Tempered reductive homogeneous spaces

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Abstract

Let G be a semisimple algebraic Lie group and H a reductive subgroup. We find geometrically the best even integer p for which the representation of G in $L^2(G/H)$ is almost L^p. As an application, we give a criterion which detects whether this representation is tempered.

Key words and phrases, Lie groups, homogeneous spaces, tempered representations, matrix coefficients, symmetric spaces

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1 Introduction

Let G be an algebraic semisimple Lie group and H a reductive subgroup. The natural unitary representation of G in $L^2(G/H)$ has been studied over years since the pioneering work of I. M. Gelfand and Harish-Chandra.

Thanks to many mathematicians including E. van den Ban, P. Delorme, M. Flensted-Jensen, S. Helgason, T. Matsuki, T. Oshima, H. Schlichtkrull, J. Sekiguchi, among others, many properties of this representation are known when G/H is a symmetric space, i.e. when H is the set of fixed points of an involution of G. Most of the preceeding works in this case are built on the fact that the ring $\mathbb{D}(G/H)$ of G-invariant differential operators is commutative, and that the disintegration of $L^2(G/H)$ (Plancherel formula) is essentially the expansion of L^2-functions into joint eigenfunctions of $\mathbb{D}(G/H)$.

This paper deals with a more general reductive subgroup H, for which we cannot expect that the ring $\mathbb{D}(G/H)$ is commutative, and a complete change of the machinery would be required in the study of $L^2(G/H)$. We address the following question: *What kind of unitary representations occur in the disintegration of G/H?* More precisely, *when are all of them tempered?*

The aim of this paper is to give an easy-to-check necessary and sufficient condition on G/H under which all these irreducible unitary representations are tempered, or equivalently under which $L^2(G/H)$ is tempered, and in particular, has a 'uniform spectral gap'. We note that irreducible tempered representations were completely classified more than 30 years ago by Knapp and Zuckerman in [14], whereas non-tempered ones are still mysterious and have not been completely understood. Our criterion singles out homogeneous spaces G/H for which irreducible non-tempered unitary representations occur in the disintegration of $L^2(G/H)$. More generally, we give, for any even integer p, a necessary and sufficient condition under which $L^2(G/H)$ is almost L^p (see Theorem 4.1).

Our criterion is new even when G/H is a reductive symmetric space where the disintegration of $L^2(G/H)$ was established up to the classification of discrete series representations for (sub)symmetric spaces ([1, 8, 20]). Indeed irreducible unitary representations that contribute to $L^2(G/H)$ in the direct integral are obtained as a parabolic induction from discrete series for subsymmetric spaces, but a subtle point arises from discrete series with singular parameter. In fact, all possible discrete series were captured in [21], however, the non-vanishing conditions of these modules are sometimes combinatorially complicated and these very modules with singular parameters would affect the worst decay of matrix coefficients if they do not vanish. (Such complication does not occur in the case of group manifolds because Harish-Chandra's discrete series do not allow singular parameters.) Algebraically, the underlying $(\mathfrak{g}, K)$-modules are certain Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ (see [13] for general theory) with possibly singular λ crossing many walls of the Weyl chambers, so that the Langlands parameter may behave in an unstable way and even the modules themselves may disappear. A necessary condition for the non-vanishing of discrete series for reductive symmetric spaces with singular parameter was proved in [18] that corrected the announcement in [21], whereas a number of general methods to verify the non-vanishing of $A_{\mathfrak{q}}(\lambda)$-modules have been developed more recently in [15, Chapters 4, 5], [22] for some classical groups, but the proof of the sufficiency of the non-vanishing condition in [18] has not been given so far.

Beyond symmetric spaces, very little has been known on the unitary representation of G in $L^2(G/H)$ (cf. [16]).

Here is an outline of the paper. As a baby case, we first study the unitary representations of a semisimple group in $L^2(V)$ where V is a finite dimensional representation. We give a necessary and sufficient condition on V under which the representation in $L^2(V)$ is tempered (Theorem 3.2), or more generally, under which this representation is almost L^p. The heart of the paper is Chapter 4 where we give a proof of the main results (Theorem 4.1) for reductive homogeneous spaces G/H. In a subsequent paper we see that this criterion suffices to give a complete classification of the pairs (G, H) of algebraic reductive groups for which the unitary representation of G on $L^2(G/H)$ is non-tempered. To give a flavor of what is possible, we collect a few applications of this criterion in Chapter 5, omitting the details of the computational verification.

2 Preliminary results

We collect in this chapter a few well-known facts on almost L^p representations, on tempered representation and on uniform decay of matrix coefficients.

