Shintani functions, real spherical manifolds, and symmetry breaking operators

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Abstract

For a pair of reductive groups $G \supset G'$, we prove a geometric criterion for the space $\mathrm{Sh}(\lambda, \nu)$ of Shintani functions to be finite-dimensional in the Archimedean case. This criterion leads us to a complete classification of the symmetric pairs (G, G') having finite-dimensional Shintani spaces. A geometric criterion for uniform boundedness of $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu)$ is also obtained. Furthermore, we prove that symmetry breaking operators of the restriction of smooth admissible representations yield Shintani functions of moderate growth, of which the dimension is determined for $(G, G') = (O(n+1, 1), O(n, 1))$.

Keywords: branching law, reductive group, symmetry breaking, real spherical variety, Shintani function

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Contents

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1 Introduction

The object of this article is to investigate Shintani functions for a pair of reductive groups $G \supset G'$ in the Archimedean case. Among others, we classify the reductive symmetric pairs (G, G') such that the Shintani spaces $\mathrm{Sh}(\lambda, \nu)$ are finite-dimensional for all $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal character (λ, ν). Explicit dimension formulae for the Shintani spaces of moderate growth are determined for the pair $(G, G') = (O(n+1,1), O(n,1))$.

Let G be a real reductive linear Lie group. We write $\mathfrak{g}$ for the Lie algebra of G, and $U(\mathfrak{g}_{\mathbb{C}})$ for the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, we set

$$(1.1) \qquad (L_X f)(g) := \frac{d}{dt}\Big|_{t=0} f(\exp(-tX)g), \quad (R_X f)(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)),$$

and extend these actions to those of $U(\mathfrak{g}_{\mathbb{C}})$.

We denote by $\mathfrak{Z}_G$ the $\mathbb{C}$-algebra of G-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})$. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$. Then any $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}$ gives rise to a $\mathbb{C}$-algebra homomorphism $\chi_\lambda : \mathfrak{Z}_G \to \mathbb{C}$ via the Harish-Chandra isomorphism $\mathfrak{Z}_G \xrightarrow{\sim} S(\mathfrak{j}_{\mathbb{C}})^{W(\mathfrak{j}_{\mathbb{C}})}$, where $W(\mathfrak{j}_{\mathbb{C}})$ is some finite group (see Section 3.3).

Suppose that G' is an algebraic reductive subgroup. Analogous notation will be applied to G'. For instance, $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_{G'}, \mathbb{C}) \simeq (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W(\mathfrak{j}'_{\mathbb{C}})$, $\chi_\nu \leftrightarrow \nu$, where $\mathfrak{j}'$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}'$ of G'.

We take a maximal compact subgroup K of G such that $K' := K \cap G'$ is a maximal compact subgroup. Following Murase–Sugano [19], we call:

Definition 1.1 (Shintani function)**.** We say $f \in C^\infty(G)$ is a *Shintani function* of $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν) if f satisfies the following three properties:

(1) $f(k'gk) = f(g)$ for any $k' \in K'$, $k \in K$.

(2) $R_u f = \chi_\lambda(u) f$ for any $u \in \mathfrak{Z}_G$.

(3) $L_v f = \chi_\nu(v) f$ for any $v \in \mathfrak{Z}_{G'}$.

We denote by $\mathrm{Sh}(\lambda, \nu)$ the space of Shintani functions of type (λ, ν).

For $G = G'$ and $\lambda = -\nu$, Shintani functions are nothing but Harish-Chandra's zonal spherical functions.

In this article, we provide the following three different realizations of the Shintani space $\mathrm{Sh}(\lambda, \nu)$:

- Matrix coefficients of symmetry breaking operators. (See Proposition 7.1.)
- $(K \times K')$-invariant functions on $(G \times G')/\operatorname{diag} G'$. (See Lemma 5.5.)
- G'-invariant functions on the Riemannian symmetric space $(G \times G')/(K \times K')$. (See Lemma 8.6.)

The first realization constructs Shintani functions having moderate growth (Definition 3.3) from the restriction of admissible smooth representations of G with respect to the subgroup G', whereas the second realization relates $\mathrm{Sh}(\lambda, \nu)$ with the theory of real spherical homogeneous spaces which was studied in [11, 12, 13, 14]. Via the third realization, we can apply powerful methods (*e.g.*, [9]) of harmonic analysis on Riemannian symmetric spaces for the study of Shintani functions.

By using these ideas, we give a characterization of the pair (G, G') for which the Shintani space $\mathrm{Sh}(\lambda, \nu)$ is finite-dimensional for all (λ, ν):

Theorem 1.2 (see Theorem 4.1)**.** *The following four conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Shintani function) $\mathrm{Sh}(\lambda, \nu)$ *is finite-dimensional for any pair* (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*

(ii) (Symmetry breaking) $\mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty)$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G' *(see* Section 3.2*).*

(iii) (Invariant bilinear form) *There exist at most finitely many linearly independent* G'*-invariant bilinear forms on* $\pi^\infty \otimes \tau^\infty$ *for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*

(iv) (Geometric property (PP)) *There exist minimal parabolic subgroups* P *and* P' *of* G *and* G'*, respectively, such that* PP' *is open in* G*.*

The dimension of the Shintani space $\mathrm{Sh}(\lambda, \nu)$ depends on λ and ν in general. We give a characterization of the uniform boundedness property:

Theorem 1.3. *The following four conditions on a pair of real reductive algebraic groups* $G \supset G'$ *are equivalent:*

(i) (Shintani function) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) \leq C$$

for any pair (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*

(ii) (Symmetry breaking) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) \leq C$$

for any pair $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*

(iii) (Invariant bilinear form) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty \otimes \tau^\infty, \mathbb{C}) \leq C$$

for any pair $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*

(iv) (Geometric property (BB)) *There exist Borel subgroups* B *and* B' *of the complex Lie groups* $G_{\mathbb{C}} \supset G'_{\mathbb{C}}$ *with Lie algebras* $\mathfrak{g}_{\mathbb{C}} \supset \mathfrak{g}'_{\mathbb{C}}$*, respectively, such that* BB' *is open in* $G_{\mathbb{C}}$*.*

By using the geometric criterion (PP), we give a complete classification of the reductive symmetric pairs (G, G') for which one of (therefore any of) the equivalent conditions in Theorem 1.2 is fulfilled. See Theorem 2.3 for the classification. Among them, those satisfying the uniform boundedness property in Theorem 1.3 are listed in Theorem 2.4.

Example 1.4 (see Theorems 2.3 and 2.4)**.** 1) If (G, G') is

$$\begin{aligned} &(GL(n+1,\mathbb{C}), GL(n,\mathbb{C}) \times GL(1,\mathbb{C})) \qquad &(n \geq 1),\\ &(O(n+1,\mathbb{C}), O(n,\mathbb{C})) &(n \geq 1), \end{aligned}$$

or any real form of them, then we have

$$\sup_{\lambda} \sup_{\nu} \dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) < \infty. \tag{1.2}$$

2) If (G, G') is

$$(Sp(n+1,\mathbb{C}), Sp(n,\mathbb{C}) \times Sp(1,\mathbb{C})) \qquad (n \geq 2),$$

or its split real form, then $\mathrm{Sh}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}'})$ is infinite-dimensional (see (3.4) for the notation). On the other hand, if (G, G') is a non-split real form, then $\mathrm{Sh}(\lambda, \nu)$ is finite-dimensional for all (λ, ν), but the dimension is not uniformly bounded, namely, (1.2) fails.

3) If (G, G') is

$$(GL(n+1,\mathbb{H}), GL(n,\mathbb{H}) \times GL(1,\mathbb{H})) \quad (n \geq 1),$$

then $\mathrm{Sh}(\lambda, \nu)$ is finite-dimensional for all (λ, ν), but (1.2) fails.

This article is organized as follows:

In Section 2, we give a complete list of the reductive symmetric pairs (G, G') such that the dimension of the Shintani space is finite/uniformly bounded.

After a brief review on basic results on continuous (infinite-dimensional) representations of real reductive Lie groups in Section 3, we enrich Theorem 1.2 by adding some more conditions that are equivalent to the finiteness of $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu)$ in Theorem 4.1.

The upper estimate of $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu)$ is proved in Section 5 by using the theory of *real spherical* homogeneous spaces which was established in [14].

In Section 7 we give a lower estimate of $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu)$ by using the intertwining operators constructed in Section 6.

In Section 8 we apply the theory of harmonic analysis on Riemannian symmetric spaces, and investigate the relationship between symmetry breaking operators of the restriction of admissible smooth representations of G to G' and Shintani functions. Section 9 provides an example for $(G, G') = (O(n+1,1), O(n,1))$ by using a recent work [16] with B. Speh on symmetry breaking operators.

2 Classification of (G, G') with $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) < \infty$

This section gives a complete classification of the reductive symmetric pairs (G, G') such that the dimension of the Shintani space $\mathrm{Sh}(\lambda, \nu)$ is finite/bounded for any $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν). Owing to the criteria in Theorems 1.2 and 1.3, the classification is reduced to that of (real) spherical homogeneous spaces of the form $(G \times G')/\operatorname{diag} G'$, which was accomplished in [13].

Definition 2.1 (Symmetric pair)**.** Let G be a real reductive Lie group. We say (G, G') is a *reductive symmetric pair* if G' is an open subgroup of the fixed point subgroup G^σ of some involutive automorphism σ of G.

Example 2.2. 1) (Group case) Let G_1 be a Lie group. Then the pair

$$(G, G') = (G_1 \times G_1, \operatorname{diag} G_1)$$

forms a symmetric pair with the involution $\sigma \in \mathrm{Aut}(G)$ defined by $\sigma(x, y) = (y, x)$. Since the homogeneous space G/G' is isomorphic to the group manifold G_1 with $(G_1 \times G_1)$-action from the left and the right, the pair $(G_1 \times G_1, \operatorname{diag} G_1)$ is sometimes referred to as the group case.
2) (Riemannian symmetric pair) Let K be a maximal compact subgroup of a real reductive linear Lie group G. Then the pair (G, K) is a symmetric pair because K is the fixed point subgroup of a Cartan involution θ of G. Since the homogeneous space G/K becomes a symmetric space with respect to the Levi-Civita connection of a G-invariant Riemannian metric on G/K, the pair (G, K) is sometimes referred to as a Riemannian symmetric pair.

The classification of reductive symmetric pairs was established by Berger [2] on the level of Lie algebras. Among them we list the pairs (G, G') such that the space of Shintani functions is finite-dimensional as follows:

Theorem 2.3. *Suppose (G, G') is a reductive symmetric pair. Then the following two conditions are equivalent:*

(i) $\mathrm{Sh}(\lambda, \nu)$ *is finite-dimensional for any $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν).*

(ii) *The pair $(\mathfrak{g}, \mathfrak{g}')$ of the Lie algebras is isomorphic (up to outer automorphisms) to the direct sum of the following pairs:*

A) Trivial case: $\mathfrak{g} = \mathfrak{g}'$.

B) Abelian case: $\mathfrak{g} = \mathbb{R}$, $\mathfrak{g}' = \{0\}$.

C) Compact case: $\mathfrak{g}$ *is the Lie algebra of a compact simple Lie group.*

D) Riemannian symmetric pair: $\mathfrak{g}'$ *is the Lie algebra of a maximal compact subgroup K of a non-compact simple Lie group G.*

E) Split rank one case ($\operatorname{rank}_{\mathbb{R}} G = 1$):

E1) $(\mathfrak{o}(p+q,1), \mathfrak{o}(p) + \mathfrak{o}(q,1))$ $\quad (p+q \geq 2)$.

