

# Residue formula for regular symmetry breaking operators

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*To Gestur Ólafsson on his 65th birthday.*

ABSTRACT. We prove an explicit residue formula for a meromorphic continuation of conformally covariant integral operators between differential forms on  $\mathbb{R}^n$  and on its hyperplane. The results provide a simple and new construction of the conformally covariant differential symmetry breaking operators between differential forms on the sphere and those on its totally geodesic hypersurface that were introduced in [Kobayashi–Kubo–Pevzner, Lect. Notes Math. (2016)]. Moreover, we determine the zeros of the matrix-valued regular symmetry breaking operators between principal series representations of  $O(n+1, 1)$  and  $O(n, 1)$ .

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## 1. Statement of the main results

Let  $\mathcal{E}^i(\mathbb{R}^n)$  be the space of (complex-valued) differential  $i$ -forms on  $\mathbb{R}^n$ , and  $\mathcal{E}_c^i(\mathbb{R}^n)$  the subspace of those having compact support. The object of study in this

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article is a meromorphic family of operators

$$\mathbb{A}_{\lambda, \nu, \pm}^{i, j} : \mathcal{E}_c^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1}),$$

which are initially defined as integral operators when  $\operatorname{Re} \lambda \gg |\operatorname{Re} \nu|$ .

The operators  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$  arise as

- matrix-valued regular *symmetry breaking operators* for principal series representations for strong Gelfand pair  $(O(n+1, 1), O(n, 1))$ ;
- conformally covariant operators on differential forms on the model space  $(S^n, S^{n-1})$

among others.

By choosing appropriate Gamma factors  $a_{\pm}(\lambda, \nu)$  (see (1.5)), we renormalize  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$  by

$$\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j} := a_{\pm}(\lambda, \nu) \mathbb{A}_{\lambda, \nu, \pm}^{i, j}$$

so that  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  depend holomorphically on  $(\lambda, \nu)$  in the entire plane  $\mathbb{C}^2$ , see Fact 1.1 below.

The goal of this paper is in twofold:

- to find the residue formula of the matrix-valued operators  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$  along  $\nu - \lambda \in \mathbb{N}$  (see Theorem 1.3);
- to determine all the (isolated) zeros of the normalized operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  (see Theorem 8.1 and Remark 8.3).

**1.1. Integral operators**  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j} : \mathcal{E}_c^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$ . To state our main results, we fix some notation.

Let  $|x| := (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$  for  $x \in \mathbb{R}^n$ . We define

$$(1.1) \quad \psi_n : \mathbb{R}^n \setminus \{0\} \rightarrow O(n), \quad x \mapsto I_n - \frac{2x \, {}^t x}{|x|^2},$$

as a matrix expression of the reflection with respect to normal vector  $x$ . Let  $\sigma^{(i)} : O(n) \rightarrow GL_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n))$  be the  $i$ th exterior representation of the natural representation of  $O(n)$  on  $\mathbb{C}^n$ .

We identify  $\mathcal{E}^i(\mathbb{R}^n)$  with  $C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n)$ , and similarly,  $\mathcal{E}^j(\mathbb{R}^{n-1})$  with  $C^\infty(\mathbb{R}^{n-1}) \otimes \bigwedge^j(\mathbb{C}^{n-1})$ . We define  $\operatorname{pr}_{i \rightarrow j} : \bigwedge^i(\mathbb{C}^n) \rightarrow \bigwedge^j(\mathbb{C}^{n-1})$  for  $j = i-1, i$  to be the first and second projections of the decomposition  $\bigwedge^i(\mathbb{C}^n) = \bigwedge^{i-1}(\mathbb{C}^{n-1}) \oplus \bigwedge^i(\mathbb{C}^{n-1})$ , respectively, so that the following linear map

$$(1.2) \quad \operatorname{Rest}_{x_n=0} \circ (\operatorname{id} \oplus \iota_{\frac{\partial}{\partial x_n}}) : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^{n-1}) \oplus \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$$

is identified with  $\operatorname{id} \otimes (\operatorname{pr}_{i \rightarrow i} \oplus \operatorname{pr}_{i \rightarrow i-1})$ . Here  $\iota_{\frac{\partial}{\partial x_n}}$  denotes the inner multiplication of the vector field  $\frac{\partial}{\partial x_n}$ .

For  $j \in \{i-1, i\}$  and  $\operatorname{Re} \lambda \gg |\operatorname{Re} \nu|$ , we define  $\operatorname{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}))$ -valued, locally integrable functions  $\mathcal{A}_{\lambda, \nu, \pm}^{i, j}$  on  $\mathbb{R}^n$  by

$$(1.3) \quad \mathcal{A}_{\lambda, \nu, +}^{i, j}(x) := (x_1^2 + \cdots + x_n^2)^{-\nu} |x_n|^{\lambda + \nu - n} \operatorname{pr}_{i \rightarrow j} \circ \sigma^{(i)}(\psi_n(x)),$$

$$(1.4) \quad \mathcal{A}_{\lambda, \nu, -}^{i, j}(x) := \mathcal{A}_{\lambda, \nu, +}^{i, j}(x) \operatorname{sgn}(x_n).$$

We introduce a normalizing factor  $a_{\pm}(\lambda, \nu)$  by

$$(1.5) \quad a_{\varepsilon(\kappa)}(\lambda, \nu)^{-1} := \Gamma\left(\frac{\lambda + \nu - n + 1 + \kappa}{2}\right) \Gamma\left(\frac{\lambda - \nu + \kappa}{2}\right)$$

for  $\kappa = 0, 1$ , where we set

$$\varepsilon: \{0, 1\} \rightarrow \{\pm\}$$

by  $\varepsilon(0) = +$  and  $\varepsilon(1) = -$ . Then  $a_{\pm}(\lambda, \nu)$  are holomorphic functions of  $(\lambda, \nu)$  in  $\mathbb{C}^2$ . We set

$$(1.6) \quad \tilde{\mathcal{A}}_{\lambda, \nu, \pm}^{i, j} := a_{\pm}(\lambda, \nu) \mathcal{A}_{\lambda, \nu, \pm}^{i, j}.$$

FACT 1.1 (see [19, 21]). *Let  $j = i - 1$  or  $i$ . Then the integral operators*

$$\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}: \mathcal{E}_c^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1}), f \mapsto \text{Rest}_{x_n=0} \circ (\tilde{\mathcal{A}}_{\lambda, \nu, \pm}^{i, j} * f),$$

*originally defined when  $\text{Re } \lambda \gg |\text{Re } \nu|$ , extend to continuous operators that depend holomorphically on  $(\lambda, \nu)$  in the entire complex plane  $\mathbb{C}^2$ . Moreover,  $\{(\lambda, \nu) \in \mathbb{C}^2 : \tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j} = 0\}$  is a discrete subset of  $\mathbb{C}^2$ .*

REMARK 1.2. A key point of (1.6) is that the Gamma factors  $a_{\pm}(\lambda, \nu)$  are chosen in an optimal way that there is no pole of  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  and that the zeros are of codimension two in  $\mathbb{C}^2$ . We note that  $a_{+}(\lambda, \nu)$  coincides with normalizing factor of the scalar-valued regular symmetry breaking operator  $\mathbb{A}_{\lambda, \nu}$  in [19, (7.8)] when  $i = j = 0$  and  $\kappa = 0$ .

**1.2. Residue formula of matrix-valued operators  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$ .** The first factor  $\Gamma(\frac{\lambda + \nu - n + 1 + \kappa}{2})$  of  $a_{\varepsilon(\kappa)}(\lambda, \nu)$  in (1.5) arises from the normalization of the distribution  $|x_n|^{\lambda + \nu - n} (\text{sgn } x_n)^{\kappa}$  of one-variable, and the corresponding residue of  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$  is easily obtained. On the other hand, the second factor  $\Gamma(\frac{\lambda - \nu + \kappa}{2})$  is more involved because it arises not only from the normalization of  $(x_1^2 + \cdots + x_n^2)^{-\nu}$  but from the whole  $\mathcal{A}_{\lambda, \nu, \pm}^{i, j}$ . The main result of this article is to give a closed formula for the residues of the operators  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$  at the places of the poles of  $\Gamma(\frac{\lambda - \nu + \kappa}{2})$  as follows:

THEOREM 1.3 (residue formula of  $\mathbb{A}_{\lambda, \nu, \pm}^{i, j}$ ). *Let  $j = i - 1$  or  $i$ , and  $\kappa \in \{0, 1\}$ . Suppose  $\nu - \lambda = 2m + \kappa$  with  $m \in \mathbb{N}$ . Then we have*

$$(1.7) \quad \tilde{\mathbb{A}}_{\lambda, \nu, \varepsilon(\kappa)}^{i, j} = \frac{(-1)^{i-j+m+\kappa} \pi^{\frac{n-1}{2}} m!}{2^{2m-1+3\kappa} \Gamma(\nu+1)} \mathbb{C}_{\lambda, \nu}^{i, j}.$$

Here  $\mathbb{C}_{\lambda, \nu}^{i, j}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$  is a matrix-valued differential operator which we introduced in [12, 13]. See [18, Def. 2.1] for instance, for the definition of differential operators between two manifolds. To review the differential operator  $\mathbb{C}_{\lambda, \nu}^{i, j}$ , we begin with a scalar-valued differential operator  $\tilde{\mathbb{C}}_{\lambda, \nu}$  which we call *Juhl's operator* from [6]. For  $l := \nu - \lambda \in \mathbb{N}$ , we set  $m := \lfloor \frac{l}{2} \rfloor$ , the largest integer that does not exceed  $\frac{l}{2}$ , and define  $\tilde{\mathbb{C}}_{\lambda, \nu}: C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$  by

$$(1.8) \quad \tilde{\mathbb{C}}_{\lambda, \nu} := \text{Rest}_{x_n=0} \circ \sum_{k=0}^m \frac{\prod_{j=1}^{m-k+1} (\nu - \frac{n-1}{2} - m + j)}{2^{2k-l} k! (l-2k)!} (\Delta_{\mathbb{R}^{n-1}})^k \left( \frac{\partial}{\partial x_n} \right)^{l-2k}.$$

More generally, we introduced in [12] the following matrix-valued differential operator  $\mathbb{C}_{\lambda, \nu}^{i, j}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$  by the formula (see also [13, (2.24) and (2.26)]):

$$\begin{aligned} \mathbb{C}_{\lambda, \nu}^{i, i} &:= \tilde{\mathbb{C}}_{\lambda+1, \nu-1} dd^* - \gamma\left(\lambda - \frac{n}{2}, \nu - \lambda\right) \tilde{\mathbb{C}}_{\lambda, \nu-1} d\iota_{\frac{\partial}{\partial x_n}} + \frac{\nu - i}{2} \tilde{\mathbb{C}}_{\lambda, \nu}, \\ \mathbb{C}_{\lambda, \nu}^{i, i-1} &:= -\tilde{\mathbb{C}}_{\lambda+1, \nu-1} dd^* \iota_{\frac{\partial}{\partial x_n}} - \gamma\left(\lambda - \frac{n-1}{2}, \nu - \lambda\right) \tilde{\mathbb{C}}_{\lambda+1, \nu} d^* + \frac{\lambda + i - n}{2} \tilde{\mathbb{C}}_{\lambda, \nu} \iota_{\frac{\partial}{\partial x_n}}, \end{aligned}$$

where the codifferential  $d^* : \mathcal{E}^{j+1}(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^n)$  is the formal adjoint of the exterior derivative  $d : \mathcal{E}^j(\mathbb{R}^n) \rightarrow \mathcal{E}^{j+1}(\mathbb{R}^n)$ . The constant  $\gamma(\mu, a) \in \mathbb{C}$  is given for  $\mu \in \mathbb{C}$  and  $a \in \mathbb{N}$  by

$$(1.9) \quad \gamma(\mu, a) := \begin{cases} 1 & \text{if } a \text{ is odd,} \\ \mu + \frac{a}{2} & \text{if } a \text{ is even.} \end{cases}$$

REMARK 1.4. (1) Theorem 1.3 gives a method to find an explicit formula of the conformally covariant differential symmetry breaking operators  $\mathbb{C}_{\lambda, \nu}^{i, j}$  for  $j = i - 1, i$ , for which we gave a different proof in [13] based on the F-method [8]. In fact Theorem 1.3 was obtained by 2015, however, we did not adopt this method in the article [13] and have postponed its publication to this article, because the residue method gives only a “construction” of differential symmetry breaking operators, whereas the F-method gives both “construction” and “exhaustion”.