2.1 Almost L^p representations

In this paper all Lie groups will be real Lie groups. Let G be a unimodular Lie group and π be a unitary representation of G in a Hilbert space $\mathcal{H}_\pi$.

Definition 2.1. *Let $p \geq 2$. The unitary representation π is said to be almost L^p if there exists a dense subset $D \subset \mathcal{H}_\pi$ for which the coefficients $c_{v_1,v_2} \colon g \mapsto \langle \pi(g)v_1, v_2\rangle$ are in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ and all v_1, v_2 in D.*

Let K be a maximal compact subgroup of G.

Lemma 2.2. *A unitary representation π is almost L^p if and only if there exists a dense subset $D_0 \subset \mathcal{H}_\pi$ of K-finite vectors for which the coefficients c_{v_1,v_2} are in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ and all v_1, v_2 in D_0.*

Proof. We first notice that for all v_1, v_2 in D and all k_1, k_2 in K the two vectors $\pi(k_1)v_1$ and $\pi(k_2)v_2$ have a coefficient with same $L^{p+\varepsilon}$-norm:

$$\|c_{\pi(k_1)v_1,\pi(k_2)v_2}\|_{L^{p+\varepsilon}} = \|c_{v_1,v_2}\|_{L^{p+\varepsilon}}.$$

Let dk be the Haar probability measure on K. For any two K-finite functions f_1 and f_2 on K, bounded by 1, the two vectors $w_1 := \int_K f_1(k)\pi(k)v_1\, dk$ and $w_2 := \int_K f_2(k)\pi(k)v_2\, dk$ have a coefficient with bounded $L^{p+\varepsilon}$-norm:

$$\|c_{w_1,w_2}\|_{L^{p+\varepsilon}} \leq \|c_{v_1,v_2}\|_{L^{p+\varepsilon}}.$$

These vectors w_i live in a dense set D_0 of K-finite vectors of $\mathcal{H}_\pi$. $\square$

2.2 Tempered representations

The following definition is due to Harish-Chandra (See also [2, Appendix F])

Definition 2.3. *The unitary representation π is said to be tempered if π is weakly contained in the regular representation λ_G of G in $L^2(G)$ i.e. if every coefficient of π is a uniform limit on every compact of G of a sequence of sums of coefficients of λ_G.*

Here are a few basic facts on tempered representations.
- Let $G' \subset G$ be a finite index subgroup. A unitary representation π of G is tempered if and only if π is tempered as a representation of G'.
- A unitary representation π of a reductive group G is tempered if and only if π is tempered as a representation of the derived subgroup $[G, G]$.

Proposition 2.4. (Cowling, Haagerup, Howe) *Let G be a semisimple connected Lie group with finite center, and m a positive integer.*

A unitary representation π of G is almost L^2 if and only if π is tempered.
More generally, π is almost L^{2m} if and only if $\pi^{\otimes m}$ is tempered.

See [7, Theorems 1, 2 and Corollary].

Remark 2.5. *When G is amenable, according to Hulanicki–Reiter Theorem (see [2, Theorem G.3.2]), every unitary representation of G is tempered. However, when G is non-compact, the trivial representation is not almost L^2.*

The following remark was used implicitly in the introduction.

Remark 2.6. *When a unitary representation π of G is a direct integral $\pi = \int^\oplus \pi_\lambda \, \mathrm{d}\mu(\lambda)$ of irreducible unitary representations π_λ, the representation π is tempered if and only if the representations π_λ are tempered for μ-almost every parameter λ.*

Proof. Indeed, π is weakly contained in the direct sum representation $\oplus_\lambda \pi_\lambda$, and conversely π_λ is weakly contained in π for μ-almost every λ.

These statements follow for instance from the following fact in [2, Theorem F.4.4] or [10, Section 18] : For two unitary representations ρ and ρ' of G, one has the equivalence:

ρ *is weakly contained in* $\rho' \iff \|\rho(f)\| \leq \|\rho'(f)\|$ *for all* f *in* $L^1(G)$,
where $\rho(f) = \int_G f(g)\rho(g)\mathrm{d}g$. Note that this condition has only to be checked for a countable dense set of functions f in $L^1(G)$, and that one has the equality $\|\pi(f)\| = \mathrm{supess}_\lambda \|\pi_\lambda(f)\|$ (see [9, Section II.2.3]). □

2.3 Uniform decay of coefficients

Let G be a linear semisimple connected Lie group and let Ξ be the Harish-Chandra spherical function on G (see [7]). A short definition for Ξ is as the coefficient of the normalized K-invariant vector of the spherical representation of the unitary principal series $\pi_o = \mathrm{Ind}_P^G(\mathbf{1}_P)$ where P is a minimal parabolic subgroup of G. In this paper we will not need the precise formula for Ξ but just the fact that this function Ξ is in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$ and the following proposition.