E2) $(\mathfrak{su}(p+q,1), \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q,1)))$ $\quad (p+q \geq 1)$.

E3) $(\mathfrak{sp}(p+q,1), \mathfrak{sp}(p) + \mathfrak{sp}(q,1))$ $\quad (p+q \geq 1)$.

E4) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8,1))$.

F) Strong Gelfand pairs and their real forms:

F1) $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$ $\quad (n \geq 2)$.

F2) $(\mathfrak{o}(n+1,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))$ $\quad (n \geq 2)$.

F3) $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$ $\quad (n \geq 1)$.

F4) $(\mathfrak{su}(p+1,q), \mathfrak{u}(p,q))$ $\quad (p+q \geq 1)$.

F5) $(\mathfrak{o}(p+1,q), \mathfrak{o}(p,q))$ $\quad (p+q \geq 2)$.

G) $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{g}_1 + \mathfrak{g}_1, \operatorname{diag} \mathfrak{g}_1)$ Group case:

G1) $\mathfrak{g}_1$ *is the Lie algebra of a compact simple Lie group.*

G2) $(\mathfrak{o}(n,1) + \mathfrak{o}(n,1), \operatorname{diag} \mathfrak{o}(n,1))$ $\quad (n \geq 2)$.

H) Other cases:

H1) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$ $\quad (n \geq 1)$.

H2) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2) + \mathfrak{su}^*(2n) + \mathbb{R})$ $\quad (n \geq 1)$.

H3) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2) + \mathfrak{o}^*(2n))$ $\quad (n \geq 1)$.

H4) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q) + \mathfrak{sp}(1))$.

H5) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) + \mathbb{R})$.

We single out those pairs (G, G') having the uniform boundedness property as follows:

Theorem 2.4. *Suppose (G, G') is a reductive symmetric pair. Then the following conditions are equivalent:*

(i) *There exists a constant such that*

$$\dim_{\mathbb{C}} \operatorname{Sh}(\lambda, \nu) \leq C$$

for any $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν).

(ii) *The pair of the Lie algebras* $(\mathfrak{g}, \mathfrak{g}')$ *is isomorphic* (*up to outer automorphisms*) *to the direct sum of the pairs in* (A), (B) *and* (F1) – (F5).

Example 2.5. In connection with branching problems, some of the pairs appeared earlier in the literatures. For instance,

1) (Strong Gelfand pairs [18]) (F1), (F2).

2) (the Gross–Prasad conjecture [4]) (F2), (F5).

3) (Finite-multiplicity for tensor products [11]) (G2).

4) (Multiplicity-free restriction [1, 21]) (F1)–(F5).

Remark 2.6. The following pairs (G, G') are non-symmetric pairs such that (G, G') satisfies the condition (i) of Theorem 2.4.

$$(G, G') = (SO(8, \mathbb{C}), Spin(7, \mathbb{C})), \quad (SO(4, 4), Spin(4, 3)).$$

In fact the Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ are symmetric pairs, but the involution of $\mathfrak{g}$ does not lift to the group G.

Proof of Theorem 2.3. Direct from Theorem 1.2 and [13, Theorem 1.3]. □

Proof of Theorem 2.4. Direct from Theorem 1.3 and [13, Proposition 1.6]. □

3 Preliminary results

We begin with a quick review of some basic results on (infinite-dimensional) continuous representations of real reductive Lie groups.

3.1 Continuous representations and the Frobenius reciprocity

By a continuous representation π of a Lie group G on a topological vector space V we shall mean that $\pi : G \to GL_{\mathbb{C}}(V)$ is a group homomorphism such that the induced map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. We say π is a (continuous) Hilbert [Banach, Fréchet, $\cdots$] representation if V is a Hilbert [Banach, Fréchet, $\cdots$] space. We note that a continuous Hilbert representation is not necessarily a unitary representation; a Hilbert representation π of G is said to be a unitary representation provided that all the operators $\pi(g)$ $(g \in G)$ are unitary.

Suppose π is a continuous representation of G on a Banach space V. A vector $v \in V$ is said to be *smooth* if the map $G \to V$, $g \mapsto \pi(g)v$ is of C^∞-class. Let V^∞ denote the space of smooth vectors of the representation (π, V). Then V^∞ carries a Fréchet topology with a family of semi-norms $\|v\|_{i_1 \cdots i_k} := \|d\pi(X_{i_1}) \cdots d\pi(X_{i_k})v\|$, where $\{X_1, \cdots, X_n\}$ is a basis of $\mathfrak{g}$. Then V^∞ is a G-invariant subspace of V, and we obtain a continuous Fréchet representation (π^∞, V^∞) of G.

Suppose that G' is another Lie group. If π and τ are Hilbert representations of G and G' on the Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\tau$, respectively, then we can define a continuous Hilbert representation $\pi \boxtimes \tau$ of the direct product group on the Hilbert completion on $\mathcal{H}_\pi \widehat{\otimes} \mathcal{H}_\tau$ of the pre-Hilbert space $\mathcal{H}_\pi \otimes \mathcal{H}_\tau$.

Suppose further that G' is a subgroup of G. Then we may regard π as a representation of G' by the restriction. The resulting representation is denoted by $\pi|_{G'}$. The restriction of the outer tensor product $\pi \boxtimes \tau$ of $G \times G'$ to the subgroup $\operatorname{diag} G' = \{(g', g') : g' \in G'\}$ is denoted by $\pi \otimes \tau$. By a *symmetry breaking operator* we mean a continuous G'-homomorphism from the representation space of π to that of τ. We write $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ for the vector space of continuous G'-homomorphisms. Analogous notation is applied to smooth representations.

For the convenience of the reader, we review some basic properties of the restriction:

Lemma 3.1. *Suppose that π and τ are Hilbert representations of G and G' on Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\tau$, respectively.*

1) *There is a canonical injective homomorphism:*

$$\operatorname{Hom}_{G'}(\pi|_{G'}, \tau) \hookrightarrow \operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty), \qquad T \mapsto T|_{\mathcal{H}_\pi^\infty}. \tag{3.1}$$

2) *Let $\tau^\vee$ be the contragredient representation of τ. Then we have a canonical isomorphism:*

$$\operatorname{Hom}_{G'}(\pi|_{G'}, \tau) \simeq \operatorname{Hom}_{G'}(\pi \otimes \tau^\vee, \mathbb{C}). \tag{3.2}$$

3) *There is a canonical injective homomorphism if G and G' are real reductive:*

$$\operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \hookrightarrow \operatorname{Hom}_{G'}(\pi^\infty \otimes (\tau^\vee)^\infty, \mathbb{C}).$$

Proof. 1) See [14, Lemma 5.1], for instance. 2) We have a canonical isomorphism between $\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}_\pi, \mathcal{H}_\tau)$ and $\operatorname{Hom}_{\mathbb{C}}(\mathcal{H}_\pi \widehat{\otimes} \mathcal{H}_\tau^\vee, \mathbb{C})$, where $\operatorname{Hom}_{\mathbb{C}}(\ ,\ )$ denotes the space of continuous linear maps. Taking G'-invariant elements, we get (3.2). 3) See [1, Lemma A.0.8], for instance. $\square$

Proposition 3.2 (Frobenius reciprocity)**.** *Let H be a closed subgroup of a Lie group G. Suppose that π is a continuous representation of G on a topological vector space V. Then there is a canonical bijection*

$$\mathrm{Hom}_H(\pi|_H, \mathbb{C}) \simeq \mathrm{Hom}_G(\pi, C(G/H)), \qquad \lambda \mapsto T \tag{3.3}$$

defined by

$$T(v)(g) = \lambda(\pi(g^{-1})v) \qquad v \in V.$$

Furthermore, if π^∞ is a smooth representation, then we have

$$\mathrm{Hom}_H(\pi^\infty|_H, \mathbb{C}) \simeq \mathrm{Hom}_G(\pi^\infty, C^\infty(G/H)).$$

Proof. The linear map $T : V \to C(G/H)$ is continuous because $G \times V \to V$, $(g, v) \mapsto \pi(g^{-1})v$ is continuous. The last statement follows because $G \to V$, $g \mapsto \pi(g)^{-1}v$ is a C^∞-map. □

3.2 Admissible representations

In this subsection we review some basic terminologies for Harish-Chandra modules.

Let G be a real reductive linear Lie group, and K a maximal compact subgroup of G. Let $\mathcal{HC}$ denote the category of Harish-Chandra modules where the objects are $(\mathfrak{g}, K)$-modules of finite length, and the morphisms are $(\mathfrak{g}, K)$-homomorphisms.

Let π be a continuous representation of G on a Fréchet space V. Suppose that π is of finite length, namely, there are at most finitely many closed G-invariant subspaces in V. We say π is *admissible* if

$$\dim \mathrm{Hom}_K(\tau, \pi|_K) < \infty$$

for any irreducible finite-dimensional representation τ of K. We denote by V_K the space of K-finite vectors. Then $V_K \subset V^\infty$ and the Lie algebra $\mathfrak{g}$ leaves V_K invariant. The resulting $(\mathfrak{g}, K)$-module on V_K is called the underlying $(\mathfrak{g}, K)$-module of π, and will be denoted by π_K.

An admissible representation (π, V) is said to be *spherical* if V contains a nonzero K-fixed vector, or equivalently, the underlying $(\mathfrak{g}, K)$-module V_K contains a nonzero K-fixed vector.

A vector $v \in V$ is said to be *cyclic* if the vector space $\mathbb{C}$-span$\{\pi(g)v : g \in G\}$ is dense in V. If W is a proper G-invariant closed subspace of V, then $v \mod W$ is a cyclic vector in the quotient representation on V/W. For a K-finite vector v, v is cyclic in π if and only if v is cyclic in the underlying $(\mathfrak{g}, K)$-module π_K in the sense that $U(\mathfrak{g}_{\mathbb{C}})v = V_K$.

3.3 Harish-Chandra isomorphism

We review the standard normalization of the Harish-Chandra isomorphism of the $\mathbb{C}$-algebra $\mathfrak{Z}_G$, where we recall from Introduction that

$$\mathfrak{Z}_G = U(\mathfrak{g}_{\mathbb{C}})^G \equiv \{u \in U(\mathfrak{g}_{\mathbb{C}}) : \operatorname{Ad}(g)u = u \quad \text{for all} \quad g \in G\}.$$

For a connected G, $\mathfrak{Z}_G$ is equal to the center $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$.

Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{j}_{\mathbb{C}} = \mathfrak{j} \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathfrak{j}_{\mathbb{C}}^{\vee} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{j}_{\mathbb{C}}, \mathbb{C})$. We set

$$W(\mathfrak{j}_{\mathbb{C}}) := N_{\widetilde{G}}(\mathfrak{j}_{\mathbb{C}})/Z_{\widetilde{G}}(\mathfrak{j}_{\mathbb{C}}),$$

where $\widetilde{G}$ is the group generated by $\operatorname{Ad}(G)$ and the group $\operatorname{Int}(\mathfrak{g}_{\mathbb{C}})$ of inner automorphisms. For a connected G, $W(\mathfrak{j}_{\mathbb{C}})$ is the Weyl group for the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$.