(2) In contrast, applications of Theorem 1.3 include a proof of the “exhaustion” of *nonlocal* symmetry breaking operators with singular parameters, and then contribute to the multiplicity-formula of conformally covariant symmetry breaking operators for differential forms on  $(S^n, S^{n-1})$  given in [20], see [21] for detailed proof.

#### Notation.

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \mathbb{N}_+ = \{1, 2, \dots\}.$$

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## 2. Symmetry breaking in conformal geometry

We discuss the operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  from two different viewpoints: representation theory of real reductive groups (Section 2.1) and conformal geometry (Section 2.2). With these perspectives, we shall explain two applications of Theorem 1.3 in Section 1. Logically, the results of this section are not used for the proof of Theorem 1.3.

**2.1. Symmetry breaking for reductive groups.** In this subsection, we discuss the operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  from the viewpoint of representation theory of real reductive Lie groups.

Let  $\Pi$  be a continuous representation of a group  $G$  on a topological vector space. If  $G'$  is a subgroup of  $G$ , we may think of  $\Pi$  as a representation of the subgroup  $G'$ , which is called the *restriction* of  $\Pi$ , to be denoted by  $\Pi|_{G'}$ . Let  $\pi$  be another representation of the subgroup  $G'$ . A *symmetry breaking operator* is a continuous linear map  $\Pi \rightarrow \pi$  that intertwines the actions of the subgroup  $G'$ .

The operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  arise as symmetry breaking operators for the pair  $(G, G') = (O(n+1, 1), O(n, 1))$  as follows. Let  $P = MAN$  be a minimal parabolic subgroup of  $G$ . For  $0 \leq i \leq n$ ,  $\delta \in \{0, 1\} \simeq \mathbb{Z}/2\mathbb{Z}$ , and  $\lambda \in \mathbb{C}$ , we take  $\Pi$  to be the principal series representation

$$I_\delta(i, \lambda) = \text{Ind}_P^G(\wedge^i(\mathbb{C}^n) \otimes \text{sgn}^\delta \otimes \mathbb{C}_\lambda),$$

given by (unnormalized) parabolic induction. Here  $\Lambda^i(\mathbb{C}^n) \otimes \text{sgn}^\delta \otimes \mathbb{C}_\lambda$  stands for an irreducible representation of  $P$  that extends the outer tensor product representation of  $MA \simeq O(n) \times O(1) \times \mathbb{R}$  with trivial  $N$ -action. Likewise, we take  $\pi$  to be the parabolic induction

$$J_\varepsilon(j, \nu) = \text{Ind}_{P'}^{G'}(\Lambda^j(\mathbb{C}^{n-1}) \otimes \text{sgn}^\varepsilon \otimes \mathbb{C}_\nu)$$

from a minimal parabolic subgroup  $P'$  of  $G'$ . By identifying the open Bruhat cell of  $G/P$  with  $\mathbb{R}^n$  and that of  $G'/P'$  with  $\mathbb{R}^{n-1}$ , we can realize  $\Pi$  and  $\pi$  in  $\mathcal{E}^i(\mathbb{R}^n)$  and  $\mathcal{E}^j(\mathbb{R}^{n-1})$  as the “ $N$ -picture” of principal series representations, respectively. Then the operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \mu}^{i, j} : \mathcal{E}_c^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$  in Fact 1.1 gives rise to symmetry breaking operators, to be denoted by the same symbol,

$$\tilde{\mathbb{A}}_{\lambda, \nu, \mu}^{i, j} : I_\delta(i, \lambda) \rightarrow J_\varepsilon(j, \nu)$$

for  $\mu = +$  ( $\delta \equiv \varepsilon \pmod{2}$ ) and  $\mu = -$  ( $\delta \not\equiv \varepsilon \pmod{2}$ ). If

$$(2.1) \quad \nu - \lambda \notin 2\mathbb{N} \text{ for } \delta \equiv \varepsilon \quad \text{or} \quad \nu - \lambda \notin 2\mathbb{N} + 1 \text{ for } \delta \not\equiv \varepsilon,$$

then the support of the distribution kernel  $\tilde{\mathcal{A}}_{\lambda, \nu, \mu}^{i, j}$  has an interior point, and the operator  $\tilde{\mathbb{A}}_{\lambda, \nu, \mu}^{i, j}$  is called a *regular* symmetry breaking operator ([19, Def. 3.3]).

The dimension of the space  $\text{Hom}_{G'}(I_\delta(i, \lambda)|_{G'}, J_\varepsilon(j, \nu))$  of symmetry breaking operators is uniformly bounded with respect to the parameters  $(\lambda, \nu, \delta, \varepsilon)$  by the general theory ([17, 21]). Moreover, it is one-dimensional and spanned by  $\tilde{\mathbb{A}}_{\lambda, \nu, \mu}^{i, j}$  if the generic condition (2.1) on parameters is satisfied, see [20].

REMARK 2.1. From the representation theory of reductive groups, an important property of the family  $\{I_\delta(i, \lambda)\}$  of the aforementioned representations is that *any* irreducible admissible smooth representation  $\varpi$  of  $G$  with trivial infinitesimal character can be obtained as an irreducible subquotient of the representation  $I_\delta(i, \lambda)$  for some  $\delta, i$  and  $\lambda$ . Furthermore, any irreducible unitary representation with nonzero  $(\mathfrak{g}, K)$ -cohomologies arises as the Hilbert completion of such  $\varpi$ , and *vice versa*, see [21, Thm. 2.16].

The dimension formula of  $\text{Hom}_{G'}(I_\delta(i, \lambda)|_{G'}, J_\varepsilon(j, \nu))$  was given in [20] where a part of the proof utilized the results of this article, namely, the vanishing criterion of  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  which will be stated in Theorem 8.1. The complete classification of  $\text{Hom}_{G'}(I_\delta(i, \lambda)|_{G'}, J_\varepsilon(j, \nu))$  with explicit generators for general  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$ ,  $\lambda, \nu \in \mathbb{C}$  and  $\delta, \varepsilon \in \{0, 1\}$  is accomplished in [21], which may be thought of as “Stage C” of the branching problem (see [10]). Among these symmetry breaking operators, the classification and construction of differential operators such as  $\mathbb{C}_{\lambda, \nu}^{i, i-1}$  and  $\mathbb{C}_{\lambda, \nu}^{i, i}$  was established in [13] by using the F-method which was initiated and developed in [8, 9, 16, 18]. A preprint [3] treats a similar problem by using also the F-method (though the formula for  $\mathbb{C}_{\lambda, \nu}^{i, i-1}$  and  $\mathbb{C}_{\lambda, \nu}^{i, i}$  *loc.cit.* is not of the present form). A generalization to the pseudo-Riemannian case and to any domains is proved in [14].

**2.2. Conformally covariant symmetry breaking operators.** In this section, we explain from the viewpoint of conformal geometry a general question on symmetry breaking with respect to a pair of conformal manifolds  $X \supset Y$ .

Let  $(X, g)$  be a Riemannian manifold. Suppose that a Lie group  $G$  acts conformally on  $X$ . This means that there exists a positive-valued function  $\Omega \in C^\infty(G \times X)$

(conformal factor) such that

$$L_h^* g_{h \cdot x} = \Omega(h, x)^2 g_x \quad \text{for all } h \in G, x \in X,$$

where we write  $L_h: X \rightarrow X, x \mapsto h \cdot x$  for the action of  $G$  on  $X$ . When  $X$  is oriented, we define a locally constant function

$$\sigma: G \times X \longrightarrow \{\pm 1\}$$

by  $\sigma(h)(x) = 1$  if  $(L_h)_{*x}: T_x X \rightarrow T_{L_h x} X$  is orientation-preserving, and  $\sigma(h)(x) = -1$  if it is orientation-reversing.

Since both the conformal factor  $\Omega$  and the orientation map  $\sigma$  satisfy cocycle conditions, we can form a family of representations  $\varpi_{u, \delta}^{(i)}$  of  $G$  with parameters  $u \in \mathbb{C}$  and  $\delta \in \mathbb{Z}/2\mathbb{Z}$  on the space  $\mathcal{E}^i(X)$  of differential  $i$ -forms on  $X$  ( $0 \leq i \leq \dim X$ ) defined by

$$(2.2) \quad \varpi_{u, \delta}^{(i)}(h)\alpha := \sigma(h)^\delta \Omega(h^{-1}, \cdot)^u L_{h^{-1}}^* \alpha, \quad (h \in G).$$

The representation  $\varpi_{u, \delta}^{(i)}$  of the conformal group  $G$  on  $\mathcal{E}^i(X)$  will be referred to as the *conformal representation* on differential  $i$ -forms ([13, (1.1)], see also [15] in the  $i = 0$  and  $\delta \equiv 0$  case).

Suppose that  $Y$  is an orientable submanifold. The submanifold  $Y$  is endowed with a Riemannian structure  $g|_Y$  by restriction, and we can define in a similar way a family of representations  $\varpi_{v, \varepsilon}^{(j)}$  on  $\mathcal{E}^j(Y)$  ( $v \in \mathbb{C}, \varepsilon \in \mathbb{Z}/2\mathbb{Z}, 0 \leq j \leq \dim Y$ ) of the conformal group of  $(Y, g|_Y)$ .

We consider the full group of conformal diffeomorphisms and its subgroup defined as

$$(2.3) \quad \begin{aligned} \text{Conf}(X) &:= \{\text{conformal diffeomorphisms of } (X, g)\}, \\ \text{Conf}(X; Y) &:= \{\varphi \in \text{Conf}(X) : \varphi(Y) = Y\}. \end{aligned}$$

Then there is a natural group homomorphism

$$(2.4) \quad \text{Conf}(X; Y) \rightarrow \text{Conf}(Y), \quad \varphi \mapsto \varphi|_Y.$$

The general question is to construct and classify continuous linear maps (*conformally covariant symmetry breaking operators*)  $\mathcal{E}^i(X) \rightarrow \mathcal{E}^j(Y)$  that intertwine the restriction  $\varpi_{u, \delta}^{(i)}|_{\text{Conf}(X; Y)}$  and the representation  $\varpi_{v, \varepsilon}^{(j)}$  of the subgroup  $\text{Conf}(X; Y)$  of  $\text{Conf}(X)$ . The integral operators  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  and the differential operators  $\mathbb{C}_{\lambda, \nu}^{i, j}$  are such operators for the model space  $(X, Y) = (S^n, S^{n-1})$ .

