})/U(p-m,q;\mathbb{F})$, Mem. Amer. Math. Soc., **462** Amer. Math. Soc., 1992. v+106 pages.

[12] Kobayashi, T., *Discrete decomposability of the restrictions of $A_q(\lambda)$ with respect to reductive subgroups and its applications*, Invent. Math. **117** (1994), 181–205.

[13] Kobayashi, T., *Multiplicity-free branching laws for unitary highest weight modules*, Proceedings of the Symposium on Representation Theory held at Saga, Kyushu 1997 (K. Mimachi, ed.), (1997), 9–17.

[14] Kobayashi, T., *Harmonic analysis on homogeneous manifolds of reductive type and unitary representation theory*, Translation, Series II, Selected Papers on Harmonic Analysis, Groups, and Invariants (K. Nomizu, ed.), Amer. Math. Soc. **183** (1998), 1–31.

[15] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups* II*—micro-local analysis and asymptotic K-support*, Ann. of Math., **147**, (1998), 709–729.

[16] Kobayashi, T., *Multiplicity-free representations and visible action on complex manifolds*, Publ. Res. Inst. Math. Sci. **41** (2005), 497–549, an special issue commemorating the fortieth anniversary of the founding of RIMS.

[17] Kobayashi, T., *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*, In Representation Theory and Automorphic Forms, Progr. Math. **255**, Birkhäuser 2007, 45–109.

[18] Kobayashi, T., *Visible actions on symmetric spaces*, Transform. Groups, **12** (2007), 671–694.

[19] Kobayashi, T. *Geometric quantization, limits, and restrictions—some examples for elliptic and nilpotent orbits*, In:Geometric Quantization in the Non-compact Setting (eds. L. Jeffrey, X. Ma and M. Vergne), Oberwolfach Reports Volume 8, Issue 1, 2011, European Mathematical Society, Publishing House. 466–469.

[20] Kobayashi, T., *Restrictions of generalized Verma modules to symmetric pairs*, Transform. Group, **17** (2012), 523–546.

[21] Kobayashi, T. and Nasrin, S., *Multiplicity one theorem in the orbit method*, Lie Groups and Symmetric Spaces: In memory of Professor F. I. Karpelevič (ed. S. Gindikin), Transl. Series 2, Amer. Math. Soc., **210** (2003), 161–169.

[22] Kobayashi, T. and Ørsted, B., *Conformal geometry and branching laws for unitary representations attached to minimal nilpotent orbits*, C. R. Acad. Sci. Paris **326** (1998), 925–930.

[23] Korányi, A. and Wolf, J. A., *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. Math. **81** (1965), 265–288.

[24] McDuff, D., *The symplectic structure of Kähler manifolds of nonpositive curvature*, J. Differential Geom. **28** (1988), 467–475.

[25] NASRIN, S. *Corwin–Greenleaf multiplicity function for Hermitian Lie groups*, thesis, The University of Tokyo, 2003.

[26] NASRIN, S., *Corwin–Greenleaf multiplicity functions for Hermitian symmetric spaces*. Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), 97–100.

[27] NASRIN, S., *Corwin–Greenleaf multiplicity function for Hermitian symmetric spaces and multiplicity-one theorem in the orbit method*, Internat. J. Math. **21** (2010), 279–296.

[28] PARADAN, P.-E., *Multiplicities of the discrete series*, preprint, arXiv:0812.0059.

[29] PARADAN, P.-E., *Quantization commutes with reduction in the non-compact setting: the case of holomorphic discrete series*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 4, 955–990.

[30] SCHMID, W., *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math. **9** (1969/1970), 61– 80.

[31] SJAMAAR, R., *Convexity properties of the moment mapping, re-examined*, Adv. Math. **138** (1998), 46–91.

[32] VOGAN, DAVID A., JR., *Irreducibility of discrete series representations for semisimple symmetric spaces*, 191–221, Adv. Stud. Pure Math., **14**, 1988.