Fix a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, and write $\mathfrak{n}_{\mathbb{C}}^+$ for the sum of the root spaces belonging to $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$, and $\mathfrak{n}_{\mathbb{C}}^-$ for $\Delta^-(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. We set

$$\rho_{\mathfrak{g}} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})} \alpha \in \mathfrak{j}_{\mathbb{C}}^{\vee}. \tag{3.4}$$

Let $\gamma' : U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{j}_{\mathbb{C}}) \simeq S(\mathfrak{j}_{\mathbb{C}})$ be the projection to the second factor of the decomposition $U(\mathfrak{g}_{\mathbb{C}}) = (\mathfrak{n}_{\mathbb{C}}^- U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}_{\mathbb{C}}^+) \oplus U(\mathfrak{j}_{\mathbb{C}})$. Then we have the Harish-Chandra isomorphism

$$\mathfrak{Z}_G = U(\mathfrak{g}_{\mathbb{C}})^G \underset{\gamma}{\xrightarrow{\sim}} S(\mathfrak{j}_{\mathbb{C}})^{W(\mathfrak{j}_{\mathbb{C}})}, \tag{3.5}$$

where $\gamma : U(\mathfrak{g}_{\mathbb{C}}) \to S(\mathfrak{j}_{\mathbb{C}})$ is defined by $\langle \gamma(u), \lambda \rangle = \langle \gamma'(u), \lambda - \rho_{\mathfrak{g}} \rangle$ for all $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}$.

Then any element $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}$ gives a $\mathbb{C}$-algebra homomorphism $\chi_\lambda : \mathfrak{Z}_G \to \mathbb{C}$ via the isomorphism (3.5), and $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda' = w\lambda$ for some $w \in W(\mathfrak{j}_{\mathbb{C}})$. This correspondence yields a bijection:

$$\operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_G, \mathbb{C}) \simeq \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}}), \quad \chi_\lambda \leftrightarrow \lambda. \tag{3.6}$$

In our normalization, the $\mathfrak{Z}_G$-infinitesimal character of the trivial representation $\mathbf{1}$ of G is given by $\rho_{\mathfrak{g}}$.

For $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}})$, we set

$$\begin{aligned} C^\infty(G; \chi_\lambda^R) &:= \{f \in C^\infty(G) : R_u f = \chi_\lambda(u) f \quad \text{for any} \quad u \in \mathfrak{Z}_G\}, \\ C^\infty(G; \chi_\lambda^L) &:= \{f \in C^\infty(G) : L_u f = \chi_\lambda(u) f \quad \text{for any} \quad u \in \mathfrak{Z}_G\}. \end{aligned}$$

Then we have $C^\infty(G; \chi_\lambda^R) = C^\infty(G; \chi_{-\lambda}^L)$.

Let H be a closed subgroup of G. Since the action of $\mathfrak{Z}_G$ on $C^\infty(G)$ via R (and via L) commutes with the right H-action, R_u and L_u ($u \in \mathfrak{Z}_G$) induce differential operators on G/H. Thus, for $\lambda \in \mathfrak{j}_{\mathbb{C}}^\vee/W(\mathfrak{j}_{\mathbb{C}})$, we can define

$$\begin{aligned} C^\infty(G/H;\chi_\lambda^R) &:= \{f \in C^\infty(G/H) : R_u f = \chi_\lambda(u) f \quad \text{for any} \quad u \in \mathfrak{Z}_G\}, \\ C^\infty(G/H;\chi_\lambda^L) &:= \{f \in C^\infty(G/H) : L_u f = \chi_\lambda(u) f \quad \text{for any} \quad u \in \mathfrak{Z}_G\}. \end{aligned}$$

3.4 Shintani functions of moderate growth

Without loss of generality, we may and do assume that a real reductive linear Lie group G is realized as a closed subgroup of $GL(n,\mathbb{R})$ such that G is stable under the transpose of matrix $g \mapsto {}^t g$ and $K = O(n) \cap G$. For $g \in G$ we define a map $\|\cdot\| : G \to \mathbb{R}$ by

$$\|g\| := \|g \oplus {}^t g^{-1}\|_{\mathrm{op}}$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm of $M(2n,\mathbb{R})$. A continuous representation π of G on a Fréchet space V is said to be of *moderate growth* if for each continuous semi-norm $|\cdot|$ on V there exist a continuous semi-norm $|\cdot|'$ on V and a constant $d \in \mathbb{R}$ such that

$$|\pi(g)u| \le \|g\|^d |u|' \quad \text{for } g \in G, u \in V.$$

For any admissible representation $(\pi, \mathcal{H})$ such that $\mathcal{H}$ is a Banach space, the smooth representation $(\pi^\infty, \mathcal{H}^\infty)$ has moderate growth. We say $(\pi^\infty, \mathcal{H}^\infty)$ is an *admissible smooth representation*. By the Casselman–Wallach globalization theory, there is a canonical equivalence of categories between the category $\mathcal{HC}$ of $(\mathfrak{g}, K)$-modules of finite length and the category of admissible smooth representations of G ([22, Chapter 11]). In particular, the Fréchet representation π^∞ is uniquely determined by its underlying $(\mathfrak{g}, K)$-module. We say π^∞ is the *smooth globalization* of $\pi_K \in \mathcal{HC}$.

For simplicity, by an *irreducible smooth representation* we shall mean an irreducible admissible smooth representation of G.

Definition 3.3. A smooth function f on G is said to have *moderate growth* if f satisfies the following three properties:

(1) f is right K-finite.

(2) f is $\mathfrak{Z}_G$-finite.

(3) There exists a constant $d \in \mathbb{R}$ (depending on f) such that if $u \in U(\mathfrak{g}_{\mathbb{C}})$ then there exists $C \equiv C(u)$ satisfying

$$|(R_u f)(x)| \le C \|x\|^d \qquad (x \in G).$$

We denote by $C^\infty_{\mathrm{mod}}(G)$ the space of all $f \in C^\infty(G)$ having moderate growth.

If (π, V) is an admissible representation of moderate growth, then the matrix coefficient $G \to \mathbb{C}$, $g \mapsto \langle \pi(g)v, u\rangle$ belongs to $C^\infty_{\mathrm{mod}}(G)$ for any $v \in V_K$ and any linear functional u of the Fréchet space V.

We define the space of Shintani functions of moderate growth by

$$\mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) := \mathrm{Sh}(\lambda, \nu) \cap C^\infty_{\mathrm{mod}}(G). \tag{3.7}$$

4 Finite-multiplicity properties of branching laws

We are ready to make a precise statement of Theorem 1.2, and enrich it by adding some more equivalent conditions. The main results of this section is Theorem 4.1.

4.1 Finite-multiplicity properties of branching laws

Theorem 4.1. *The following twelve conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (PP) *There exist minimal parabolic subgroups P and P' of G and G', respectively, such that PP' is open in G.*

(ii) (Sh) $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) < \infty$ *for any pair (λ, ν) of $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters.*

(iii) ($\mathrm{Sh}_{\mathrm{mod}}$) $\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) < \infty$ *for any pair (λ, ν) of $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters.*

(iv) $(\mathrm{Sh}_{\mathrm{mod}})_{\mathbf{1}}$ $\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}'}) < \infty$.

(v) $(\infty \downarrow)$ $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty$ *for any pair $(\pi^\infty, \tau^\infty)$ of admissible smooth representations of G and G'.*

(vi) $(\infty \downarrow)_K$ $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty$ *for any pair $(\pi^\infty, \tau^\infty)$ of admissible smooth representations of G and G' such that π^∞ and $(\tau^\infty)^\vee$ have cyclic spherical vectors.*

(vii) $(\mathcal{H} \downarrow)$ $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ *for any pair (π, τ) of admissible Hilbert representations of G and G'.*

(viii) $(\mathcal{H} \downarrow)_K$ $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) < \infty$ *for any pair (π, τ) of admissible Hilbert representations of G and G' such that π and $\tau^\vee$ have cyclic spherical vectors.*

(ix) $(\infty\otimes)$ $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi^\infty \otimes \tau^\infty, \mathbb{C}) < \infty$ *for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*

(x) $(\infty\otimes)_K$ $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi^\infty \otimes \tau^\infty, \mathbb{C}) < \infty$ *for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G' *such that* π^∞ *and* τ^∞ *have cyclic spherical vectors.*

(xi) $(\mathcal{H}\otimes)$ $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi \otimes \tau, \mathbb{C}) < \infty$ *for any pair* (π, τ) *of admissible Hilbert representations of* G *and* G'*.*

(xii) $(\mathcal{H}\otimes)_K$ $\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi \otimes \tau, \mathbb{C}) < \infty$ *for any pair* (π, τ) *of admissible Hilbert representations of* G *and* G' *such that* π *and* τ *have cyclic spherical vectors.*

4.2 Outline of the proof of Theorem 4.1

The following implications are obvious:

$$\text{(ii) (Sh)} \Rightarrow \text{(iii) } (\mathrm{Sh}_{\mathrm{mod}}) \Rightarrow \text{(iv) } (\mathrm{Sh}_{\mathrm{mod}})_{\mathbf{1}}.$$

By Lemma 3.1, we have the following inclusive relations and isomorphism.

$$\operatorname{Hom}_{G'}(\pi^\infty \otimes (\tau^\vee)^\infty, \mathbb{C}) \supset \operatorname{Hom}_{G'}(\pi^\infty, \tau^\infty) \supset \operatorname{Hom}_{G'}(\pi, \tau) \simeq \operatorname{Hom}_{G'}(\pi \otimes \tau^\vee, \mathbb{C}).$$

In turn, we have the obvious implications and equivalences as below.

$$\begin{array}{ccccccc}
\text{(ix) } (\infty\otimes) & \Longrightarrow & \text{(v) } (\infty\downarrow) & \Longrightarrow & \text{(vii) } (\mathcal{H}\downarrow) & \Longleftrightarrow & \text{(xi) } (\mathcal{H}\otimes) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{(x) } (\infty\otimes)_K & \Longrightarrow & \text{(vi) } (\infty\downarrow)_K & \Longrightarrow & \text{(viii) } (\mathcal{H}\downarrow)_K & \Longleftrightarrow & \text{(xii) } (\mathcal{H}\otimes)_K.
\end{array}$$

The remaining non-trivial implications are

$$\begin{array}{c}
\text{(viii) } (\mathcal{H}\downarrow)_K \text{ or (iv) } (\mathrm{Sh}_{\mathrm{mod}})_{\mathbf{1}} \\
\Downarrow \\
\text{(i) (PP)} \\
\Downarrow \\
\text{(ii) (Sh) and (ix) } (\infty\otimes).
\end{array}$$

We discuss the geometric property (PP) in Section 5.1. Then the implications

$$\text{(i) (PP)} \Rightarrow \text{(ii) (Sh) and (ix) } (\infty\otimes)$$

are given in Propositions 5.6 and 5.7, respectively.

The implication

$$\text{(viii)}\ \ (\mathcal{H}\downarrow)_K \Rightarrow \text{(i)}\ \ \text{(PP)}$$

is proved in Proposition 6.5, and the implication

$$\text{(iv)}\ \ (\mathrm{Sh}_{\mathrm{mod}})_{\mathbf{1}} \Rightarrow \text{(i)}\ \ \text{(PP)}$$

is proved in Corollary 7.3.

The relationship of $\mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty)$ (symmetry breaking operators) and $\mathrm{Sh}(\lambda, \nu)$ (Shintani functions) will be discussed in Sections 7 and 8.