In fact, for the model space

$$(2.5) \quad (X, Y) = (S^n, S^{n-1}),$$

where  $Y$  is embedded as a totally geodesic hypersurface of the standard sphere  $X$ , the pair  $(\text{Conf}(X), \text{Conf}(X; Y))$  of conformal groups is locally isomorphic to the pair of the Lorentz groups:

$$(2.6) \quad (G, G') = (O(n+1, 1), O(n, 1)).$$

In turn, the geometry  $(X, Y)$  can be recovered as the real flag manifolds of the real reductive Lie groups  $G$  and  $G'$ . Furthermore, the conformal representations are naturally isomorphic to the principal series representations given in Section 2.1 as follows:

LEMMA 2.2 (see [13, Prop. 2.3]). *Let  $0 \leq i \leq n$ ,  $\delta \in \{\pm\}$ ,  $u \in \mathbb{C}$ . Then there is a natural isomorphism*

$$\mathcal{E}^i(S^n)_{u,\delta} \simeq I_{i+\delta}(i, u+i)$$

as representations of  $G = O(n+1, 1)$ . Here  $i+\delta$  is understood as an element of  $\mathbb{Z}/2\mathbb{Z}$ .

Thus the conformal representation  $\varpi_{u,\delta}^{(i)}$  of the group  $\text{Conf}(S^n)$  on the space  $\mathcal{E}^i(S^n)$  of differential  $i$ -forms may be identified with the so-called “ $K$ -picture” of the principal series representation  $I_\delta(i, \lambda)$  of  $G = O(n+1, 1)$  after some shift of parameters. Similarly, the conformal representation  $(\varpi_{v,\varepsilon}^{(j)}, \mathcal{E}^j(S^{n-1}))$  of the subgroup  $\text{Conf}(S^n; S^{n-1})$  is identified with  $J_\varepsilon(j, \nu)$ . Hence the integral operator  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  and its holomorphic continuation give rise to conformally covariant, symmetry breaking operators  $\mathcal{E}^i(S^n) \rightarrow \mathcal{E}^j(S^{n-1})$ .

**2.3. Applications.** As mentioned briefly in Remark 1.4, Theorem 1.3 in Section 1 leads us to two applications:

- (1) (a necessary and sufficient condition for  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  to vanish) The matrix-valued symmetry breaking operators  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  are defined as the holomorphic continuation of integral operators, and it is nontrivial to find the precise location of the zeros. By the residue formula (Theorem 1.3), we can determine the zeros of  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  (see Theorem 8.1 and Remark 8.3). This plays a crucial role in the classification problem of symmetry breaking operators ([21]).
- (2) (another approach to construct conformally covariant differential operators) It is easy to see that the integral transforms (and its analytic continuation)  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}: I_\delta(i, \lambda) \rightarrow J_\varepsilon(j, \nu)$  respect the actions of the subgroup  $G' = O(n, 1)$  of  $G = O(n+1, 1)$  ([19, 21]). Equivalently,  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  give conformally covariant, symmetry breaking operators from  $\mathcal{E}^i(S^n)$  to  $\mathcal{E}^j(S^{n-1})$ , and also from  $\mathcal{E}_c^i(\mathbb{R}^n)$  to  $\mathcal{E}^j(\mathbb{R}^{n-1})$ , as is seen in Section 2.2, and in turn, so do the residues. Thus the residue formula (Theorem 1.3) gives a proof that the differential operator  $\mathbb{C}_{\lambda,\nu}^{i,j}$  ( $j = i, i-1$ ) is a conformally covariant symmetry breaking operator from  $\mathcal{E}^i(\mathbb{R}^n)$  to  $\mathcal{E}^j(\mathbb{R}^{n-1})$ . In [13] we gave another proof for this result based on the F-method ([8, 9, 16, 18]). In contrast, the argument of the present article does not use the F-method. Since the operator  $\mathbb{C}_{\lambda,\nu}^{i,j}$  is recovered from its matrix coefficients by an elementary computation in differential geometry (*cf.* Facts 7.2 and 7.3), we may think of the residue formula (1.7) in Theorem 1.3 as yet another method to find an explicit formula for the conformally covariant, differential symmetry breaking operators  $\mathbb{C}_{\lambda,\nu}^{i,j}$  when  $j = i, i-1$ .

REMARK 2.3. (scalar-valued case) In the case where  $i = j = 0$ , the matrix-valued symmetry breaking operator  $\mathbb{C}_{\lambda,\nu}^{i,j}$  reduces to a scalar-valued one (Juhl’s operator), and we have

$$\mathbb{C}_{\lambda,\nu}^{0,0} = \frac{1}{2}\nu\tilde{\mathbb{C}}_{\lambda,\nu},$$

see [13, p. 23]. Thus Theorem 1.3 in this case coincides with [19, Thm. 12.2 (2)], see Fact 6.3. Actually, our proof of Theorem 1.3 uses the results in the scalar case.

### 3. Some identities in the Weyl algebra

A key technique in our proof of the matrix-valued residue formula (Theorem 1.3) relies on an algebraic manipulation in the Weyl algebra, for which we give a basic set-up in this section. We shall develop it for Juhl's operators in Section 5.

Let  $\mathcal{D}'(\mathbb{R}^n)$  be the space of distributions on  $\mathbb{R}^n$ , and  $\mathcal{D}'_{\{0\}}(\mathbb{R}^n)$  the subspace consisting of distributions supported at the origin. Then the Weyl algebra

$$\mathbb{C}[x, \frac{\partial}{\partial x}] \equiv \mathbb{C}[x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]$$

acts naturally on  $\mathcal{D}'(\mathbb{R}^n)$  and leaves the subspace  $\mathcal{D}'_{\{0\}}(\mathbb{R}^n)$  invariant. Let  $\mathcal{J}$  be the annihilator of the Dirac delta function  $\delta(x) = \delta(x_1, \dots, x_n)$ , namely, the kernel of the following  $\mathbb{C}[x, \frac{\partial}{\partial x}]$ -homomorphism:

$$(3.1) \quad \Psi : \mathbb{C}[x, \frac{\partial}{\partial x}] \rightarrow \mathcal{D}'(\mathbb{R}^n), \quad P \mapsto P\delta.$$

Then  $\mathcal{J}$  is the left ideal generated by the coordinate functions  $x_1, \dots, x_n$ , and  $\Psi$  induces an isomorphism of  $\mathbb{C}[x, \frac{\partial}{\partial x}]$ -modules.

$$(3.2) \quad \bar{\Psi} : \mathbb{C}[x, \frac{\partial}{\partial x}] / \mathcal{J} \xrightarrow{\sim} \mathcal{D}'_{\{0\}}(\mathbb{R}^n).$$

Our strategy is to reduce (rather complicated) computations in  $\mathcal{D}'_{\{0\}}(\mathbb{R}^n)$  to simpler algebraic ones via the isomorphism (3.2) by preparing systematically certain identities in the Weyl algebra  $\mathbb{C}[x, \frac{\partial}{\partial x}]$  modulo  $\mathcal{J}$  (see Lemmas 3.2, 4.4 and 5.2).

Before entering this part, we give the following observation:

LEMMA 3.1. *If  $P$  is a differential operator on  $\mathbb{R}^n$  with constant coefficients, then*

$$(3.3) \quad \Psi(P) * f = Pf \quad \text{for all } f \in C^\infty(\mathbb{R}^n).$$

PROOF. Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a multi-index, and we write  $P = \sum_{\alpha} a_{\alpha} \frac{\partial^{|\alpha|}}{\partial x^{\alpha}}$ . Then

$$\begin{aligned} \Psi(P) * f(x) &= \int \sum_{\alpha} a_{\alpha} \frac{\partial^{|\alpha|} \delta}{\partial y^{\alpha}}(y) f(x-y) dy \\ &= \sum_{\alpha} a_{\alpha} \int \delta(y) (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial y^{\alpha}} f(x-y) dy \\ &= \sum_{\alpha} a_{\alpha} \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} f(x). \end{aligned}$$

Hence  $\Psi(P) * f(x) = Pf$ . □

We set

$$\Delta \equiv \Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}.$$

LEMMA 3.2. *Let  $k \in \mathbb{N}$ , and  $1 \leq p, q \leq n$  with  $p \neq q$ . Then the following identities hold in the Weyl algebra  $\mathbb{C}[x, \frac{\partial}{\partial x}]$  modulo  $\mathcal{J}$ .*

- (1)  $x_p \Delta^k \equiv -2k \frac{\partial}{\partial x_p} \Delta^{k-1} \pmod{\mathcal{J}}$ .
- (2)  $x_p x_q \Delta^k \equiv 4k(k-1) \frac{\partial^2}{\partial x_p \partial x_q} \Delta^{k-2} \pmod{\mathcal{J}}$ .



$$(3) \quad x_p^2 \Delta^k \equiv 4k(k-1) \frac{\partial^2}{\partial x_p^2} \Delta^{k-2} + 2k \Delta^{k-1} \pmod{\mathcal{J}}.$$

PROOF. We denote by  $[P, Q] := PQ - QP$  the bracket of  $P, Q \in \mathbb{C}[x, \frac{\partial}{\partial x}]$  as usual. Then the assertions are derived from the following commutation relations:

$$(3.4) \quad [x_p, \Delta^k] = -2k \frac{\partial}{\partial x_p} \Delta^{k-1},$$

$$(3.5) \quad [x_p x_q, \Delta^k] = 4k(k-1) \frac{\partial^2}{\partial x_p \partial x_q} \Delta^{k-2} - 2k \left( \frac{\partial}{\partial x_p} \Delta^{k-1} x_q + \frac{\partial}{\partial x_q} \Delta^{k-1} x_p \right),$$

$$(3.6) \quad [x_p^2, \Delta^k] = 4k(k-1) \frac{\partial^2}{\partial x_p^2} \Delta^{k-2} + 2k \Delta^{k-1} - 4k \frac{\partial}{\partial x_p} \Delta^{k-1} x_p.$$

The first equation (3.4) is verified easily by induction on  $k$ . In turn, the second and third ones (3.5) and (3.6) follow from the iterated use of (3.4) and from the identity  $[AB, C] = A[B, C] + [A, C]B$ .  $\square$

#### 4. Residue formulæ of the matrix-valued Knapp–Stein intertwining operators

In this section we consider a baby-case (*i.e.*  $G = G'$  case), and apply the machinery in the previous section to find a residue formula for the matrix-valued Knapp–Stein intertwining operator  $\widetilde{\mathbb{T}}_{\lambda, n-\lambda}^i$ . Since the principal series representation  $I_\delta(i, \lambda)$  is realized in the space  $\mathcal{E}^i(\mathbb{R}^n)$  of differential forms on  $\mathbb{R}^n$ , the residue formula should be given by some familiar operators known in differential geometry. Actually, we shall see that the residue formula is proportional to Branson’s conformal covariant differential operator [2].

Our proof here illustrates an idea of the more complicated argument in later sections, where we give a proof of our main results on symmetry breaking operators  $\widetilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$ .