Proposition 2.7. (Cowling, Haagerup, Howe) *Let p be an even integer. A unitary representation π of G is almost L^p if and only if, for every K-finite vectors v, w in $\mathcal{H}_\pi$, for every g in G, one has*

$$|\langle \pi(g)v, w\rangle| \leq \Xi(g)^{2/p} \|v\| \|w\| (\dim\langle Kv\rangle)^{\frac{1}{2}} (\dim\langle Kw\rangle)^{\frac{1}{2}}.$$

See [7, Corollary p.108]

This proposition tells us that once an almost L^p-norm condition is checked for the coefficients of a dense set of vectors of $\mathcal{H}_\pi$, one gets a UNIFORM estimate for the coefficients of ALL the K-finite vectors of $\mathcal{H}_\pi$.

In this proposition, the assumption that the real number $p \geq 2$ is an even integer can probably be dropped. If this is the case, the same assumption can also be dropped in our Theorems 3.2 and 4.1.

The set of p for which π is almost L^p is an interval $[p_\pi, \infty[$ with $p_\pi \geq 2$ or $p_\pi = \infty$. Even though we will not use them, we recall the following two important properties of these constant p_π.

When G is quasisimple of higher rank and $\mathcal{H}_\pi$ does not contain G-invariant vectors, this real number p_π is bounded by a constant $p_G < \infty$ (see [19]).

According to Harish-Chandra, when G is semisimple and π is irreducible with finite kernel, this real p_π is finite (see [12, Theorem 8.48]).

2.4 Representations in $L^2(X)$

Let X be a locally compact space endowed with a continuous action of G preserving a Radon measure vol on X. One has a natural representation π of G in $L^2(X)$ given by, $(\pi(g)\varphi)(x) = \varphi(g^{-1}x)$ for g in G, φ in $L^2(X)$ and x in X.

Lemma 2.8. *Let G be a semisimple linear connected Lie group, p a positive even integer, and X a locally compact space endowed with a continuous action of G preserving a Radon measure* vol.

The representation of G in $L^2(X)$ is almost L^p if and only if, for any compact subset C of X and any $\varepsilon > 0$ $\mathrm{vol}(gC \cap C) \in L^{p+\varepsilon}(G)$.

Proof. If the representation of G in $L^2(X)$ is almost L^p then, according to Proposition 2.7, for all K-invariant compact set B of X, the function $g \mapsto \mathrm{vol}(gB \cap B) = \langle \pi(g)\mathbf{1}_B, \mathbf{1}_B\rangle$ belongs to $L^{p+\varepsilon}(G)$. Since any compact set C of X is included in such a K-invariant compact set B, the function $g \mapsto \mathrm{vol}(gC \cap C)$ belongs also to $L^{p+\varepsilon}(G)$.

Conversely, let $D \subset L^2(X)$ be the dense subspace of continuous compactly supported functions on X. For any two continuous functions $\varphi_1, \varphi_2 \in D$, the coefficient $\langle \pi(g)\varphi_1, \varphi_2\rangle$ is bounded by $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \,\mathrm{vol}(gC \cap C)$ where $C := \mathrm{Supp}(\varphi_1) \cup \mathrm{Supp}(\varphi_2)$, and hence this coefficient belongs to $L^{p+\varepsilon}(G)$. □

3 Representations in $L^2(V)$

In this chapter we study the representation of a semisimple Lie group in $L^2(V)$ where V is a finite dimensional representation.

3.1 Function ρ_V

Let H be a reductive algebraic Lie group, and $\tau\colon H \to SL_\pm(V)$ a finite dimensional algebraic representation over $\mathbb{R}$ preserving the Lebesgue measure on V. We write $d\tau\colon \mathfrak{h} \to \mathrm{End}(V)$ for the differential representation of τ. Let $\mathfrak{a} = \mathfrak{a}_\mathfrak{h}$ be a maximal split abelian subspace in $\mathfrak{h}$.