4.3 Invariant trilinear forms

Suppose that π_i^∞ are admissible smooth representations of a Lie group G on Fréchet spaces $\mathcal{H}_i^\infty$ $(i = 1, 2, 3)$. A continuous trilinear form

$$T : \mathcal{H}_1^\infty \times \mathcal{H}_2^\infty \times \mathcal{H}_3^\infty \to \mathbb{C}$$

is *invariant* if

$$T(\pi_1^\infty(g)u_1, \pi_2^\infty(g)u_2, \pi_3^\infty(g)u_3) = T(u_1, u_2, u_3) \quad \text{for all} \ \ g \in G \ \ \text{and} \ \ u_i \in \mathcal{H}_i^\infty \ (i = 1, 2, 3).$$

Corollary 4.2. 1) *Suppose G is a real reductive Lie group. Then the following four conditions on G are equivalent:*

(i) $(G \times G \times G)/\operatorname{diag} G$ *is real spherical as a* $(G \times G \times G)$*-space.*

(ii) (Shintani functions in the group case) *The space* $\mathrm{Sh}((\lambda_1, \lambda_2), \lambda_3)$ *of Shintani functions for* $(G \times G, \operatorname{diag} G)$ *is finite-dimensional for any triple of* $\mathfrak{Z}_G$*-infinitesimal characters* λ_1*,* λ_2 *and* λ_3*.*

(iii) (Symmetry breaking for the tensor product) *For any triple of admissible smooth representations* π_1^∞*,* π_2^∞*, and* π_3^∞ *of* G*,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi_1^\infty \otimes \pi_2^\infty, \pi_3^\infty) < \infty.$$

(iv) (Invariant trilinear form) *For any triple of admissible smooth representations* π_1^∞*,* π_2^∞ *and* π_3^∞ *of* G*, the space of invariant trilinear forms is finite-dimensional.*

2) *Suppose that G is a simple Lie group. Then one of (therefore any of) the above four equivalent conditions is fulfilled if and only if either G is compact or $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(n, 1)$ $(n \geq 2)$.*

Proof. The first and second statements are special cases of Theorems 4.1 and 2.3, respectively. □

Remark 4.3. As in (vi) and (viii) of Theorem 4.1, the conditions (iii) and (iv) of Corollary 4.2 are equivalent to the analogous statements by replacing π_j^∞ ($j = 1, 2, 3$) with spherical ones.

Remark 4.4. The equivalence (i) $\Leftrightarrow$ (ii) was first formulated in [11] with a sketch of proof.

Example 4.5. For $G = O(n, 1)$, a meromorphic family of invariant trilinear forms for spherical principal series representations was constructed in [3].

5 Real spherical manifolds and Shintani functions

In this section, we regard Shintani functions as smooth functions on the homogeneous space $(G \times G')/\operatorname{diag} G'$, and apply the theory of real spherical homogeneous spaces [14]. In particular, we give a proof of the implication (i) (PP) $\Rightarrow$ (ii) (Sh) and (ix) ($\infty\otimes$) in Theorem 4.1 (see Proposition 5.6).

5.1 Real spherical homogeneous spaces and (PP)

A complex manifold $X_{\mathbb{C}}$ with action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In the real setting, in search of a good framework for global analysis on homogeneous spaces which are broader than the usual (*e.g.* symmetric spaces), we proposed to call:

Definition 5.1 ([11])**.** Let G be a real reductive Lie group. We say a smooth manifold X with G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X.

The significance of this geometric property is its application to the finite-multiplicity property in the regular representation of G on $C^\infty(X)$, which was proved by using the theory of hyperfunctions and regular singularities of a system of partial differential equations:

Proposition 5.2 ([14, Theorem A and Theorem 2.2])**.** *Suppose G is a real reductive linear Lie group, and H is a closed subgroup. If the homogeneous space G/H is real* *spherical, then the regular representation of G on the Fréchet space $C^\infty(G/H;\chi^L_\lambda)$ is admissible for any $\mathfrak{Z}_G$-infinitesimal character $\lambda \in \mathfrak{j}^\vee_{\mathbb{C}}/W(\mathfrak{j}_{\mathbb{C}})$. In particular,*

$$\mathrm{Hom}_G(\pi^\infty, C^\infty(G/H)) \text{ is finite-dimensional}$$

for any smooth admissible representation π^∞ of G.

Suppose that G' is an algebraic reductive subgroup of G. Let P' be a minimal parabolic subgroup of G'.

Definition 5.3 ([14])**.** We say the pair (G, G') satisfies (PP) if one of the following five equivalent conditions is satisfied.

(PP1) $(G \times G')/\operatorname{diag} G'$ is real spherical as a $(G \times G')$-space.

(PP2) G/P' is real spherical as a G-space.

(PP3) G/P is real spherical as a G'-space.

(PP4) G has an open orbit in $G/P \times G/P'$ via the diagonal action.

(PP5) There are finitely many G-orbits in $G/P \times G/P'$ via the diagonal action.

The above five equivalent conditions are determined only by the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. Therefore we also say that the pair $(\mathfrak{g}, \mathfrak{g}')$ of Lie algebras satisfies (PP).

Next we consider another property, to be denoted by (BB), which is stronger than (PP). Let $G_{\mathbb{C}}$ be a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and $G'_{\mathbb{C}}$ a subgroup of $G_{\mathbb{C}}$ with complexified Lie algebra $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{g}' \otimes_{\mathbb{R}} \mathbb{C}$. Let B and B' be Borel subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$, respectively.

Definition 5.4. We say the pair (G, G') (or the pair $(\mathfrak{g}, \mathfrak{g}')$) satisfies (BB) if one of the following five equivalent conditions is satisfied:

(BB1) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ is spherical as a $(G_{\mathbb{C}} \times G'_{\mathbb{C}})$-space.

(BB2) $G_{\mathbb{C}}/B'$ is spherical as a $G_{\mathbb{C}}$-space.

(BB3) $G_{\mathbb{C}}/B$ is real spherical as a $G'_{\mathbb{C}}$-space.

(BB4) $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/B \times G_{\mathbb{C}}/B'$ via the diagonal action.

(BB5) There are finitely many $G_{\mathbb{C}}$-orbits in $G_{\mathbb{C}}/B \times G_{\mathbb{C}}/B'$ via the diagonal action.

The above five equivalent conditions are determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$. It follows from [14, Lemmas 4.2 and 5.3] that we have an implication

$$\text{(BB)} \Rightarrow \text{(PP)}.$$

5.2 Shintani functions and real spherical homogeneous spaces

We return to Shintani functions for the pair $G \supset G'$. Let $(\lambda, \nu) \in \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}}) \times (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W(\mathfrak{j}'_{\mathbb{C}})$. We begin with an elementary and useful point of view:

Lemma 5.5. *The multiplication map*

$$\varphi : G \times G' \to G, \qquad (g, h) \mapsto gh^{-1}$$

induces the following linear isomorphism

$$\varphi^* : \mathrm{Sh}(\lambda, \nu) \xrightarrow{\sim} \mathrm{Hom}_{K \times K'}(\mathbf{1} \boxtimes \mathbf{1}, C^\infty((G \times G')/\operatorname{diag} G'; \chi^L_{\lambda,\nu})),$$

where $\mathbf{1}$ *denotes the trivial one-dimensional representation of the group* K *(or that of* K'*).*

Proof. The pull-back of functions

$$\varphi^* : C^\infty(G) \xrightarrow{\sim} C^\infty((G \times G')/\operatorname{diag} G')$$

satisfies

$$L_X L_Y(\varphi^* f) = \varphi^*(L_X R_Y f) \qquad \text{for all } X \in \mathfrak{g},\ Y \in \mathfrak{g}' \text{ and } f \in C^\infty(G).$$

Hence φ^* maps $\mathrm{Sh}(\lambda, \nu)$ onto the space of $(K \times K')$-invariant functions of $C^\infty((G \times G')/\operatorname{diag} G'; \chi^L_{\lambda,\nu})$. $\square$

Proposition 5.6. *If* (G, G') *satisfies* (PP)*, then* $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) < \infty$ *for any pair* (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*

Proof. Since (G, G') satisfies (PP1), the regular representation on the Fréchet space $C^\infty((G \times G')/\operatorname{diag} G'; \chi^L_{\lambda,\nu})$ is admissible as a representation of the direct product group $G \times G'$ by Proposition 5.2. Therefore, Proposition 5.6 follows from Lemma 5.5. $\square$

Proposition 5.7. *If* (G, G') *satisfies* (PP)*, then* $\mathrm{Hom}_{G'}(\pi^\infty \otimes \tau^\infty, \mathbb{C})$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*

Proof. Since $(G \times G')/\operatorname{diag} G'$ is real spherical,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G \times G'}(\pi^\infty \boxtimes \tau^\infty, C^\infty(G \times G'/\operatorname{diag} G')) < \infty$$

by Proposition 5.2. Therefore Proposition 5.7 follows from the Frobenius reciprocity (Proposition 3.2). $\square$

6 Construction of intertwining operators

In this section we give lower bounds of the dimension of the space of symmetry breaking operators for the restriction of admissible Hilbert representations.

6.1 A generalization of the Poisson integral transform

We fix some general notation. Let H be a closed subgroup of G. Given a finite-dimensional representation τ of H on a vector space W_τ, we denote by $\mathcal{W}_\tau$ the G-equivariant vector bundle $G \times_H W_\tau$ over the homogeneous space G/H. Then we have a representation of G naturally on the space of sections

$$\mathcal{F}(G/H;\tau) \equiv \mathcal{F}(G/H;\mathcal{W}_\tau) \simeq \{f \in \mathcal{F}(G) \otimes W : f(\cdot h) = \tau(h)^{-1} f(\cdot) \ \text{ for } \ h \in H\},$$

where $\mathcal{F} = \mathcal{A}$, C^∞, $\mathcal{D}'$, or $\mathcal{B}$ denote the sheaves of analytic functions, smooth functions, distributions, or hyperfunctions, respectively.

Remark 6.1. We shall regard distributions as generalized functions à la Gelfand (or a special case of hyperfunctions à la Sato) rather than continuous linear forms on $C_c^\infty(G/H, \mathcal{W}_\tau)$.

We define a one-dimensional representation of H by

$$\chi_{G/H} : H \to \mathbb{R}^\times, \quad h \mapsto |\det(\mathrm{Ad}_\#(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1},$$

where $\mathrm{Ad}_\#(h)$ is the quotient representation of the adjoint representation $\mathrm{Ad}(h) \in GL_{\mathbb{R}}(\mathfrak{g})$. The bundle of volume densities of $X = G/H$ is given as a G-homogeneous line bundle $\Omega_X \simeq G \times_H \chi_{G/H}$. Then the dualizing bundle of $\mathcal{W}_\tau$ is given, as a homogeneous vector bundle, by

$$\mathcal{W}_\tau^* := (G \times_H W_\tau^\vee) \otimes \Omega_X \simeq G \times_H \tau^*,$$

where $(\tau^\vee, W_\tau^\vee)$ denotes the contragredient representation of (τ, W_τ), and τ^* is a complex representation of H given by

$$\tau^* := \tau^\vee \otimes \chi_{G/H}. \tag{6.1}$$

Suppose now that Q is a parabolic subgroup of a real reductive Lie group G, and $Q = LN$ a Levi decomposition. By an abuse of notation we write $\mathbb{C}_{2\rho}$ for $\chi_{G/Q}$. Then $\mathbb{C}_{2\rho}$ is trivial on the nilpotent subgroup N, and the restriction of $\mathbb{C}_{2\rho}$ to the Levi part L coincides with the one-dimensional representation defined by

$$L \to \mathbb{R}^\times \quad l \mapsto |\det(\mathrm{Ad}(l) : \mathfrak{n} \to \mathfrak{n})|.$$

In view of the isomorphism $K/(Q \cap K) \xrightarrow{\sim} G/Q$, $\mathcal{W}_\tau$ may be regarded as a K-equivariant vector bundle over $K/(Q \cap K)$. Then there exist a K-invariant Hermitian vector bundle structure on $\mathcal{W}_\tau$ and a K-invariant Radon measure on $K/(Q \cap K)$, and we can define a Hilbert representation of G on the Hilbert space $L^2(G/Q; \tau)$ of square integrable sections of $\mathcal{W}_\tau$. The underlying $(\mathfrak{g}, K)$-module of $\mathcal{F}(G/Q; \tau)$ does not depend on the choice of $\mathcal{F} = \mathcal{A}, C^\infty$, $\mathcal{D}'$, $\mathcal{B}$, or L^2, and will be denoted by $E(G/Q; \tau)$.