**4.1. Matrix-valued Knapp–Stein intertwining operators.** Suppose  $0 \leq i \leq n$ . For  $\text{Re } \lambda \gg 0$ , we define  $\widetilde{\mathcal{T}}_\lambda^i(x)$  to be an  $\text{End}_{\mathbb{C}}(\wedge^i(\mathbb{C}^n))$ -valued, locally integrable function on  $\mathbb{R}^n$  by the following formula

$$(4.1) \quad \widetilde{\mathcal{T}}_\lambda^i(x) := \frac{1}{\Gamma(\lambda - \frac{n}{2})} |x|^{2(\lambda-n)} \sigma^{(i)}(\psi_n(x)).$$

The Knapp–Stein intertwining operator [7] between principal series representations of  $G = O(n+1, 1)$ ,

$$\widetilde{\mathbb{T}}_{\lambda, n-\lambda}^i : I_\delta(i, \lambda) \rightarrow I_\delta(i, n-\lambda) \quad \text{for } \delta \in \{0, 1\},$$

is defined in the  $N$ -picture as the analytic continuation of the convolution map

$$\mathcal{E}_c^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^n), \quad f \mapsto \widetilde{\mathcal{T}}_\lambda^i * f.$$

Next, we recall that Branson–Gover’s conformally covariant differential operator (see [2])

$$\mathcal{D}_{2l}^{(i)} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^n)$$

is given by

$$(4.2) \quad \mathcal{D}_{2l}^{(i)} := \begin{cases} -(\frac{n}{2} - i) \Delta^l + l(d^* d - d d^*) \Delta^{l-1} & \text{for } l \in \mathbb{N}_+, \\ -(\frac{n}{2} - i) \text{id} & \text{for } l = 0, \end{cases}$$

where  $\Delta = -(dd^* + d^*d)$  is the Laplace–Bertrami operator acting on differential forms. We adopt the normalization of  $\mathcal{D}_{2l}^{(i)}$  given by [13, (12.1)]. In particular,  $\mathcal{D}_{2l}^{(i)}$  vanishes when  $i = \frac{n}{2}$  and  $l = 0$ . The conformally covariant property of Branson’s operator is reformulated as the intertwining property between two principal series representations in their  $N$ -picture:

$$\mathcal{D}_{2l}^{(i)} : I_\delta(i, \frac{n}{2} - l) \rightarrow I_\delta(i, \frac{n}{2} + l) \quad \text{for } \delta \in \{0, 1\}.$$

See [13, Thm. 12.2] for instance, for the classification of such operators.

Here is a relationship between  $\text{End}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n))$ -valued Knapp–Stein intertwining operators  $\widetilde{\mathbb{T}}_{\lambda, n-\lambda}^i$  and Branson’s conformally covariant operators  $\mathcal{D}_{2l}^{(i)}$ :

**THEOREM 4.1.** *Let  $0 \leq i \leq n$  and  $\lambda \in \mathbb{C}$ .*

- (1) *The matrix-valued Knapp–Stein intertwining operator  $\widetilde{\mathbb{T}}_{\lambda, n-\lambda}^i$  reduces to a differential operator if and only if  $n - 2\lambda \in 2\mathbb{N}$ .*
- (2) *Suppose  $l \in \mathbb{N}_+$ . Then we have*

$$(4.3) \quad \widetilde{\mathbb{T}}_{\lambda, n-\lambda}^i|_{\lambda=\frac{n}{2}-l} = \frac{(-1)^{l+1} \pi^{\frac{n}{2}}}{2^{2l} \Gamma(\frac{n}{2} + l + 1)} \mathcal{D}_{2l}^{(i)}.$$

Theorem 4.1 will be proved in Section 4.4 after preparing some basic results.

A different approach to prove Theorem 4.1 may be seen in a recent preprint [4, Cor. 4.4], which utilizes the Fourier transform of the distribution kernel (*cf.*, the F-method [9] for finding a residue formula).

**4.2. Residue formula of the Riesz potential  $|x|^\mu$ .** We review a classical result on the Riesz potential

$$|x|^\mu = (x_1^2 + \cdots + x_n^2)^{\frac{\mu}{2}}.$$

This is a meromorphic family of distributions on  $\mathbb{R}^n$ , and has simple poles at  $\mu = -n - 2l$  ( $l \in \mathbb{N}$ ). Thus the normalized Riesz potential on  $\mathbb{R}^n$  defined by

$$\widetilde{\mathcal{T}}_\lambda(x) := \frac{1}{\Gamma(\lambda - \frac{n}{2})} |x|^{2(\lambda-n)}$$

depends holomorphically on  $\lambda$  in the entire plane  $\mathbb{C}$ . The residue formula is classically known (see [5, Chap. (2.2)], [7] for example):

**FACT 4.2** (residue of  $|x|^\mu$ ). *Suppose  $l \in \mathbb{N}$ . Then we have:*

$$\widetilde{\mathcal{T}}_{\frac{n}{2}-l}(x) = C(l, n) \Delta^l \delta(x),$$

where we set

$$C(l, n) := \frac{(-1)^l \pi^{\frac{n}{2}}}{2^{2l} \Gamma(\frac{n}{2} + l)}.$$

**4.3. Index set  $\mathfrak{J}_{n,i}$ .** In what follows, we use the convention of index sets as below. For  $0 \leq i \leq n$ , we define

$$\mathfrak{J}_{n,i} := \{I \subset \{1, \dots, n\} : \#I = i\}.$$

For  $I = \{k_1, \dots, k_i\} \in \mathfrak{J}_{n,i}$  with  $k_1 < \cdots < k_i$ , we set

$$e_I := e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_i},$$

$$dx_I := dx_{k_1} \wedge dx_{k_2} \wedge \cdots \wedge dx_{k_i}.$$

Then  $\{e_I\}$  forms a basis of the vector space  $\Lambda^i(\mathbb{C}^n)$ , and  $\{dx_I\}$  forms a basis of  $\mathcal{E}^i(\mathbb{R}^n)$  as a  $C^\infty(\mathbb{R}^n)$ -module. We then have a natural isomorphism as  $C^\infty(\mathbb{R}^n)$ -modules:

$$(4.4) \quad C^\infty(\mathbb{R}^n) \otimes \Lambda^i(\mathbb{C}^n) \xrightarrow{\sim} \mathcal{E}^i(\mathbb{R}^n), \quad \sum f_I \otimes e_I \mapsto \sum f_I dx_I.$$

We introduce a family of quadratic polynomials  $S_{IJ}(x)$  indexed by  $I, J \in \mathfrak{J}_{n,i}$  of  $x = (x_1, \dots, x_n)$  as follows:

$$(4.5) \quad S_{IJ}(x) = \begin{cases} \sum_{k=1}^n \varepsilon_I(k) x_k^2 & \text{if } I = J, \\ \text{sgn}(I, I') x_p x_q & \text{if } \#(I \setminus J) = 1, \\ 0 & \text{if } \#(I \setminus J) \geq 2. \end{cases}$$

Here we set

$$\varepsilon_I(k) := \begin{cases} 1 & \text{for } k \in I, \\ -1 & \text{for } k \notin I. \end{cases}$$

For  $K \subset \{1, \dots, n\}$  and  $1 \leq p, q \leq n$ ,  $\text{sgn}(K; p, q) \in \{\pm 1\}$  is defined by

$$\text{sgn}(K; p, q) := (-1)^{\#\{r \in K : \min(p, q) < r < \max(p, q)\}}.$$

For  $I, I' \in \mathfrak{J}_{n,i}$  with  $\#(I \setminus I') = 1$ , we set

$$(4.6) \quad \text{sgn}(I, I') := \text{sgn}(I \cap I'; p, q),$$

where  $p \in I \setminus I'$  and  $q \in I' \setminus I$ .

We have the following.

**LEMMA 4.3.** *Suppose  $0 \leq i \leq n$ . Then the minor determinant of  $\psi_n(x) \in O(n)$  for  $I, J \in \mathfrak{J}_{n,i}$  is given by*

$$(4.7) \quad (\det \psi_n(x))_{IJ} = -\frac{1}{|x|^2} S_{IJ}(x).$$

**4.4. Proof of Theorem 4.1.** We complete the proof of Theorem 4.1 by comparing the matrix components of the both sides of the equation (4.3).

For  $I, J \in \mathfrak{J}_{n,i}$ , we define the  $(I, J)$ -component of the matrix-valued distribution  $\tilde{\mathcal{T}}_\lambda^i$  by

$$(\tilde{\mathcal{T}}_\lambda^i)_{IJ} := \langle \tilde{\mathcal{T}}_\lambda^i(e_I), e_J^\vee \rangle,$$

where  $\{e_J^\vee\}$  is the dual basis. By (4.1) and (4.7), we have

$$\begin{aligned} (\tilde{\mathcal{T}}_\lambda^i)_{IJ} &= \frac{1}{\Gamma(\lambda - \frac{n}{2})} |x|^{2(\lambda-n)} (\det \psi_n(x))_{JI} \\ &= \frac{-1}{\Gamma(\lambda - \frac{n}{2})} |x|^{2(\lambda-n-1)} S_{JI}(x) \\ &= \frac{1}{\frac{n}{2} - \lambda + 1} \tilde{\mathcal{T}}_{\lambda-1}^i(x) S_{IJ}(x). \end{aligned}$$

Applying Fact 4.2, we get the first statement of Theorem 4.1, and the following equality in  $\mathcal{D}'_{\{0\}}(\mathbb{R}^n)$ :

$$(4.8) \quad (\tilde{\mathcal{T}}_\lambda^i)_{IJ}|_{\lambda=\frac{n}{2}-l} = \frac{1}{l+1} C(l+1, n) S_{IJ}(x) \Delta^{l+1} \delta(x).$$

In order to compute the right-hand side of (4.8), we work with the Weyl algebra by using Lemma 3.2.

LEMMA 4.4. *Suppose  $l \in \mathbb{N}_+$  and  $I, J \in \mathfrak{I}_{n,i}$ . Then the following equalities hold in  $\mathbb{C}[x, \frac{\partial}{\partial x}]/\mathcal{J}$ .*

(1) *Suppose  $I = J$ .*

$$S_{IJ}(x)\Delta^{l+1} \equiv 4(l+1)l\left(\sum_{p \in I} \varepsilon_I(p) \frac{\partial^2}{\partial x_p^2}\right)\Delta^{l-1} + 2(l+1)(2i-n)\Delta^l.$$

(2) *Suppose  $\#(I - J) = 1$ . Let  $\{p, q\} := (I \cup J) \setminus (I \cap J)$ .*

$$S_{IJ}(x)\Delta^{l+1} \equiv 8l(l+1) \operatorname{sgn}(I, J) \frac{\partial^2}{\partial x_p \partial x_q} \Delta^{l-1}.$$

PROOF. Direct from the definition (4.5) of  $S_{IJ}(x)$  and from Lemma 3.2.  $\square$

By Lemma 4.4, the equation (4.8) gives the following.