For an element Y in $\mathfrak{a}$, we denote by V_+ the sum of eigenspaces of $\tau(Y)$ having positive eigenvalues, and set

$$\rho_V(Y) := \mathrm{Trace}_{V_+}(d\tau(Y)). \tag{3.1}$$

Since this function $\rho_V \colon \mathfrak{a} \to \mathbb{R}_{\geq 0}$ will be very important in our analysis, we begin by a few trivial but useful comments. We notice first that, since H is volume preserving, for any $Y \in \mathfrak{a}$,

$$\rho_V(-Y) = \rho_V(Y), \tag{3.2}$$

$$\rho_V(Y) = 0 \Leftrightarrow d\tau(Y) = 0. \tag{3.3}$$

This function ρ_V is invariant under the finite group $W_H := N_H(\mathfrak{a})/Z_H(\mathfrak{a})$. This group is isomorphic to the Weyl group of the restricted root system $\Sigma(\mathfrak{h}, \mathfrak{a})$ if H is connected. This function ρ_V is continuous and is piecewise linear i.e. there exist finitely many convex polyhedral cones which cover $\mathfrak{a}$ and on which ρ_V is linear.

Example 3.1. *For $(\tau, V) = (\mathrm{Ad}, \mathfrak{h})$, $\rho_{\mathfrak{h}}$ coincides with twice the usual 'ρ' on the positive Weyl chamber $\mathfrak{a}_+$ with respect to a positive system $\Sigma^+(\mathfrak{h}, \mathfrak{a})$.*

$$\rho_{\mathfrak{h}} = \sum_{\alpha \in \Sigma^+(\mathfrak{h}, \mathfrak{a})} \dim \mathfrak{h}_\alpha \, \alpha \quad \text{on } \mathfrak{a}_+,$$

where $\mathfrak{h}_\alpha \subset \mathfrak{h}$ is the root subspace associated to α.

For other representations (τ, V), the maximal convex polyhedral cones on which ρ_V is linear are most often much smaller than the Weyl chambers.

3.2 Criterion for temperedness of $L^2(V)$

Since the Lebesgue measure on V is H-invariant, we have a natural unitary representation of H on $L^2(V)$ as in Section 2.4.

Theorem 3.2. *Let H an algebraic semisimple Lie group, $\tau \colon H \to SL_\pm(V)$ an algebraic representation and p a positive even integer. Then, one has the equivalences :*
a) $L^2(V)$ is tempered $\iff$ $\rho_{\mathfrak{h}}(Y) \leq 2\,\rho_V(Y)$ for any $Y \in \mathfrak{a}$.
b) $L^2(V)$ is almost L^p $\iff$ $\rho_{\mathfrak{h}}(Y) \leq p\,\rho_V(Y)$ for any $Y \in \mathfrak{a}$.

Remark 3.3. *Inequality* $\rho_{\mathfrak{h}} \leq p\,\rho_V$ *holds on* $\mathfrak{a}$ *if and only if it holds on* $\mathfrak{a}_+$.

Since all the maximal split abelian subspaces of $\mathfrak{h}$ *are* H*-conjugate, it is clear that this condition does not depend on the choice of* $\mathfrak{a}$.

Example 3.4. *Let* $H = \mathrm{SL}(2,\mathbb{R})^d$ *with* $d \geq 1$. *The unitary representation in* $L^2(V)$ *is tempered if and only if the kernel of* τ *is finite.*

Example 3.5. *Let* $H = \mathrm{SL}(3,\mathbb{R})$. *The unitary representation in* $L^2(V)$ *is tempered if and only if* $\dim(V/V^H) > 3$ *where* $V^H = \{v \in V : Hv = v\}$.

For $h \in H$, $x \in V$ and a measurable subset $C \subset V$, we write hx for $\tau(h)x$ and we set $hC := \{hx \in V : x \in C\}$. Similarly, for $a > 0$ we set $aC := \{ax \in V : x \in C\}$. We write $\mathrm{vol}(C)$ for the volume of C with respect to the Lebesgue measure.

Proof of Theorem 3.2. When the kernel of τ is noncompact, both sides of the equivalence are false. Hence we may assume that the kernel of τ is compact. Since H is semisimple, according to Proposition 2.4 and Lemma 2.8, it is sufficient to prove the following equivalence:

$$\begin{array}{lcl} \rho_{\mathfrak{h}}(Y) \;\leq\; p\,\rho_V(Y) & \multirow{2}{*}{$\Longleftrightarrow$} & \mathrm{vol}(hC \cap C) \in L^{p+\varepsilon}(H) \textit{ for any compact} \\ \textit{for any } Y \in \mathfrak{a} & \Longleftrightarrow & \textit{subset } C \textit{ in } V \textit{ and any } \varepsilon > 0. \end{array}$$

This statement is a special case of Proposition 3.6 below. □

3.3 L^p-norm of $\mathrm{vol}(hC \cap C)$

Suppose now that the kernel of τ is compact. According to (3.3), one has $\rho_V(Y) > 0$ as soon as $Y \neq 0$. Hence the real number

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} \tag{3.4}$$

is finite.