We denote by $\widehat{G}_f$ and $\widehat{L}_f$ the sets of equivalence classes of finite-dimensional irreducible representations over $\mathbb{C}$ of the groups G and L, respectively. Then there is an injective map

$$\widehat{G}_f \hookrightarrow \widehat{L}_f, \qquad \sigma \mapsto \lambda(\sigma)$$

such that σ is the unique quotient of the $(\mathfrak{g}, K)$-module $E(G/Q; \lambda(\sigma))$. We note that $\lambda(\mathbf{1}) = \mathbb{C}_{2\rho}$.

Here is a Hilbert space analog of [14, Theorem 3.1] which was formulated in the category of $(\mathfrak{g}, K)$-modules (and was proved in the case where Q is a minimal parabolic subgroup of G).

Proposition 6.2. *Let Q be a parabolic subgroup of G, and H a closed subgroup of G. Suppose that there are m disjoint H-invariant open subsets in the real generalized flag variety G/Q. Then*

$$\dim \operatorname{Hom}_G(L^2(G/Q; \lambda(\sigma)), C(G/H; \tau)) \geq m \dim \operatorname{Hom}_H(\sigma|_H, \tau),$$

for any finite-dimensional representations σ and τ of G and H, respectively. In particular, we have

$$\dim \operatorname{Hom}_G(L^2(G/Q, \Omega_{G/Q}), C(G/H)) \geq m.$$

A key of the proof is the construction of integral intertwining operators formulated as follows:

Proposition 6.3. *Let τ and ζ be finite-dimensional representations of H and Q, respectively. We set $\zeta^* = \zeta^\vee \otimes \mathbb{C}_{2\rho}$. Let $(\mathcal{F}, \mathcal{F}')$ be one of the pairs*

$$(\mathcal{A}, \mathcal{B}),\ (C^\infty, \mathcal{D}'),\ (L^2, L^2),\ (\mathcal{D}', C^\infty), \quad \text{or} \quad (\mathcal{B}, \mathcal{A}).$$

Then there is a canonical injective map

$$\Phi : (\mathcal{F}'(G/Q; \zeta^*) \otimes \tau)^H \hookrightarrow \operatorname{Hom}_G(\mathcal{F}(G/Q; \zeta), C(G/H; \tau)).$$

Proof. The proof is essentially the same with that of [14, Lemma 3.2] which treated the case where $(\mathcal{F}, \mathcal{F}') = (\mathcal{A}, \mathcal{B})$ and where Q is a minimal parabolic subgroup of G. For the sake of completeness, we repeat the proof with appropriate modifications.

The natural G-invariant non-degenerate bilinear form

$$\langle\ ,\ \rangle : \mathcal{F}(G/Q;\zeta) \times \mathcal{F}'(G/Q;\zeta^*) \to \mathbb{C}$$

induces an injective G-homomorphism

$$\Psi : \mathcal{F}'(G/Q;\zeta^*) \hookrightarrow \mathrm{Hom}_G(\mathcal{F}(G/Q;\zeta), C(G))$$

by

$$\Psi(\chi)(u)(g) := \langle \pi(g)^{-1}u, \chi\rangle \quad \text{for} \quad \chi \in \mathcal{F}'(G/Q;\zeta^*) \quad \text{and} \quad u \in \mathcal{F}(G/Q;\zeta),$$

where π is the regular representation of G on $\mathcal{F}(G/Q;\zeta)$.

Taking the tensor product with the finite-dimensional representation τ followed by collecting H-invariant elements, we get the linear map Φ in Proposition 6.3. $\square$

Example 6.4 (Poisson integral transform)**.** We apply Proposition 6.3 in the following setting:

$(\mathcal{F}, \mathcal{F}') = (\mathcal{B}, \mathcal{A})$,

$H = K$,

Q : a minimal parabolic subgroup of G,

τ : the trivial one-dimensional representation $\mathbf{1}$ of K,

ζ : a one-dimensional representation of Q such that $\zeta|_{Q\cap K}$ is trivial.

Then $\mathcal{A}(G/Q;\zeta^*)$ is identified with $\mathcal{A}(K/(Q\cap K))$ as a K-module, and the constant function $\mathbf{1}_K$ on $K/(Q\cap K)$ gives rise to an element of $(\mathcal{A}(G/Q;\zeta^*) \otimes \tau)^K$. Then $\mathcal{P}_\mu := \Phi(\mathbf{1}_K)$ in Proposition 6.3 coincides with the Poisson integral transform for the Riemannian symmetric space G/K ([8, Chapter 2]):

$$\mathcal{P}_\mu : \mathcal{B}(G/Q;\zeta) \to C(G/K), \quad f \mapsto (\mathcal{P}_\mu f)(g) = \int_K f(gk)dk.$$

See Proposition 8.5 for the preceding results on the image of $\mathcal{P}_\mu$.

Proof of Proposition 6.2. The proof is parallel to that of [14, Theorem 3.17].

Let U_i $(i = 1, 2, \cdots, m)$ be disjoint H-invariant open subsets in G/Q. We define

$$\chi_i(g) := \begin{cases} 1 & \text{if } g \in U_i, \\ 0 & \text{if } g \notin U_i. \end{cases}$$

Then $\chi_i \in L^2(G/Q) \simeq L^2(K/Q\cap K)$ $(i = 1, \cdots, m)$, and they are H-invariant and linearly independent.

We take linearly independent elements $u_1, \cdots, u_n$ in $\mathrm{Hom}_H(\sigma|_H, \tau)$. Taking the dual of the surjective $(\mathfrak{g}, K)$-homomorphism $E(G/Q; \lambda(\sigma)) \twoheadrightarrow \sigma$, we have an injective $(\mathfrak{g}, K)$-homomorphism $\sigma^\vee \hookrightarrow E(G/Q; \lambda(\sigma)^*) \subset \mathcal{A}(G/Q; \lambda(\sigma)^*)$. Hence we may regard $u_j \in \mathrm{Hom}_H(\sigma|_H, \tau) \simeq (\sigma^\vee \otimes \tau)^H$ as H-invariant elements of $\mathcal{A}(G/Q; \lambda(\sigma)^*) \otimes \tau$. Then $\chi_i u_j \in (L^2(G/Q; \lambda(\sigma)^*) \otimes \tau)^H$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ are linearly independent.

Proposition 6.2 now follows from Proposition 6.3 with $(\mathcal{F}, \mathcal{F}') = (L^2, L^2)$. $\square$

Proposition 6.5. *Let Q and Q' be parabolic subgroups of G and G'. Suppose that there are m disjoint Q'-invariant open sets in G/Q. Then*

$$\dim \mathrm{Hom}_{G'}(L^2(G/Q, \Omega_{G/Q}), L^2(G'/Q')) \geq m.$$

Proof. We apply Proposition 6.2 to $(G \times G', \operatorname{diag} G', \mathbf{1}, \mathbf{1}, Q \times Q')$ for (G, H, σ, τ, Q). Then we have

$$\dim \mathrm{Hom}_{G \times G'}(\pi \boxtimes \tau, C(G \times G' / \operatorname{diag} G')) \geq m,$$

where π is the Hilbert representation of G on $L^2(G/Q, \Omega_{G/Q})$ and τ is that of G' on $L^2(G'/Q', \Omega_{G'/Q'})$.

By Proposition 3.2, we have

$$\dim \mathrm{Hom}_{G'}((\pi \boxtimes \tau)|_{\operatorname{diag} G'}, \mathbb{C}) \geq m.$$

By Lemma 3.1 (2), we get the required lower bound. $\square$

6.2 Realization of small representations

We end this section with a refinement of [14, Theorem A (2)] which was formulated originally in the category of $(\mathfrak{g}, K)$-modules and was proved when Q is a minimal parabolic subgroup of G.

Definition 6.6. Let Q be a parabolic subgroup of a real reductive Lie group G. Let π be an irreducible admissible representation of G, and π_K the underlying $(\mathfrak{g}, K)$-module. We say π (or π_K) belongs to *Q-series* if π_K occurs as a subquotient of the induced $(\mathfrak{g}, K)$-module $E(G/Q; \tau)$ for some finite-dimensional representation τ of Q.

By Harish-Chandra's subquotient theorem [5], all irreducible admissible representations of G belong to P-series where P is a minimal parabolic subgroup of G. Loosely speaking, the larger a parabolic subgroup Q is, the "smaller" a representation belonging to Q-series becomes, as the following lemma indicates:

Lemma 6.7. *If π_K belongs to Q-series, then its Gelfand–Kirillov dimension, to be denoted by* $\mathrm{DIM}(\pi_K)$*, satisfies*

$$\mathrm{DIM}(\pi_K) \le \dim G/Q.$$

The following result formulates that if a subgroup H is "small enough" then the space $(\pi^{-\infty})^H$ of H-invariant distribution vectors of π can be of infinite dimension even for a "small" admissible representations π:

Corollary 6.8. *Let H be an algebraic subgroup of G, and Q a parabolic subgroup of G. Assume that H does not have an open orbit in G/Q. Then for any algebraic finite-dimensional representation τ of H, there exists an irreducible admissible Hilbert representation π of G such that π satisfies the following two properties:*

- *π belongs to Q-series,*
- $\dim \mathrm{Hom}_G(\pi, C(G/H;\tau)) = \infty$.

In particular, $\dim \mathrm{Hom}_G(\pi^\infty, C^\infty(G/H;\tau)) = \dim \mathrm{Hom}_{\mathfrak{g},K}(\pi_K, \mathcal{A}(G/H;\tau)) = \infty$.

Proof. There exist infinitely many disjoint H-invariant open sets in G/Q if H does not have an open orbit in G/Q (see [14, Lemma 3.5]). Hence Corollary 6.8 follows from Proposition 6.2 because there exist at most finitely many irreducible subquotients in the Hilbert representation of G on $L^2(G/Q, \Omega_{G/Q})$. □

Corollary 6.9. *Let $G \supset G'$ be algebraic real reductive Lie groups and Q and Q' parabolic subgroups of G and G', respectively. Assume that Q' does not have an open orbit in G/Q. Then there exist irreducible admissible Hilbert representations π and τ of G and G', respectively, such that (π, τ) satisfies the following two properties:*

- *π belongs to Q-series, τ belongs to Q'-series.*
- $\dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) = \infty$.

In particular, $\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) = \infty$.

Proof. Corollary 6.9 follows from Proposition 6.5. Since the argument is similar to the proof of Corollary 6.8, and we omit it. □

7 Symmetry breaking operators and construction of Shintani functions

In this section we construct Shintani functions of moderate growth from symmetry breaking operators of the restriction of admissible smooth representations.