LEMMA 4.5. *Suppose  $l \in \mathbb{N}_+$ .*

$$\begin{aligned} & (\tilde{\mathcal{T}}_\lambda^{(i)})_{IJ}|_{\lambda=\frac{n}{2}-l} \\ &= 4C(l+1, n) \times \begin{cases} l\left(\sum_{p \in I} \varepsilon_I(p) \frac{\partial^2}{\partial x_p^2}\right)\Delta^{l-1} + (i - \frac{n}{2})\Delta^l & \text{if } I = J, \\ 2l \operatorname{sgn}(I, J) & \text{if } \#(I \setminus J) = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, Branson's conformally covariant differential operators  $\mathcal{D}_{2l}^{(i)}$  take the following form:

LEMMA 4.6. *Suppose  $0 \leq i \leq n$  and  $l \in \mathbb{N}_+$ . For any  $\alpha = f dx_I \in \mathcal{E}^i(\mathbb{R}^n)$  with  $f \in C^\infty(\mathbb{R}^n)$  and  $I \in \mathfrak{I}_{n,i}$ , we have*

$$\begin{aligned} \mathcal{D}_{2l}^{(i)} \alpha &= \left(i - \frac{n}{2}\right)\Delta^l f + l\left(\sum_{p \in I} \varepsilon_I(p) \frac{\partial^2}{\partial x_p^2}\right)\Delta^{l-1} f dx_I \\ &\quad + 2l \sum_{\substack{p \in I \\ q \notin I}} \operatorname{sgn}(I; p, q) \frac{\partial^2}{\partial x_p \partial x_q} (\Delta^{l-1} f) dx_{I \setminus \{p\} \cup \{q\}}. \end{aligned}$$

PROOF. The formula is direct from the definition (4.2) of  $\mathcal{D}_{2l}^{(i)}$  and from the following elementary computations:

$$\begin{aligned} dd^*(f dx_I) &= - \sum_{p \in I} \frac{\partial^2 f}{\partial x_p^2} dx_I - \sum_{\substack{p \in I \\ q \notin I}} \operatorname{sgn}(I; p, q) \frac{\partial^2 f}{\partial x_p \partial x_q} dx_{I \setminus \{p\} \cup \{q\}}, \\ d^*d(f dx_I) &= - \sum_{q \notin I} \frac{\partial^2 f}{\partial x_q^2} dx_I + \sum_{\substack{p \in I \\ q \notin I}} \operatorname{sgn}(I; p, q) \frac{\partial^2 f}{\partial x_p \partial x_q} dx_{I \setminus \{p\} \cup \{q\}}. \end{aligned}$$

$\square$

PROOF OF THEOREM 4.1. We identify  $\mathcal{E}^i(\mathbb{R}^n)$  with  $C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n)$  via (4.4). Comparing the  $(I, J)$ -component of Branson's conformally covariant operators with that of the matrix-valued Knapp–Stein intertwining operator  $\tilde{\mathbb{T}}_{\lambda, n-\lambda}^i$  in Lemmas 4.5 and 4.6, we get Theorem 4.1.  $\square$

**4.5. Vanishing condition of  $\tilde{\mathbb{T}}_{\lambda, n-\lambda}^i$ .** As a biproduct of the residue formula (4.3), we obtain the vanishing condition of the matrix-valued Knapp–Stein intertwining operator  $\tilde{\mathbb{T}}_{\lambda, n-\lambda}^i$  as below.

**COROLLARY 4.7.**  $\tilde{\mathbb{T}}_{\lambda, n-\lambda}^i = 0$  if and only if  $n$  is even and  $i = \lambda = \frac{n}{2}$ .

**PROOF.** Since both  $\tilde{\mathcal{T}}_{\lambda}(x)$  and  $\sigma^{(i)}(\psi_n(x))$  are smooth in  $\mathbb{R}^n \setminus \{0\}$ , the distribution kernel  $\tilde{\mathcal{T}}_{\lambda}^i$  vanishes only when  $\text{Supp} \tilde{\mathcal{T}}_{\lambda}^i \subset \{0\}$ , or equivalently,  $n - 2\lambda \in 2\mathbb{N}$ . Suppose now  $n - 2\lambda = 2l$  for some  $l \in \mathbb{N}$ . Then it follows from Theorem 4.1 that  $\tilde{\mathcal{T}}_{\lambda, n-\lambda}^{(i)}$  vanishes if and only if  $\mathcal{D}_{2l}^{(i)} = 0$ . In turn, this happens exactly when  $i = \frac{n}{2}$  and  $l = 0$  by (4.2). Thus the corollary is proved.  $\square$

**REMARK 4.8.** When  $n$  is even, a special attention is required at the middle degree  $i = \frac{n}{2}$  with  $\lambda = i$ . See [21, Chap. Sec. 4] for representation theoretic properties of the *renormalization* of  $\tilde{\mathbb{T}}_{\lambda, n-\lambda}^i$  in this case.

## 5. Juhl's operator in the Weyl algebra

In order to prove our main results (Theorem 1.3), we need some further identities in  $\mathbb{C}[x, \frac{\partial}{\partial x}]/\mathcal{J}$  where  $\mathcal{J}$  is the left ideal generated by  $x_1, \dots, x_n$  as in Section 3.

For  $l \in \mathbb{N}$ , we define a finite-dimensional vector space of polynomials of one variable by

$$\text{Pol}_l[z]_{\text{even}} := \mathbb{C}\text{-span}\langle z^{l-2j} : 0 \leq 2j \leq l \rangle.$$

We inflate a polynomial  $g(z) \in \text{Pol}_l[z]_{\text{even}}$  to a polynomial of two variables  $s$  and  $t$  by

$$(5.1) \quad (I_l g)(s, t) := s^{\frac{l}{2}} g\left(\frac{t}{\sqrt{s}}\right).$$

If  $g(z)$  is given as  $g(z) = \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} a_j z^{l-2j}$ , then the definition (5.1) yields

$$(5.2) \quad (I_l g)(s, t) = s^{\frac{l}{2}} g\left(\frac{t}{\sqrt{s}}\right) = \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} a_j s^j t^{l-2j}.$$

We note that  $(I_l g)(s^2, t)$  is a homogeneous polynomial of  $s$  and  $t$  of degree  $l$ . The following example reveals how Juhl's conformally covariant differential operators [6, 16] arise.

**EXAMPLE 5.1.** Let  $\tilde{C}_l^\alpha(z)$  be the renormalized Gegenbauer polynomial defined by

$$\tilde{C}_l^\alpha(z) = \frac{1}{\Gamma(\alpha + \lfloor \frac{l+1}{2} \rfloor)} \sum_{k=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^k \frac{\Gamma(l-k+\alpha)}{k!(l-2k)!} (2z)^{l-2k}.$$

We adopt the normalization given in [13, (14.3)] so that  $\tilde{C}_l^\alpha(z)$  is a nonzero element in  $\text{Pol}_l[z]_{\text{even}}$  for any  $\alpha \in \mathbb{C}$ . Suppose  $\nu - \lambda \in \mathbb{N}$ . Then Juhl's conformally covariant operator  $\tilde{\mathcal{C}}_{\lambda, \nu}$  (see (1.8)) is expressed as

$$\tilde{\mathcal{C}}_{\lambda, \nu} = \text{Rest}_{x_n=0} \circ P,$$

where  $P$  is a homogeneous differential operator of order  $\nu - \lambda$  on  $\mathbb{R}^n$  given by

$$P := (I_{\nu-\lambda} \tilde{C}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}})(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}).$$

We also define a distribution on  $\mathbb{R}^n$  supported at the origin by

$$(5.3) \quad \tilde{C}_{\lambda,\nu} := (I_{\nu-\lambda} \tilde{C}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}})(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \delta(x_1, \dots, x_n).$$

This is the distribution kernel of the scalar-valued, differential symmetry breaking operator  $\tilde{C}_{\lambda,\nu}$  (see (1.8)), and was denoted by  $\tilde{K}_{\lambda,\nu}^{\mathbb{C}}$  in [19, (10.3)] when  $\nu - \lambda \in 2\mathbb{N}$ . In terms of the map  $\Psi: \mathbb{C}[x, \frac{\partial}{\partial x}] \rightarrow \mathcal{D}_{\{0\}}(\mathbb{R}^n)$  defined in (3.1), Lemma 3.1 shows

$$\begin{aligned} \tilde{C}_{\lambda,\nu} &= \Psi(P), \\ \tilde{C}_{\lambda,\nu} &= \text{Rest}_{x_n=0} \circ \tilde{C}_{\lambda,\nu} * . \end{aligned}$$

We begin with some basic computations in the Weyl algebra  $\mathbb{C}[x, \frac{\partial}{\partial x}]$  modulo the left ideal  $\mathcal{J}$ .

LEMMA 5.2. *Suppose  $g(z) \in \text{Pol}_l[z]_{\text{even}}$ . Let  $\theta = z \frac{d}{dz}$ .*

(1) *For any  $p$  with  $1 \leq p \leq n-1$ , we have*

$$x_p(I_l g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \equiv \frac{\partial}{\partial x_p} I_{l-2}((l-\theta)g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \pmod{\mathcal{J}}.$$

(2)  *$x_n(I_l g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \equiv -(I_{l-1} g')(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \pmod{\mathcal{J}}$ .*

PROOF. Applying  $\frac{\partial}{\partial s}$  and  $\frac{\partial}{\partial t}$  to the equation (5.2), we get

$$(5.4) \quad \frac{\partial}{\partial s}(I_l g)(s, t) = \frac{1}{2} I_{l-2}((l-\theta)g)(s, t) = \sum_j a_j j s^{j-1} t^{l-2j},$$

$$(5.5) \quad \frac{\partial}{\partial t}(I_l g)(s, t) = (I_{l-1} g')(s, t) = \sum_j a_j (l-2j) s^j t^{l-2j-1}.$$

(1) By Lemma 3.2,

$$(5.6) \quad x_p(I_l g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \equiv 2 \frac{\partial}{\partial x_p} \sum_j a_j j (-\Delta_{\mathbb{R}^{n-1}})^{j-1} (\frac{\partial}{\partial x_n})^{l-2j} \pmod{\mathcal{J}}.$$

By (5.4), the right-hand side of (5.6) amounts to

$$\frac{\partial}{\partial x_p} I_{l-2}((l-\theta)g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \pmod{\mathcal{J}}.$$

(2) Since  $x_n(\frac{\partial}{\partial x_n})^{l-2j} \equiv -(l-2j)(\frac{\partial}{\partial x_n})^{l-2j-1} \pmod{\mathcal{J}}$ , we have

$$(5.7) \quad x_n(I_l g)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \equiv - \sum_j a_j (l-2j) (-\Delta_{\mathbb{R}^{n-1}})^j (\frac{\partial}{\partial x_n})^{l-2j} \pmod{\mathcal{J}}.$$

By (5.5), we get the desired formula.  $\square$

LEMMA 5.3. *Suppose  $\lambda, \nu \in \mathbb{C}$  with  $\nu - \lambda \in \mathbb{N}$ . Let  $\tilde{\mathcal{C}}_{\lambda, \nu}$  be the distribution on  $\mathbb{R}^{n-1}$  defined in (5.3). Then the following equations hold.*

$$(5.8) \quad x_p \tilde{\mathcal{C}}_{\lambda, \nu} = -2 \frac{\partial}{\partial x_p} \tilde{\mathcal{C}}_{\lambda+1, \nu-1} \quad \text{for } 1 \leq p \leq n-1,$$

$$(5.9) \quad x_n \tilde{\mathcal{C}}_{\lambda, \nu} = -2\gamma \left( \lambda - \frac{n-1}{2}, \nu - \lambda \right) \tilde{\mathcal{C}}_{\lambda+1, \nu}.$$

PROOF. We recall from [1], or [13, (14.8)] and [16, (A.13), (A.14)] that the normalized Gegenbauer polynomial  $\tilde{C}_l^\alpha(z)$  satisfies the following relations

$$(5.10) \quad \left( l - z \frac{d}{dz} \right) \tilde{C}_l^\alpha(z) = -2 \tilde{C}_{l-2}^{\alpha+1}(z),$$

$$(5.11) \quad \frac{d}{dz} \tilde{C}_l^\alpha(z) = 2\gamma(\alpha, l) \tilde{C}_{l-1}^{\alpha+1}(z),$$

for  $\alpha \in \mathbb{C}$  and  $l \in \mathbb{N}_+$ .

Applying Lemma 5.2 (1) to  $g(z) = \tilde{C}_l^\alpha(z)$ , we get from (5.10) the following identity in  $\mathbb{C}[x, \frac{\partial}{\partial x}]/\mathcal{J}$ :

$$x_p (I_l \tilde{C}_l^\alpha)(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \equiv -2 \frac{\partial}{\partial x_p} (I_{l-2} \tilde{C}_{l-2}^{\alpha+1})(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \pmod{\mathcal{J}}.$$

Hence (5.8) is verified by putting  $\alpha = \lambda - \frac{n-1}{2}$  and  $l = \nu - \lambda$ . Likewise, applying Lemma 5.2 (2) to  $g(z) = \tilde{C}_{\nu-\lambda}^{\lambda - \frac{n-1}{2}}(z)$ , we get (5.9) from (5.11).  $\square$

PROPOSITION 5.4. *Suppose  $1 \leq p, q \leq n-1$ , and  $\lambda, \nu \in \mathbb{C}$ . Then we have the following equations in  $\mathcal{D}'(\mathbb{R}^n)$ .*

$$(5.12) \quad x_p x_n \tilde{\mathcal{C}}_{\lambda-1, \nu+1} = 4\gamma \left( \lambda - \frac{n-1}{2}, \nu - \lambda \right) \frac{\partial}{\partial x_p} \tilde{\mathcal{C}}_{\lambda+1, \nu}.$$

$$(5.13) \quad x_p x_q \tilde{\mathcal{C}}_{\lambda-1, \nu+1} = 4 \frac{\partial^2}{\partial x_p \partial x_q} \tilde{\mathcal{C}}_{\lambda+1, \nu-1}.$$

$$(5.14) \quad x_p^2 \tilde{\mathcal{C}}_{\lambda-1, \nu+1} = 4 \frac{\partial^2}{\partial x_p^2} \tilde{\mathcal{C}}_{\lambda+1, \nu-1} + 2 \tilde{\mathcal{C}}_{\lambda, \nu}.$$

$$(5.15) \quad \begin{aligned} x_n^2 \tilde{\mathcal{C}}_{\lambda-1, \nu+1} &= 4 \left( \lambda - \frac{n-1}{2} + \left[ \frac{\nu - \lambda + 1}{2} \right] \right) \tilde{\mathcal{C}}_{\lambda+1, \nu+1} \\ &= 2(2\nu - n + 1) \tilde{\mathcal{C}}_{\lambda, \nu} - 4 \Delta_{\mathbb{R}^{n-1}} \tilde{\mathcal{C}}_{\lambda+1, \nu-1}. \end{aligned}$$

$$(5.16) \quad Q_{n-1}(x) \tilde{\mathcal{C}}_{\lambda-1, \nu+1} = 4\nu \tilde{\mathcal{C}}_{\lambda, \nu} - 4 \left( \lambda - \frac{n-1}{2} + \left[ \frac{\nu - \lambda + 1}{2} \right] \right) \tilde{\mathcal{C}}_{\lambda+1, \nu+1}.$$

Here  $Q_{n-1}(x) = x_1^2 + \dots + x_{n-1}^2$  is a quadratic polynomial of  $x = (x_1, \dots, x_n)$ .

PROOF. The iterated use of Lemma 5.3 yields these identities except for the second equality in (5.15). The second equality is nothing but the three-term relation [13, (9.10)]

$$\left( \lambda - \frac{n-1}{2} + \left[ \frac{\nu - \lambda + 1}{2} \right] \right) \tilde{\mathcal{C}}_{\lambda+1, \nu+1} = \left( \nu - \frac{n-1}{2} \right) \tilde{\mathcal{C}}_{\lambda, \nu} - \Delta_{\mathbb{R}^{n-1}} \tilde{\mathcal{C}}_{\lambda+1, \nu-1},$$

which is derived from the following three-term relation of the Gegenbauer polynomials:

$$(5.17) \quad (\mu + a) \tilde{C}_a^\mu(t) + \tilde{C}_{a-1}^{\mu+1}(t) = \left( \mu + \left[ \frac{a+1}{2} \right] \right) \tilde{C}_a^{\mu+1}(t).$$

□

## 6. Reduction to the scalar-valued case

Our strategy to find the residue of the matrix-valued operators  $\mathbb{A}_{\lambda,\nu,\pm}^{i,j}$  is to reduce the problem to the one in the scalar-valued case by proving the following key decomposition formula:

$$(6.1) \quad \mathbb{A}_{\lambda,\nu,\varepsilon}^{i,j} = Q(\lambda, \nu)^{-1} H(x) \tilde{\mathbb{A}}_{\lambda+a,\nu-b,+}$$

see Lemma 6.2 below. The primary features of the three factors in the right-hand side are as follows:

- $Q(\lambda, \nu)$  is a polynomial of  $(\lambda, \nu)$ ;
- $H(x)$  is a  $\text{Hom}_{\mathbb{C}}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}))$ -valued polynomial of  $x$ ;
- $\tilde{\mathbb{A}}_{\lambda+a,\nu-b,+}$  is the *scalar-valued* regular symmetry breaking operator with parameter *shifted* by  $(a, b) \in \mathbb{N}^2$ .

The decomposition (6.1) is a technical heart of our trick to overcome some difficulties arising from representation theory.

**6.1. Polynomial  $g_{IJ}(x)$ .** For  $x = (x_1, \dots, x_n) \neq 0$ , we recall  $\psi_n(x) = I_n - 2|x|^{-2}x^t x$  where  $|x|^2 = x_1^2 + \dots + x_n^2$ . To find the matrix-valued polynomial  $H(x)$  in (6.1), we introduce the following polynomials  $g_{IJ}(x)$  for  $I \in \mathfrak{I}_{n,i}$ ,  $J \in \mathfrak{I}_{n-1,j}$  with  $j \in \{i-1, i\}$ :

$$(6.2) \quad g_{IJ}(x) := |x|^2 \langle \text{pr}_{i \rightarrow j} \circ \sigma^{(i)}(\psi(x)) e_I, e_J^\vee \rangle.$$

LEMMA 6.1.  $g_{IJ}(x)$  is a polynomial of  $x = (x_1, \dots, x_{n-1}, x_n)$  given by

$$(6.3) \quad g_{IJ} = \begin{cases} -S_{JI} & \text{for } j = i, \\ (-1)^i S_{J \cup \{n\}, I} & \text{for } j = i-1. \end{cases}$$

PROOF. The right-hand side of (6.2) amounts to

$$|x|^2 \sum_{K \in \mathfrak{I}_{n,i}} (\det \psi_n(x))_{KI} \langle \text{pr}_{i \rightarrow j}(e_K), e_J^\vee \rangle.$$

The projection  $\text{pr}_{i \rightarrow j}: \wedge^i(\mathbb{C}^n) \rightarrow \wedge^j(\mathbb{C}^{n-1})$  in Section 1.1 sends the basis  $\{e_I\}$  as follows.

$$\text{pr}_{i \rightarrow i}(e_I) = \begin{cases} e_I & \text{for } n \notin I, \\ 0 & \text{for } n \in I, \end{cases}$$

$$\text{pr}_{i \rightarrow i-1}(e_I) = \begin{cases} 0 & \text{for } n \notin I, \\ (-1)^{i-1} e_{I \setminus \{n\}} & \text{for } n \in I. \end{cases}$$

Therefore, we get the desired result by (4.7). □

**6.2. Matrix components of  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$ .** For  $I \in \mathfrak{I}_{n,i}$  and  $J \in \mathfrak{I}_{n-1,j}$ , we define  $(\tilde{\mathcal{A}}_{\lambda,\nu,\pm}^{i,j})_{IJ} \in \mathcal{D}'(\mathbb{R}^n)$  as the  $(I, J)$ -component of the distribution kernel  $\tilde{\mathcal{A}}_{\lambda,\nu,\pm}^{i,j}$  of the matrix-valued symmetry breaking operator  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  by

$$(\tilde{\mathcal{A}}_{\lambda,\nu,\pm}^{i,j})_{IJ} := \langle \tilde{\mathcal{A}}_{\lambda,\nu,\pm}^{i,j}(e_I), e_J^\vee \rangle.$$



We compare the matrix-valued symmetry breaking operator with the *shifted* scalar-valued one. We set

$$(6.4) \quad \tilde{\mathcal{A}}_{\lambda, \nu, +} := a_+(\lambda, \nu) |x|^{-2\nu} |x_n|^{\lambda + \nu - n}.$$

This is a distribution with holomorphic parameter  $(\lambda, \nu) \in \mathbb{C}^2$ , and is identified with  $\tilde{\mathcal{A}}_{\lambda, \nu, +}^{0,0}$ .

LEMMA 6.2. *The matrix component  $(\tilde{\mathcal{A}}_{\lambda, \nu, \pm}^{i,j})_{IJ}$  takes the following form:*

$$(6.5) \quad (\tilde{\mathcal{A}}_{\lambda, \nu, +}^{i,j})_{IJ} = \frac{2}{\lambda - \nu - 2} g_{IJ} \tilde{\mathcal{A}}_{\lambda-1, \nu+1, +},$$

$$(6.6) \quad (\tilde{\mathcal{A}}_{\lambda, \nu, -}^{i,j})_{IJ} = \frac{2}{(\lambda + \nu - n)(\lambda - \nu - 1)(\lambda - \nu - 3)} x_n g_{IJ} \tilde{\mathcal{A}}_{\lambda-2, \nu+1, +}.$$

PROOF. If  $\operatorname{Re} \lambda \gg |\operatorname{Re} \nu|$ , then the definition (1.6) of  $\tilde{\mathcal{A}}_{\lambda, \nu, +}^{i,j}$  shows that

$$\begin{aligned} (\tilde{\mathcal{A}}_{\lambda, \nu, +}^{i,j})_{IJ} &= a_+(\lambda, \nu) |x|^{-2\nu} |x_n|^{\lambda + \nu - n} \langle \operatorname{pr}_{i \rightarrow j} \circ \sigma^{(i)}(\psi(x))(e_I), e_J^\vee \rangle \\ &= a_+(\lambda, \nu) |x|^{-2\nu-2} |x_n|^{\lambda + \nu - n} g_{IJ}(x), \end{aligned}$$

where the second equality follows from the definition (6.2) of  $g_{IJ}(x)$ . In view of the definition (6.4) of the scalar-valued distribution  $\tilde{\mathcal{A}}_{\lambda, \nu, +}$ , the identity (6.5) holds for  $\operatorname{Re} \lambda \gg |\operatorname{Re} \nu|$ . A similar computation tells that the identity (6.6) holds for  $\operatorname{Re} \lambda \gg |\operatorname{Re} \nu|$ . Since  $g_{IJ}$  is a polynomial, the right-hand sides of (6.5) and (6.6) are well-defined distributions on  $\mathbb{R}^n$  that depend meromorphically on  $(\lambda, \nu) \in \mathbb{C}^2$ . On the other hand, the left-hand sides of (6.5) and (6.6) depend holomorphically on  $(\lambda, \nu) \in \mathbb{C}^2$  by Fact 1.1. Therefore, the identities (6.5) and (6.6) are proved for all  $(\lambda, \nu) \in \mathbb{C}^2$  by analytic continuation.  $\square$