Proposition 7.1. *Let π^∞ be a spherical, admissible smooth representation of G, and τ^∞ that of G'. Suppose that π^∞ and τ^∞ have $\mathfrak{Z}_G$ and $\mathfrak{Z}_{G'}$-infinitesimal characters λ and $-\nu$, respectively.*
1) *Let $\mathbf{1}_\pi$ and $\mathbf{1}_{\tau^\vee}$ be non-zero spherical vectors of π_K and $\tau^\vee_{K'}$, respectively. Then there is a natural linear map*

$$\mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) \to \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu), \qquad T \mapsto F \tag{7.1}$$

defined by

$$F(g) := \langle T \circ \pi^\infty(g)\mathbf{1}_\pi, \mathbf{1}_{\tau^\vee}\rangle \quad \text{for} \ \ g \in G.$$

2) *Assume that the spherical vectors $\mathbf{1}_\pi$ and $\mathbf{1}_{\tau^\vee}$ are cyclic in π_K and $\tau^\vee_{K'}$, respectively. Then* (7.1) *is injective. In particular, if both π^∞ and τ^∞ are irreducible,* (7.1) *is injective.*

Remark 7.2. In the setting of Proposition 7.1, if we drop the assumption that $\mathbf{1}_\pi$ is cyclic, then the homomorphism (7.1) may not be injective. In fact, we shall see in Section 9 that there is a countable set of (λ, ν) for which the following three conditions are satisfied:

- $\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) = 2$,
- $\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) = 1$,
- $\mathbf{1}_{\tau^\vee}$ is cyclic in $\tau^\vee$.

Proof of Proposition 7.1. 1) Since $T \in \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty)$, the function $F \in C^\infty(G)$ satisfies

$$\begin{aligned} F(hg) =& \langle \tau^\infty(h) \circ \tau \circ \pi^\infty(g)\mathbf{1}_\pi, \mathbf{1}_{\tau^\vee}\rangle \\ =& \langle T \circ \pi^\infty(g)\mathbf{1}_\pi, (\tau^\vee)^\infty(h^{-1})\mathbf{1}_{\tau^\vee}\rangle \end{aligned} \tag{7.2}$$

for all $h \in G'$ and $g \in G$. Therefore we have

$$\begin{aligned} F(k'gk) =& F(g) \quad \text{for} \ \ k' \in K' \ \ \text{and} \ \ k \in K, \\ (L_Y F)(g) =& \langle T \circ \pi^\infty(g)\mathbf{1}_\pi, d\tau^\vee(Y)\mathbf{1}_{\tau^\vee}\rangle \quad \text{for} \ \ Y \in \mathfrak{g}' \subset U(\mathfrak{g}'_{\mathbb{C}}), \\ (R_X F)(g) =& \langle T \circ \pi^\infty(g)d\pi(X)\mathbf{1}_\pi, \mathbf{1}_{\tau^\vee}\rangle \quad \text{for} \ \ X \in \mathfrak{g} \subset U(\mathfrak{g}_{\mathbb{C}}). \end{aligned}$$

Since $u \in \mathfrak{Z}_G$ acts on π^∞ as the scalar multiple of $\chi_\lambda(u)$, we have $d\pi^\infty(u)\mathbf{1}_\pi = \chi_\lambda(u)\mathbf{1}_\pi$, and therefore $R_u F = \chi_\lambda(u)F$. Likewise, for $v \in \mathfrak{Z}_{G'}$, we have $d(\tau^\vee)^\infty(v)\mathbf{1}_{\tau^\vee} = \chi_\nu(v)\mathbf{1}_{\tau^\vee}$, and thus $L_v F = \chi_\nu(v)F$. Hence $F \in \mathrm{Sh}(\lambda, \nu)$.

Let V_π^∞ and W_τ^∞ be the representation spaces of π^∞ and τ^∞, respectively. First we find a continuous seminorm $|\cdot|_1$ on W_τ^∞ and a constant C_1 such that

$$|\langle w, \mathbf{1}_{\tau^\vee}\rangle| \le C_1 |w|_1 \quad \text{for any} \;\; w \in W_\tau^\infty.$$

Second, since $T : V_\pi^\infty \to W_\tau^\infty$ is continuous, there exist a continuous seminorm $|\cdot|_2$ on V_π^∞ and a constant C_2 such that

$$|Tv|_1 \le C_2 |v|_2 \quad \text{for any} \;\; v \in V_\pi^\infty.$$

Third, since π^∞ has moderate growth, there exist constants $C_3 > 0$, $d \in \mathbb{R}$ and a continuous seminorm $|\cdot|_3$ on V_π^∞ such that

$$|\pi^\infty(g) d\pi(u)\mathbf{1}_\pi|_2 \le C_3 |d\pi^\infty(u)\mathbf{1}_\pi|_3 \|g\|^d \quad \text{for any } g \in G \text{ and for any } u \in U(\mathfrak{g}_\mathbb{C}).$$

Therefore $(R_u F)(g) = \langle T \circ \pi^\infty(g) d\pi(u)\mathbf{1}_\pi, \mathbf{1}_{\tau^\vee}\rangle$ satisfies the following inequality:

$$|(R_u F)(g)| \le C_1 C_2 C_3 |d\pi^\infty(u)\mathbf{1}_\pi|_3 \|g\|^d \quad \text{for any} \;\; g \in G.$$

Hence $F \in C^\infty(G)$ has moderate growth.
2) Suppose $F \equiv 0$. Since $\mathbf{1}_{\tau^\vee}$ is a cyclic vector, we have $T \circ \pi^\infty(g)\mathbf{1}_\pi = 0$ for any $g \in G$ by (7.2). Since $\mathbf{1}_\pi$ is a cyclic vector, we have $T = 0$. Therefore the map (7.1) is injective. □

Corollary 7.3. *Suppose that there are m disjoint P'-invariant open sets in G/P. Then*

$$\dim_\mathbb{C} \mathrm{Sh}_{\mathrm{mod}}(\rho_\mathfrak{g}, \rho_{\mathfrak{g}'}) \ge m. \tag{7.3}$$

In particular, if $\mathrm{Sh}_{\mathrm{mod}}(\rho_\mathfrak{g}, \rho_{\mathfrak{g}'})$ is finite-dimensional, then the pair (G, G') of reductive groups satisfies (PP).

Proof. We denote by π the Hilbert representation of G on $L^2(G/P, \Omega_{G/P})$, and by τ that of G' on $L^2(G'/P')$. By Proposition 6.5, we have

$$\dim_\mathbb{C} \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \ge m.$$

On the other hand, since both π and $\tau^\vee$ contain spherical cyclic vectors, we have

$$\dim_\mathbb{C} \mathrm{Sh}_{\mathrm{mod}}(\rho_\mathfrak{g}, -\rho_{\mathfrak{g}'}) \ge \dim_\mathbb{C} \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$$

from Proposition 7.1. Combining these inequalities with (3.1), we have obtained

$$\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\rho_{\mathfrak{g}}, -\rho_{\mathfrak{g}'}) \geq m.$$

Since $-\rho_{\mathfrak{g}'}$ is conjugate to $\rho_{\mathfrak{g}'}$ by the longest element of the Weyl group $W(\mathfrak{j}'_{\mathbb{C}})$, we have proved (7.3).

Finally, if (G, G') does not satisfy (PP), then there exist infinitely many disjoint P'-invariant open sets in G/P, and therefore we get $\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}'}) = \infty$ from (7.3). □

8 Boundary values of Shintani functions

In this section we realize Shintani functions as joint eigenfunctions of invariant differential operators on the Riemannian symmetric space $X = (G \times G')/(K \times K')$, and then as hyperfunctions on the minimal boundary $Y = (G \times G')/(P \times P')$ of the compactification of X. The main results of this section are Theorems 8.1 and 8.2. We prove these theorems in Sections 8.4 and 8.5, respectively, after giving a brief summary of the preceding results of harmonic analysis on Riemannian symmetric spaces in Sections 8.2 and 8.3.

8.1 Symmetry breaking of principal series representations

Denote by θ the Cartan involution of the Lie algebra $\mathfrak{g}$ corresponding to the maximal compact subgroup K of G. We take a maximal abelian subspace $\mathfrak{a}$ in the vector space $\{X \in \mathfrak{g} : \theta X = -X\}$, and set

$$W(\mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

We fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$, and define a minimal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{l} + \mathfrak{n},$$

where $\mathfrak{l} := Z_{\mathfrak{g}}(\mathfrak{a}) = \{X \in \mathfrak{a} : [H, X] = 0 \ \text{ for all } \ H \in \mathfrak{a}\}$, $\mathfrak{m} := \mathfrak{l} \cap \mathfrak{k}$, and $\mathfrak{n}$ is the sum of the root spaces for all $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let $P = MAN$ be the minimal parabolic subgroup of G with Lie algebra $\mathfrak{p}$.

We take a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{m}$. Then $\mathfrak{j} := \mathfrak{t} + \mathfrak{a}$ is a maximally split Cartan subalgebra of $\mathfrak{g}$. We fix a positive system $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let $\rho_{\mathfrak{n}} \in \mathfrak{a}^\vee$ be half the sum of the elements in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ counted with multiplicities, and $\rho_{\mathfrak{l}} \in \mathfrak{t}_{\mathbb{C}}^\vee$ that of $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

The positive systems $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ determine naturally a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then we have

$$\rho_{\mathfrak{g}} = \rho_{\mathfrak{l}} + \rho_{\mathfrak{n}} \in \mathfrak{j}_{\mathbb{C}}^{\vee} = \mathfrak{t}_{\mathbb{C}}^{\vee} + \mathfrak{a}_{\mathbb{C}}^{\vee},$$

where we regard $\mathfrak{t}_{\mathbb{C}}^{\vee}$ and $\mathfrak{a}_{\mathbb{C}}^{\vee}$ as subspaces of $\mathfrak{j}_{\mathbb{C}}^{\vee}$ via the direct sum decomposition $\mathfrak{j} = \mathfrak{t}+\mathfrak{a}$. Then $\rho_{\mathfrak{l}} + \mathfrak{a}_{\mathbb{C}}^{\vee} = \rho_{\mathfrak{g}} + \mathfrak{a}_{\mathbb{C}}^{\vee}$ is an affine subspace of $\mathfrak{j}_{\mathbb{C}}^{\vee}$.

Analogous notation is applied to the reductive subgroup G'. In particular, $\mathfrak{j}' = \mathfrak{t}'+\mathfrak{a}'$ is a maximally split Cartan subalgebra of $\mathfrak{g}'$.

We recall $(\lambda, \nu) \in \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}}) \times (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W(\mathfrak{j}'_{\mathbb{C}}) \simeq \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_G \times \mathfrak{Z}_{G'}, \mathbb{C})$. Let us begin with a non-vanishing condition for $\mathrm{Sh}(\lambda, \nu)$.

Theorem 8.1. *If* $\mathrm{Sh}(\lambda, \nu) \neq \{0\}$*, then*

$$\lambda \in W(\mathfrak{j}_{\mathbb{C}})(\rho_{\mathfrak{l}} + \mathfrak{a}_{\mathbb{C}}^{\vee}) \quad \textit{and} \quad \nu \in W(\mathfrak{j}'_{\mathbb{C}})(\rho_{\mathfrak{l}'} + (\mathfrak{a}'_{\mathbb{C}})^{\vee}). \tag{8.1}$$

We shall give a proof of Theorem 8.1 in Section 8.4.

Next we consider a construction of Shintani functions under the assumption (8.1). Suppose $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}$ satisfies $\lambda - \rho_{\mathfrak{l}} \in \mathfrak{a}_{\mathbb{C}}^{\vee}$. Then there exists $\lambda_+ \in \mathfrak{a}_{\mathbb{C}}^{\vee}$ such that λ_+ satisfies the following two conditions:

$$\lambda_+ - \rho_{\mathfrak{n}} = w(\lambda - \rho_{\mathfrak{l}}) \quad \text{for some} \quad w \in W(\mathfrak{a}). \tag{8.2}$$

$$\mathrm{Re}\langle \lambda_+ - \rho_{\mathfrak{n}}, \alpha \rangle \geq 0 \quad \text{for any } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}). \tag{8.3}$$

Similarly, suppose $\nu \in (\mathfrak{j}'_{\mathbb{C}})^{\vee}$ satisfies $\nu - \rho_{\mathfrak{l}'} \in (\mathfrak{a}'_{\mathbb{C}})^{\vee}$. Then there exists $\nu_- \in (\mathfrak{a}'_{\mathbb{C}})^{\vee}$ satisfying the following two conditions:

$$\nu_- - \rho_{\mathfrak{n}'} = w'(-\nu + \rho_{\mathfrak{l}'}) \quad \text{for some} \quad w' \in W(\mathfrak{a}').$$

$$\mathrm{Re}\langle \nu_- - \rho_{\mathfrak{n}'}, \alpha \rangle \leq 0 \quad \text{for any } \alpha \in \Sigma^+(\mathfrak{g}', \mathfrak{a}'). \tag{8.4}$$

Theorem 8.2. *Suppose that* $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}$ *and* $\nu \in (\mathfrak{j}'_{\mathbb{C}})^{\vee}$ *satisfy* $\lambda+\rho_{\mathfrak{l}} \in \mathfrak{a}_{\mathbb{C}}^{\vee}$ *and* $\nu+\rho_{\mathfrak{l}'} \in (\mathfrak{a}'_{\mathbb{C}})^{\vee}$*. Let* λ_+ *and* ν_- *be defined as above.*

1) *There is a natural injective linear map*

$$\mathrm{Hom}_{G'}(C^\infty(G/P; \lambda_+), C^\infty(G'/P'; \nu_-)) \hookrightarrow \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu). \tag{8.5}$$

2) *If* G*,* G' *are classical groups, then* (8.5) *is a bijection:*

$$\mathrm{Hom}_{G'}(C^\infty(G/P; \lambda_+), C^\infty(G'/P'; \nu_-)) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu). \tag{8.6}$$

We shall prove Theorem 8.2 in Section 8.5.

Remark 8.3. As the proof shows, the bijection (8.6) holds for generic (λ, ν) even when G or G' are exceptional groups.

8.2 Invariant differential operators

In this and next subsections, we give a quick review of the preceding results of harmonic analysis on Riemannian symmetric spaces. We denote by $\mathbb{D}(G/K)$ the $\mathbb{C}$-algebra consisting of all G-invariant differential operators on the Riemannian symmetric space G/K. It is isomorphic to a polynomial ring of $(\dim_{\mathbb{R}} \mathfrak{a})$-generators. More precisely, let $\gamma' : U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}})$ be the projection to the second factor of the decomposition $U(\mathfrak{g}_{\mathbb{C}}) = (\mathfrak{k}_{\mathbb{C}} U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}_{\mathbb{C}}) \oplus U(\mathfrak{a}_{\mathbb{C}})$. Then we have the Harish-Chandra isomorphism

$$\mathbb{D}(G/K) \underset{R}{\overset{\sim}{\leftarrow}} U(\mathfrak{g}_{\mathbb{C}})^K / U(\mathfrak{g}_{\mathbb{C}})^K \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}} \underset{\gamma}{\overset{\sim}{\rightarrow}} S(\mathfrak{a}_{\mathbb{C}})^{W(\mathfrak{a})}, \tag{8.7}$$

where $\gamma : U(\mathfrak{g}_{\mathbb{C}}) \to S(\mathfrak{a}_{\mathbb{C}})$ is defined by

$$\langle \gamma(u), \lambda \rangle = \langle \gamma'(u), \lambda - \rho_{\mathfrak{n}} \rangle \qquad \text{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^{\vee},$$

which is a generalization of (3.5), see [8, Chapter II]. Through (8.7), we have a bijection

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}(G/K), \mathbb{C}) \simeq \mathfrak{a}_{\mathbb{C}}^{\vee}/W(\mathfrak{a}), \qquad \psi_\mu \leftrightarrow \mu, \tag{8.8}$$

given by $\psi_\mu(R_v) = \langle \gamma(v), \mu \rangle = \langle \gamma'(v), \lambda - \rho_{\mathfrak{n}} \rangle$ for $v \in U(\mathfrak{g}_{\mathbb{C}})^K$.

Comparing the two bijections $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_G, \mathbb{C}) \simeq \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}})$ (see (3.6)) and (8.8) via the $\mathbb{C}$-algebra homomorphism

$$\mathfrak{Z}_G \subset U(\mathfrak{g}_{\mathbb{C}})^K \overset{R}{\to} \mathbb{D}(G/K), \tag{8.9}$$

we have

$$\psi_\mu \circ R = \chi_{\mu+\rho_{\mathfrak{n}}} \quad \text{ on } \mathfrak{Z}_G \text{ for all } \mu \in \mathfrak{a}_{\mathbb{C}}^{\vee}. \tag{8.10}$$

By (8.10), we have

$$C^\infty(G/K; \mathcal{M}_\mu) \subset C^\infty(G/K; \chi^R_{\mu+\rho_{\mathfrak{l}}}). \tag{8.11}$$

For a simple Lie group G, it is known [7] that the $\mathbb{C}$-algebra homomorphism (8.9) is surjective if and only if $(\mathfrak{g}, \mathfrak{k})$ is not one of the following pairs:

$$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R}),\ (\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-52)}),\ (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-78)} + \mathbb{R}),\ (\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-133)} + \mathfrak{su}(2)).$$

For $\mathcal{F} = \mathcal{A}, \mathcal{B}, C^\infty$, or $\mathcal{D}'$, we denote by $\mathcal{F}(G/K; \mathcal{M}_\mu)$ the space of all $F \in \mathcal{F}(G/K)$ such that F satisfies the system of the following partial differential equations:

$$DF = \psi_\mu(D)F \quad \text{ for all } \ D \in \mathbb{D}(G/K). \tag{$\mathcal{M}_\mu$}$$

Since the Laplacian Δ on the Riemannian symmetric space G/K is an elliptic differential operator and belongs to $\mathbb{D}(G/K)$, we have

$$\mathcal{A}(G/K; \mathcal{M}_\mu) = \mathcal{B}(G/K; \mathcal{M}_\mu) = C^\infty(G/K; \mathcal{M}_\mu) = \mathcal{D}'(G/K; \mathcal{M}_\mu)$$

by the elliptic regularity theorem.

8.3 Poisson transform and boundary maps

Given $\mu \in \mathfrak{a}_{\mathbb{C}}^{\vee}$, we lift and extend it to a one-dimensional representation of the minimal parabolic subgroup $P = MAN$ by

$$P \to \mathbb{C}^{\times}, \quad m \exp Hn \mapsto e^{\langle \mu, H \rangle}$$

for $m \in M$, $H \in \mathfrak{a}$, and $n \in N$.

Remark 8.4. In the field of harmonic analysis on symmetric spaces people sometimes adopt the opposite signature of the (normalized) parabolic induction which is used in the representation theory of real reductive groups. Since our definition of parabolic induction does not involve the "ρ-shift" (*i.e.*, unnormalized parabolic induction where $\sqrt{-1}\mathfrak{a}^{\vee} + \rho_{\mathfrak{n}}$ is the unitary axis), the G-module $C^{\infty}(G/P;\mu)$ in our notation corresponds to $C^{\infty}(G/P;\mathcal{L}_{\rho_{\mathfrak{n}}-\mu})$ in [8, 20].

With this remark in mind, we summarize some known results that we need:

Proposition 8.5. 1) *The* $(\mathfrak{g}, K)$*-module* $E(G/P;\mu)$ *has* $\mathfrak{Z}_G$*-infinitesimal character* $\mu + \rho_{\mathfrak{g}} = \mu + \rho_{\mathfrak{n}} + \rho_{\mathfrak{l}} \in \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}})$.

2) *The* $(\mathfrak{g}, K)$*-module* $E(G/P;\mu)$ *is spherical for all* $\mu \in \mathfrak{a}_{\mathbb{C}}^{\vee}$. *Furthermore, the unique* (*up to scalar*) *non-zero spherical vector is cyclic if* μ *satisfies*

$$\operatorname{Re}\langle \mu - \rho_{\mathfrak{n}}, \alpha \rangle \geq 0 \quad \textit{for any} \quad \alpha \in \Sigma^{+}(\mathfrak{g}, \mathfrak{a}).$$

3) *For all* $\mu \in \mathfrak{a}_{\mathbb{C}}^{\vee}$, *the Poisson transform* $\mathcal{P}_{\mu}$ *maps into the space of joint eigenfunctions of the* $\mathbb{C}$*-algebra* $\mathbb{D}(G/K)$*:*

$$\mathcal{P}_{\mu} : \mathcal{B}(G/P;\mu) \to \mathcal{A}(G/K;\mathcal{M}_{\rho_{\mathfrak{n}}-\mu}). \tag{8.12}$$

4) *The Poisson transform* (8.12) *is bijective if* μ *satisfies*

$$\operatorname{Re}\langle \mu - \rho_{\mathfrak{n}}, \alpha \rangle \leq 0 \quad \textit{for any} \quad \alpha \in \Sigma^{+}(\mathfrak{g}, \mathfrak{a}). \tag{8.13}$$

5) *The Poisson transform* $\mathcal{P}_{\mu}$ *induces a bijection*

$$\mathcal{P}_{\mu} : \mathcal{D}'(G/P;\mu) \to \mathcal{A}_{\mathrm{mod}}(G/K;\mathcal{M}_{\rho_{\mathfrak{n}}-\mu})$$

if (8.13) *is satisfied.*

Proof. The first statement is elementary. See Kostant [17] for (2), and Helgason [6] for (3). The fourth statement was proved in Kashiwara *et.al.* [9] by using the theory of regular singularity of a system of partial differential equations. For the proof of the fifth statement, see Oshima–Sekiguchi [20] or Wallach [22, Theorem 11.9.4]. We note that for $f \in \mathcal{A}(G/K; \mathcal{M}_{\rho_{\mathfrak{n}}-\mu})$, f has moderate growth (Definition 3.3) if and only if f has at most exponential growth in the sense that there exist constants $d \in \mathbb{R}$ and $C > 0$ such that $|f(x)| \leq C\|x\|^d$ for all $x \in G$. $\square$

8.4 Proof of Theorem 8.1

In Lemma 5.5, we realized the Shintani space $\mathrm{Sh}(\lambda, \nu)$ in $C^\infty((G \times G')/\operatorname{diag} G')$. We give another realization of $\mathrm{Sh}(\lambda, \nu)$:

Lemma 8.6. *The multiplication map*

$$\psi : G \times G' \to G, \quad (g, g') \mapsto (g')^{-1} g$$

induces the following bijection:

$$\psi^* : \mathrm{Sh}(\lambda, \nu) \xrightarrow{\sim} C^\infty((G \times G')/(K \times K'); \chi^R_{\lambda,\nu})^{\operatorname{diag} G'}. \tag{8.14}$$

Proof. We set $C^\infty(K' \backslash G/K) := \{f \in C^\infty(G) : f(k'gk) = f(g) \quad \text{for all} \quad k' \in K' \text{ and } k \in K\}$. The pull-back ψ^* of functions induces a bijective linear map

$$\begin{array}{ccc} C^\infty(G) & \xrightarrow{\sim} & C^\infty(G \times G')^{\operatorname{diag} G'} \\ \cup & & \cup \\ C^\infty(K' \backslash G/K) & \underset{\psi^*}{\xrightarrow{\sim}} & C^\infty((G \times G')/(K \times K'))^{\operatorname{diag} G'}. \end{array}$$

On the other hand, for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}'$, we have

$$R_X R_Y(\psi^* f) = \psi^*(R_X L_Y f).$$

Thus Lemma 8.6 is proved. $\square$

Proof of Theorem 8.1. Suppose $\mathrm{Sh}(\lambda, \nu) \neq \{0\}$. Then, by Lemma 8.6, we have

$$V_{\lambda,\nu} := C^\infty((G \times G')/(K \times K'); \chi^R_{\lambda,\nu}) \neq \{0\}.$$

Since $R(\mathfrak{Z}_{G \times G'})$ is an ideal in $\mathbb{D}((G \times G')/(K \times K'))$ of finite-codimension, we can take the boundary values of $V_{\lambda,\nu}$ inductively to the hyperfunction-valued principal series representations of $G \times G'$ as in [14, Section 2]. To be more precise, there exist $\mu_1, \cdots, \mu_N \in \mathfrak{a}_{\mathbb{C}}^{\vee} \times (\mathfrak{a}'_{\mathbb{C}})^{\vee}$ and $(G \times G')$-invariant subspaces

$$\{0\} = V(0) \subset V(1) \subset \cdots \subset V(N) = V_{\lambda,\nu}$$

such that the quotient space $V(j)/V(j-1)$ is isomorphic to a subrepresentation of the spherical principal series representation $\mathcal{B}((G \times G')/(P \times P'); \mu_j)$ as $(G \times G')$-modules.