**6.3. Review on the scalar-valued case.** In the case  $i = j = 0$ , the operator  $\tilde{\mathbb{A}}_{\lambda, \nu, +}^{0,0}$  is identified with scalar-valued one

$$\tilde{\mathbb{A}}_{\lambda, \nu, +} : C_c^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$$

with the distribution kernel  $\tilde{\mathcal{A}}_{\lambda, \nu, +} \equiv \tilde{\mathcal{A}}_{\lambda, \nu, +}^{0,0}$ , which was thoroughly studied in [9, 19]. In particular, we determined the residue formula for the *scalar-valued* symmetry breaking operators  $\tilde{\mathbb{A}}_{\lambda, \nu, +}$  in [9] (see also [19, Thm. 12.2 (2)]) as follows. For  $(\lambda, \nu) \in \mathbb{C}^2$  with  $\nu - \lambda \in 2\mathbb{N}$ , we set

$$(6.7) \quad q_C^A(\lambda, \nu) := \frac{(-1)^{\frac{\nu-\lambda}{2}} \left(\frac{\nu-\lambda}{2}\right)! \pi^{\frac{n-1}{2}}}{2^{\nu-\lambda} \Gamma(\nu)}.$$

FACT 6.3 (residue formula in the scalar-valued case). *Suppose  $(\lambda, \nu) \in \mathbb{C}^2$  satisfies  $\nu - \lambda \in 2\mathbb{N}$ . Then we have*

$$\tilde{\mathbb{A}}_{\lambda, \nu, +} = q_C^A(\lambda, \nu) \tilde{\mathbb{C}}_{\lambda, \nu}.$$

There are two approaches to the proof of the above residue formula. One is by using the Fourier transform of the distribution kernel (F-method) given in [9], and the other is by using the  $(K, K')$ -spectrum given in [19, Thm. 12.2 (2)].

**6.4. Reduction to the scalar-valued case.** We are ready to formulate an intermediate step for the proof of Theorem 1.3 on the residue formula of the matrix-valued regular symmetry breaking operators  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$ . We recall that  $g_{IJ}(x)$  is a quadratic polynomial of  $x$  (see Lemma 6.1) and  $\tilde{\mathcal{C}}_{\lambda,\nu}$  is a distribution on  $\mathbb{R}^n$  supported at the origin (see (5.3)). Then we have the following.

PROPOSITION 6.4. (1) *If  $\nu - \lambda = 2m$  with  $m \in \mathbb{N}$ , then*

$$(\tilde{\mathcal{A}}_{\lambda,\nu,+}^{i,j})_{IJ} = \frac{(-1)^m m! \pi^{\frac{n-1}{2}}}{2^{2m+2} \Gamma(\nu+1)} g_{IJ}(x) \tilde{\mathcal{C}}_{\lambda-1,\nu+1}.$$

(2) *If  $\nu - \lambda = 2m + 1$  with  $m \in \mathbb{N}$ , then*

$$(\tilde{\mathcal{A}}_{\lambda,\nu,-}^{i,j})_{IJ} = \frac{(-1)^m m! \pi^{\frac{n-1}{2}}}{2^{2m+5} (\lambda + \nu - n) \Gamma(\nu+1)} x_n g_{IJ}(x) \tilde{\mathcal{C}}_{\lambda-2,\nu+1}.$$

PROOF OF PROPOSITION 6.4. (1) By Fact 6.3, we have

$$\tilde{\mathcal{A}}_{\lambda-1,\nu+1,+} = q_C^A(\lambda-1, \nu+1) \tilde{\mathcal{C}}_{\lambda-1,\nu+1}$$

if  $\nu - \lambda \in \{-2, 0, 2, 4, \dots\}$ . Combining this with (6.5), we obtain

$$(\tilde{\mathcal{A}}_{\lambda,\nu,+}^{i,j})_{IJ} = \frac{2}{\lambda - \nu - 2} q_C^A(\lambda-1, \nu+1) \tilde{\mathcal{C}}_{\lambda-1,\nu+1} g_{IJ}.$$

Now a simple computation shows the first statement.

(2) Suppose  $\nu - \lambda = 2m + 1$  with  $m \in \mathbb{N}$ . By Fact 6.3, we have

$$\tilde{\mathcal{A}}_{\lambda-2,\nu+1,+} = q_C^A(\lambda-2, \nu+1) \tilde{\mathcal{C}}_{\lambda-2,\nu+1}.$$

Then (6.6) and Fact 6.3 tell that

$$\begin{aligned} (\tilde{\mathcal{A}}_{\lambda,\nu,-}^{i,j})_{IJ} &= \frac{x_n g_{IJ} \tilde{\mathcal{A}}_{\lambda-2,\nu+1,+}}{2(\lambda + \nu - n)(m+1)(m+2)} \\ &= \frac{q_C^A(\lambda-2, \nu+1)}{2(\lambda + \nu - n)(m+1)(m+2)} x_n g_{IJ} \tilde{\mathcal{C}}_{\lambda-2,\nu+1}. \end{aligned}$$

Thus the second statement is shown.  $\square$

In order to find a closed expression of the right-hand sides of the formulæ in Proposition 6.4, we shall apply in the next section the identities in  $\mathbb{C}[x, \frac{\partial}{\partial x}]/\mathcal{J}$  proved in Section 5. In particular, we shall see in Proposition 7.1 that a  $\text{Hom}_{\mathbb{C}}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}))$ -valued distribution on  $\mathbb{R}^n$  ( $j = i-1, i$ ) whose  $(I, J)$ -component is equal to  $g_{IJ}(x) \tilde{\mathcal{C}}_{\lambda-1,\nu+1}$  recovers the differential symmetry breaking operator

$$\mathbb{C}_{\lambda,\nu}^{i,j} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$$

given in Section 1.

## 7. Proof of Theorem 1.3

In this section we complete the proof of the residue formula of the *matrix-valued* symmetry breaking operator  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  by comparing the  $(I, J)$ -component of the equation (1.7).

**7.1. Matrix components of matrix-valued differential symmetry breaking operators  $\mathbb{C}_{\lambda,\nu}^{i,j}$ .** We recall the matrix-valued differential operator from Section 1:

$$\mathbb{C}_{\lambda,\nu}^{i,j}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1}) \quad (j = i - 1, i).$$

For  $I \in \mathcal{I}_{n,i}$  and  $J \in \mathcal{I}_{n-1,j}$ , we define a linear map

$$(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}: C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$$

as the  $(I, J)$ -component of the differential operator  $\mathbb{C}_{\lambda,\nu}^{i,j}$ , which is characterized by the formula

$$\mathbb{C}_{\lambda,\nu}^{i,j}(f(x)dx_I) = \sum_{J \in \mathcal{I}_{n-1,j}} ((\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}f)dx_J.$$

Then there exist uniquely distributions  $(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}$  on  $\mathbb{R}^n$  supported at the origin such that

$$(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ} = \text{Rest}_{x_n=0} \circ (\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ} *.$$

The explicit formulæ of  $(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}$  (or equivalently, of  $(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}$ ) are given in Facts 7.2 and 7.3 below.

Both the distributions  $(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}$  and  $g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1}$  are distributions on  $\mathbb{R}^n$  supported at the origin. We give its relationship as follows.

**PROPOSITION 7.1.** *Let  $j = i$  or  $i - 1$ . For any  $I \in \mathcal{I}_{n,i}$  and  $J \in \mathcal{I}_{n-1,j}$ , we have*

$$(7.1) \quad g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1} = 8(-1)^{i-j}(\mathbb{C}_{\lambda,\nu}^{i,j})_{IJ}.$$

The next two subsections will be devoted to the proof of Proposition 7.1 by using the identities in Section 5. The cases  $j = i$  and  $j = i - 1$  are treated separately in Sections 7.2 and 7.3. In Proposition 6.4, we have related the left-hand side of (7.1) with the  $(I, J)$ -component of the regular symmetry breaking operator  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$ . Thus we shall complete the proof of Theorem 1.3 based on Proposition 7.1. The last step will be given in Section 7.4.

**7.2. Proof of Proposition 7.1 for  $j = i$ .** Let  $j = i$ . Suppose  $I \in \mathcal{I}_{n,i}$  and  $J \in \mathcal{I}_{n-1,i}$ . The main cases will be the following.

**Case 1.**  $n \notin I, J = I$ .

**Case 2.**  $n \notin I, \#(J \setminus I) = 1$ . We may write  $I = K \cup \{p\}, J = K \cup \{q\}$ .

**Case 3.**  $n \in I, \#(J \setminus I) = 1$ . We may write  $I = K \cup \{n\}, J = K \cup \{q\}$ .

**FACT 7.2** ([13, Lem. 9.6]). *The  $(I, J)$ -component  $(\mathbb{C}_{\lambda,\nu}^{i,i})_{IJ}$  of the differential operator  $\mathbb{C}_{\lambda,\nu}^{i,i}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^{n-1})$  is equal to*

$$\text{Case 1.} \quad -\tilde{\mathcal{C}}_{\lambda+1,\nu-1}(\sum_{p \in I} \frac{\partial^2}{\partial x_p^2}) + \frac{1}{2}(\nu - i)\tilde{\mathcal{C}}_{\lambda,\nu},$$

$$\text{Case 2.} \quad -\text{sgn}(I; p, q)\tilde{\mathcal{C}}_{\lambda+1,\nu-1} \frac{\partial^2}{\partial x_p \partial x_q},$$

$$\text{Case 3.} \quad -\text{sgn}(I; q, n)\gamma(\lambda - \frac{n-1}{2}, \nu - \lambda)\tilde{\mathcal{C}}_{\lambda+1,\nu} \frac{\partial}{\partial x_q},$$

and is zero otherwise.

**PROOF OF PROPOSITION 7.1 FOR  $j = i$ .** It is readily seen that the both sides of (7.1) vanish unless  $(I, J)$  belongs to Cases 1–3. From now, we focus on Cases 1–3.

**Case 1.** By (4.5), the polynomial  $g_{IJ}(x)$  in (6.3) amounts to  $g_{IJ}(x) = x_n^2 - \sum_{k=1}^{n-1} \varepsilon_I(k)x_k^2$ . Then a small computation using (5.14) and (5.15) shows the following equalities in  $\mathcal{D}'_{\{0\}}(\mathbb{R}^n)$ :

$$g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1} = 4(\nu-i)\tilde{\mathcal{C}}_{\lambda,\nu} - 8\left(\sum_{k \in I} \frac{\partial^2}{\partial x_k^2}\right)\tilde{\mathcal{C}}_{\lambda+1,\nu-1}.$$

By Fact 7.2 in Case 1 and (3.3), we get

$$(7.2) \quad \text{Rest}_{x_n=0} \circ (g_{IJ}\tilde{\mathcal{C}}_{\lambda-1,\nu+1})^* = 8(\mathbb{C}_{\lambda,\nu}^{i,i})_{IJ}.$$

**Case 2.** In this case,  $g_{IJ}(x) = -2 \operatorname{sgn}(K; p, q)x_p x_q$ . By (5.13),

$$\begin{aligned} g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1} &= -2 \operatorname{sgn}(K; p, q)x_p x_q \tilde{\mathcal{C}}_{\lambda-1,\nu+1} \\ &= -8 \operatorname{sgn}(K; p, q) \frac{\partial^2}{\partial x_p \partial x_q} \tilde{\mathcal{C}}_{\lambda+1,\nu-1}. \end{aligned}$$

Comparing this with Fact 7.2 in Case 2, we get (7.2) in this case.

**Case 3.** Suppose  $n \in I$  and  $I = K \cup \{n\}$ ,  $J = K \cup \{q\}$ . Then

$$g_{IJ}(x) = 2 \operatorname{sgn}(K; q, n)x_q x_n.$$

By (5.12), we have

$$\begin{aligned} g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1} &= 2 \operatorname{sgn}(K; q, n)x_q x_n \tilde{\mathcal{C}}_{\lambda-1,\nu+1} \\ &= -8\gamma\left(\lambda - \frac{n-1}{2}, \nu - \lambda\right) \operatorname{sgn}(K; q, n) \frac{\partial}{\partial x_q} \tilde{\mathcal{C}}_{\lambda+1,\nu}. \end{aligned}$$

Again by Fact 7.2, we get (7.2) in this case.  $\square$

**7.3. Proof of Proposition 7.1 for  $j = i - 1$ .** In this section, we give a proof of Proposition 7.1 for  $j = i - 1$ . Suppose  $I \in \mathcal{I}_{n,i}$  and  $J \in \mathcal{I}_{n-1,i-1}$ . The main cases will be the following.

**Case 1.**  $n \in I$ ,  $J = I \setminus \{n\}$ .

**Case 2.**  $n \in I$ ,  $\#(J \setminus I) = 1$ . We may write  $I = K \cup \{p, n\}$ ,  $J = K \cup \{q\}$ .

**Case 3.**  $n \notin I$ ,  $J \subset I$ . We may write  $I = J \cup \{p\}$ .

FACT 7.3 ([13, Lem. 9.5]). *The  $(I, J)$ -component  $(\mathbb{C}_{\lambda,\nu}^{i,i-1})_{IJ}$  of the differential operator  $\mathbb{C}_{\lambda,\nu}^{i,i-1}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$  is equal to*

$$\text{Case 1. } (-1)^{i-1}(-\tilde{\mathcal{C}}_{\lambda+1,\nu-1}(\sum_{p \notin I} \frac{\partial^2}{\partial x_p^2}) + \frac{\nu+i-n}{2}\tilde{\mathcal{C}}_{\lambda,\nu}),$$

$$\text{Case 2. } (-1)^{i-1} \operatorname{sgn}(I; p, q)\tilde{\mathcal{C}}_{\lambda+1,\nu-1} \frac{\partial^2}{\partial x_p \partial x_q},$$

$$\text{Case 3. } \operatorname{sgn}(I; p)\gamma\left(\lambda - \frac{n-1}{2}, \nu - \lambda\right)\tilde{\mathcal{C}}_{\lambda+1,\nu} \frac{\partial}{\partial x_p},$$

and is zero otherwise.

PROOF OF PROPOSITION 7.1 FOR  $j = i - 1$ .

**Case 1.** By (4.5), we have  $g_{IJ}(x) = (-1)^i(x_n^2 + \sum_{k=1}^{n-1} \varepsilon_I(k)x_k^2)$ . Then (5.14) and (5.15) tell that

$$(-1)^i g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1,\nu+1} = 4(\nu-n+i)\tilde{\mathcal{C}}_{\lambda,\nu} - 8\sum_{k \notin I} \frac{\partial^2}{\partial x_k^2} \tilde{\mathcal{C}}_{\lambda+1,\nu-1}.$$

Comparing this with Fact 7.3 in Case 1, we get

$$(7.3) \quad \text{Rest}_{x_n=0} \circ (g_{IJ}\tilde{\mathcal{C}}_{\lambda-1,\nu+1})^* = -8(\mathbb{C}_{\lambda,\nu}^{i,i-1})_{IJ}.$$

**Case 2.** In this case,  $g_{IJ}(x) = 2(-1)^i \operatorname{sgn}(K; p, q)x_p x_q$ . By (5.13), we have

$$g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1, \nu+1} = 8(-1)^i \operatorname{sgn}(K; p, q) \frac{\partial^2}{\partial x_p \partial x_q} \tilde{\mathcal{C}}_{\lambda+1, \nu-1}.$$

Hence, Fact 7.3 in Case 2 implies (7.3).

**Case 3.** In this case,  $g_{IJ}(x) = -2 \operatorname{sgn}(I; p)x_p x_n$ . Then (5.12) implies

$$g_{IJ}(x)\tilde{\mathcal{C}}_{\lambda-1, \nu+1} = -8\gamma\left(\lambda - \frac{n-1}{2}, \nu - \lambda\right) \operatorname{sgn}(I; p) \frac{\partial}{\partial x_p} \tilde{\mathcal{C}}_{\lambda+1, \nu}.$$

Hence Fact 7.3 in Case 3 shows (7.3).  $\square$

**7.4. Proof of Theorem 1.3.** We are ready to complete the proof of Theorem 1.3.

**PROOF OF THEOREM 1.3.** Let  $\gamma = 0$ . Theorem 1.3 in this case follows from Propositions 6.4 and 7.1.

Next, let  $\gamma = 1$ . By (5.9) and Proposition 7.1,

$$\begin{aligned} x_n g_{IJ} \tilde{\mathcal{C}}_{\lambda-2, \nu+1} &= (n - \lambda - \nu) g_{IJ} \tilde{\mathcal{C}}_{\lambda-1, \nu+1} \\ &= 8(-1)^{i-j+1} (\lambda + \nu - n) (\mathcal{C}_{\lambda, \nu}^{i, j})_{IJ}. \end{aligned}$$

Hence Theorem 1.3 for  $\gamma = 1$  is shown.  $\square$

## 8. Vanishing condition of the symmetry breaking operator $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$

As a corollary of Theorem 1.3, we can determine the (isolated) zeros of the analytic continuation  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  of the integral operators. This generalizes one of the most important discoveries of [19] about the isolated zeros of the normalized operator  $\tilde{\mathbb{A}}_{\lambda, \nu, \pm}^{i, j}$  in the special case  $i = j = 0$  ([19, Thm. 1.5]). However, our method is based on the residue formula, and is different from the previous approach based on the analysis of  $(K, K')$ -spectrum. Keeping the same notation as in [19, Chap. 1], we define two subsets in  $\mathbb{Z}^2$  as below:

$$\begin{aligned} L_{\text{even}} &:= \{(-i, -j) : 0 \leq j \leq i \text{ and } i \equiv j \pmod{2}\}, \\ L_{\text{odd}} &:= \{(-i, -j) : 0 \leq j \leq i \text{ and } i \equiv j + 1 \pmod{2}\}. \end{aligned}$$

**THEOREM 8.1.** (1) Suppose  $\nu - \lambda \in 2\mathbb{N}$ .

$\tilde{\mathbb{A}}_{\lambda, \nu, +}^{i, i} = 0$  if and only if

$$(\lambda, \nu) \in \begin{cases} L_{\text{even}} & \text{for } i = 0, \\ (L_{\text{even}} \setminus \{\nu = 0\}) \cup \{(i, i)\} & \text{for } 1 \leq i \leq n - 1. \end{cases}$$

$\tilde{\mathbb{A}}_{\lambda, \nu, +}^{i, i-1} = 0$  if and only if

$$(\lambda, \nu) \in \begin{cases} (L_{\text{even}} \setminus \{\nu = 0\}) \cup \{(n - i, n - i)\} & \text{for } 1 \leq i \leq n - 1, \\ L_{\text{even}} & \text{for } i = n. \end{cases}$$

(2) Suppose  $\nu - \lambda \in 2\mathbb{N} + 1$ .

$\tilde{\mathbb{A}}_{\lambda, \nu, -}^{i, i} = 0$  if and only if

$$(\lambda, \nu) \in \begin{cases} L_{\text{odd}} & \text{for } i = 0, \\ L_{\text{odd}} \setminus \{\nu = 0\} & \text{for } 1 \leq i \leq n - 1. \end{cases}$$

$$\begin{aligned} \tilde{\mathbb{A}}_{\lambda,\nu,-}^{i,i-1} &= 0 \text{ if and only if} \\ (\lambda, \nu) &\in \begin{cases} L_{\text{odd}} \setminus \{\nu = 0\} & \text{for } 1 \leq i \leq n-1, \\ L_{\text{odd}} & \text{for } i = n. \end{cases} \end{aligned}$$

As we have already mentioned, the strategy of our proof is to apply the residue formula (Theorem 1.3). Then we can reduce the proof of Theorem 8.1 to an easier question, that is, to find a necessary and sufficient condition for the matrix-valued differential symmetry breaking operators  $\mathbb{C}_{\lambda,\nu}^{i,j}$  to vanish. The latter condition is verified immediately by the formula for the  $(I, J)$ -component of  $\mathbb{C}_{\lambda,\nu}^{i,j}$  (see Facts 7.2 and 7.3), and is described as follows:

LEMMA 8.2 ([13, Prop. 1.4 and p. 23]). *Suppose  $(\lambda, \nu) \in \mathbb{C}^2$  with  $\nu - \lambda \in \mathbb{N}$ .*

- (1) *Let  $1 \leq i \leq n$ . Then  $\mathbb{C}_{\lambda,\nu}^{i,i}$  vanishes if and only if  $\lambda = \nu = i$  or  $\nu = i = 0$ .*
- (2) *Let  $1 \leq i \leq n-1$ . Then  $\mathbb{C}_{\lambda,\nu}^{i,i-1}$  vanishes if and only if  $\lambda = \nu = n-i$  or  $\nu = n-i = 0$ .*

PROOF OF THEOREM 8.1. Suppose

$$\nu - \lambda \in 2\mathbb{N} \quad (\varepsilon = +) \quad \text{or} \quad \nu - \lambda \in 2\mathbb{N} + 1 \quad (\varepsilon = -).$$

In either case, the residue formula in Theorem 1.3 asserts that

$$\tilde{\mathbb{A}}_{\lambda,\nu,\varepsilon}^{i,j} = \frac{c}{\Gamma(\nu+1)} \mathbb{C}_{\lambda,\nu}^{i,j},$$

for some  $c \neq 0$ . Therefore we have

$$\tilde{\mathbb{A}}_{\lambda,\nu,\varepsilon}^{i,j} = 0 \quad \text{if and only if} \quad \nu \in -\mathbb{N}_+ \quad \text{or} \quad \mathbb{C}_{\lambda,\nu}^{i,j} = 0.$$

In light of Lemma 8.2, we conclude Theorem 8.1.  $\square$

REMARK 8.3. By the general results (see [21]),  $\tilde{\mathbb{A}}_{\lambda,\nu,\varepsilon}^{i,j}$  vanishes only if

$$\nu - \lambda \in 2\mathbb{N} \quad (\varepsilon = +) \quad \text{or} \quad \nu - \lambda \in 2\mathbb{N} + 1 \quad (\varepsilon = -).$$

Hence Theorem 8.1 determines precisely when the regular symmetry breaking operator  $\tilde{\mathbb{A}}_{\lambda,\nu,\pm}^{i,j}$  vanishes.

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