Comparing the $\mathfrak{Z}_{G\times G'}$-infinitesimal characters of $V_{\lambda,\nu}$ and $\mathcal{B}((G \times G')/(P \times P'); \mu_j)$, we get Theorem 8.1. □

8.5 Proof of Theorem 8.2

Proof of Theorem 8.2. 1) Since λ_+ satisfies (8.3), the $(\mathfrak{g}, K)$-module $E(G/P; \lambda_+)$ contains a cyclic spherical vector by Proposition 8.5. Similarly, the $(\mathfrak{g}', K')$-module

$$E(G'/P'; \nu_-)_{K'}^{\vee} \simeq E(G'/P'; \nu_-^*)_{K'}$$

has a cyclic vector because $\nu_-^* = -\nu_- + 2\rho_{\mathfrak{n}'}$ satisfies

$$\mathrm{Re}\langle \nu_-^* - \rho_{\mathfrak{n}'}, \alpha \rangle \geq 0 \quad \text{for any } \alpha \in \Sigma^+(\mathfrak{g}', \mathfrak{a}')$$

by (8.4). Hence the first statement follows from Proposition 7.1.

2) In view of the definition of moderate growth (Definition 3.3), we see that the bijection ψ^* in (8.14) induces the following bijection:

$$\mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) \xrightarrow{\sim} C^{\infty}_{\mathrm{mod}}((G \times G')/(K \times K'); \chi^R_{\lambda,\nu})^{\mathrm{diag}\, G'}. \tag{8.15}$$

Since the $\mathbb{C}$-algebra homomorphism $R : \mathfrak{Z}_{G\times G'} \to \mathbb{D}(G \times G'/K \times K')$ is surjective for classical groups G and G', the isomorphism (8.15) implies the following bijection

$$\mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) \simeq C^{\infty}_{\mathrm{mod}}((G \times G')/(K \times K'); \mathcal{M}_{(\lambda+\rho_{\mathfrak{l}}, \nu+\rho_{\mathfrak{l}'})})^{\mathrm{diag}\, G'}$$

by (8.11). In turn, combining with the Poisson transform, we have obtained the following natural isomorphism by Proposition 8.5 (5):

$$\mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) \simeq \mathcal{D}'((G \times G')/(P \times P'); \lambda_+^* \boxtimes \nu_-)^{\mathrm{diag}\, G'}.$$

By [16, Proposition 3.2], we proved the following natural bijection:

$$\mathrm{Hom}_{G'}(C^{\infty}(G/P; \lambda_+), C^{\infty}(G'/P'; \nu_-)) \simeq \mathcal{D}'((G \times G')/(P \times P'); \lambda_+^* \boxtimes \nu_-)^{\mathrm{diag}\, G'}.$$

Hence we have completed the proof of Theorem 8.2. □

9 Shintani functions for $(O(n+1,1), O(n,1))$

It has been an open problem to find $\dim_{\mathbb{C}} \mathrm{Sh}(\lambda,\nu)$ in the Archimedean case ([19, Remark 5.6]). In this section, by using a classification of symmetry breaking operators between spherical principal series representations of the pair $(G,G') = (O(n+1,1), O(n,1))$ in a recent work [16] with B. Speh, we determine the dimension of $\mathrm{Sh}(\lambda,\nu)$ in this case.

We denote by $[x]$ the greatest integer that does not exceed x. For the pair $(G,G') = (O(n+1,1), O(n,1))$, $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ,ν) are parametrized by

$$\mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}}) \times (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W(\mathfrak{j}'_{\mathbb{C}}) \simeq \mathbb{C}^{[\frac{n+2}{2}]}/W_{[\frac{n+2}{2}]} \times \mathbb{C}^{[\frac{n+1}{2}]}/W_{[\frac{n+1}{2}]}$$

in the standard coordinates, where $W_k := \mathfrak{S}_k \ltimes (\mathbb{Z}/2\mathbb{Z})^k$.

Theorem 9.1. *Let $(G,G') = (O(n+1,1), O(n,1))$.*

1) *The following three conditions on (λ,ν) are equivalent:*

   (i) $\mathrm{Sh}(\lambda,\nu) \neq \{0\}$.

   (ii) $\mathrm{Sh}_{\mathrm{mod}}(\lambda,\nu) \neq \{0\}$.

   (iii) *In the standard coordinates*

$$\lambda = w(\frac{n}{2}+t, \frac{n}{2}-1, \frac{n}{2}-2, \cdots, \frac{n}{2}-[\frac{n}{2}]), \tag{9.1}$$
$$\nu = w'(\frac{n-1}{2}+s, \frac{n-1}{2}-1, \cdots, \frac{n-1}{2}-[\frac{n-1}{2}]),$$

   for some t, $s \in \mathbb{C}$, $w \in W_{[\frac{n+2}{2}]}$, and $w' \in W_{[\frac{n+1}{2}]}$.

2) *If (λ,ν) satisfies* (iii) *in* (1), *then*

$$\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\lambda,\nu) = 1.$$

Proof. It is sufficient to prove the implication (i) $\Rightarrow$ (iii) and (2).

For $\lambda \in \mathfrak{j}_{\mathbb{C}}^{\vee}/W(\mathfrak{j}_{\mathbb{C}}) \simeq \mathbb{C}^{[\frac{n+1}{2}]}/W_{[\frac{n+1}{2}]}$, λ belongs to $\rho_{\mathfrak{l}} + \mathfrak{a}_{\mathbb{C}}^{\vee} \mod W(\mathfrak{j}_{\mathbb{C}})$ if and only if λ is of the form (9.1) for some $t \in \mathbb{C}$ and $w \in W_{[\frac{n+2}{2}]}$. Similarly for $\nu \in (\mathfrak{j}'_{\mathbb{C}})^{\vee}/W(\mathfrak{j}'_{\mathbb{C}})$. Hence the implication (i) $\Rightarrow$ (iii) holds as a special case of Theorem 8.1.

Next, suppose that (λ,ν) satisfies (iii). Without loss of generality, we may and do assume $\mathrm{Re}\, t \geq \frac{n}{2}$ and $\mathrm{Re}\, s \leq \frac{n-1}{2}$. In this case the unique element $\lambda_+ \in \mathfrak{a}_{\mathbb{C}}$ satisfying (8.2) and (8.3) is equal to t if we identify $\mathfrak{a}_{\mathbb{C}}^{\vee}$ with $\mathbb{C}$ via the standard basis $\{e_1\}$ of $\mathfrak{a}^{\vee}$ such that $\Sigma(\mathfrak{g},\mathfrak{a}) = \{\pm e_1\}$. Similarly, $\nu_- = s$ via $(\mathfrak{a}'_{\mathbb{C}})^{\vee} \simeq \mathbb{C}$.

We define a discrete subset of $\mathfrak{a}_{\mathbb{C}}^{\vee} \oplus (\mathfrak{a}'_{\mathbb{C}})^{\vee} \simeq \mathbb{C}^2$ by

$$L_{\text{even}} := \{(a,b) \in \mathbb{Z}^2 : a \le b \le 0,\ a \equiv b \mod 2\}.$$

According to [16, Theorem 1.1], we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(C^\infty(G/P;a), C^\infty(G'/P';b)) \simeq \begin{cases} 1 & \text{if } \ (a,b) \in \mathbb{C}^2 \setminus L_{\text{even}}, \\ 2 & \text{if } \ (a,b) \in L_{\text{even}}. \end{cases}$$

Since $(\lambda_+, \nu_-) = (t,s) \notin L_{\text{even}}$, we conclude that

$$\dim_{\mathbb{C}} \mathrm{Sh}_{\mathrm{mod}}(\lambda,\nu) = \dim_{\mathbb{C}} \operatorname{Hom}_{G'}(C^\infty(G/P;\lambda_+), C^\infty(G'/P';\nu_-)) = 1$$

by Theorem 8.2 (2). Thus Theorem 9.1 is proved. $\square$

10 Concluding remarks

We raise the following two related questions:

Problem 10.1. Find a condition on a pair of real reductive linear Lie groups $G \supset G'$ such that the following properties (A) and (B) are satisfied.

(A) All Shintani functions have moderate growth (Definition 3.3), namely, $\mathrm{Sh}_{\mathrm{mod}}(\lambda,\nu) = \mathrm{Sh}(\lambda,\nu)$ for all $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ,ν).

(B) The natural injective homomorphism

$$\operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \hookrightarrow \operatorname{Hom}_{\mathfrak{g}',K'}(\pi_K, \tau_{K'}) \tag{10.1}$$

is bijective for any admissible smooth representations π^∞ and τ^∞ of G and G', respectively.

Remark 10.2. 1) If $G' = \{e\}$ then neither (A) nor (B) holds.

2) If $G = G'$ then (A) holds by the theory of asymptotic behaviors of Harish-Chandra's zonal spherical functions and (B) holds by the Casselman–Wallach theory of the Fréchet globalization [22, Chapter 11].

3) If $G' = K$ then both (A) and (B) hold.

4) It is plausible that if (G, G') satisfies the geometric condition (PP) (Definition 5.3) then both (A) and (B) hold.

By using the argument in Sections 7 and 8 on the construction of Shintani functions from symmetry breaking operators, we have the following:

Proposition 10.3. *For a pair of real reductive classical Lie groups $G \supset G'$,* (B) *implies* (A)*.*

Proof. Let λ_+ and ν_- be as in Theorem 8.2. We denote by π^∞ the admissible smooth representation of G on $C^\infty(G/P; \lambda_+)$ and τ^∞ the admissible smooth representation of G' on $C^\infty(G'/P'; \nu_-)$. Then by Theorem 8.2 (2), we have the following linear isomorphism:

$$\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu).$$

Similarly to the proof of Theorem 8.2 (2), we have the natural bijection

$$\mathrm{Hom}_{G'}(\pi^\omega|_{G'}, \tau^\omega) \simeq \mathrm{Sh}(\lambda, \nu),$$

where π^ω is a continuous representation of G on the space of real analytic vectors of π^∞, and τ^ω that of τ^∞.

In view of the canonical injective homomorphisms

$$\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \hookrightarrow \mathrm{Hom}_{G'}(\pi^\omega|_{G'}, \tau^\omega) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}', K'}(\pi_K, \tau_K),$$

we see that if (B) holds, then the inclusion

$$\mathrm{Sh}_{\mathrm{mod}}(\lambda, \nu) \hookrightarrow \mathrm{Sh}(\lambda, \nu)$$

is bijective. Hence the implication (B) $\Rightarrow$ (A) is proved. $\square$

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