Proposition 3.6. *Let* H *be an algebraic reductive Lie group, and* $\tau\colon H \to SL_{\pm}(V)$ *a volume preserving algebraic representation with compact kernel. For any real* $p > 0$, *one has the equivalence :*

$$p > p_V \Longleftrightarrow \mathrm{vol}(hC \cap C) \in L^p(H) \textit{ for any compact set } C \textit{ in } V.$$

In this section we will show how to deduce Proposition 3.6 from a volume estimate that we will prove in the next section.

Proof of Proposition 3.6. Let H_K be a maximal compact subgroup of H such that $H = H_K(\exp \mathfrak{a})H_K$ is a Cartan decomposition of H.

The Haar measure dh of H is given as

$$\int_H f(h)dh = \int_{\mathfrak{a}} f(e^Y)D_{\mathfrak{h}}(Y)dY \tag{3.5}$$

for any H_K-biinvariant measurable function f on H, where

$$D_{\mathfrak{h}}(Y) := \prod_{\alpha\in\Sigma^+(\mathfrak{h},\mathfrak{a})} |\sinh\langle\alpha, Y\rangle|^{\dim \mathfrak{h}_\alpha} \quad \text{for } Y \in \mathfrak{a}.$$

We also introduce the function on $\mathfrak{a}$

$$\widetilde{D}_{\mathfrak{h}}(Y) := \int_{\|Z\|\le 1} D_{\mathfrak{h}}(Y+Z)dZ.$$

We shall prove successively the following equivalences

$$\begin{array}{lll} & (i) & \mathrm{vol}(hC \cap C) \in L^p(H), \text{ for any compact } C \subset V, \\ \Longleftrightarrow & (ii) & \mathrm{vol}(e^Y C \cap C)^p\, D_{\mathfrak{h}}(Y) \in L^1(\mathfrak{a}), \text{ for any compact } C \subset V, \\ \Longleftrightarrow & (iii) & \mathrm{vol}(e^Y C \cap C)^p\, \widetilde{D}_{\mathfrak{h}}(Y) \in L^1(\mathfrak{a}), \text{ for any compact } C \subset V, \\ \Longleftrightarrow & (iv) & \mathrm{vol}(e^Y C \cap C)^p\, e^{\rho_{\mathfrak{h}}(Y)} \in L^1(\mathfrak{a}), \text{ for any compact } C \subset V, \\ \Longleftrightarrow & (v) & e^{\rho_{\mathfrak{h}}(Y) - p\,\rho_V(Y)} \in L^1(\mathfrak{a}), \\ \Longleftrightarrow & (vi) & p\,\rho_V(Y) - \rho_{\mathfrak{h}}(Y) > 0, \text{ for any } Y \in \mathfrak{a} \smallsetminus 0. \end{array}$$

$(i) \Longleftrightarrow (ii)$ We may choose C to be H_K-invariant by expanding C if necessary. We apply then the integration formula (3.5) to the H_K-biinvariant function $\mathrm{vol}(hC \cap C)$.
$(ii) \Longleftrightarrow (iii)$ Replace C by a larger compact $C' := e^{\mathfrak{a}(1)}C$ where $\mathfrak{a}(1)$ is the unit ball $\{Z \in \mathfrak{a} \mid \|Z\| \le 1\}$. Since $\mathrm{vol}(e^{Y-Z}C \cap C) \le \mathrm{vol}(e^Y C' \cap C')$ for any $Z \in \mathfrak{a}(1)$, one has, by using the change of variables $Y' := Y - Z$,

$$\begin{aligned} \int_{\mathfrak{a}} \mathrm{vol}(e^Y C \cap C)^p \widetilde{D}_{\mathfrak{h}}(Y)dY &= \int_{\mathfrak{a}}\int_{\|Z\|\le 1} \mathrm{vol}(e^Y C \cap C)^p D_{\mathfrak{h}}(Y+Z)dY\,dZ \\ &\le \mathrm{vol}(\mathfrak{a}(1)) \int_{\mathfrak{a}} \mathrm{vol}(e^Y C' \cap C')^p D_{\mathfrak{h}}(Y)dY, \\ \int_{\mathfrak{a}} \mathrm{vol}(e^{Y'} C \cap C)^p D_{\mathfrak{h}}(Y')dY' &\le \int_{\mathfrak{a}}\int_{\|Z\|\le 1} \mathrm{vol}(e^{Y-Z} C' \cap C')^p D_{\mathfrak{h}}(Y)dY\,dZ \\ &= \mathrm{vol}(\mathfrak{a}(1))^{-1} \int_{\mathfrak{a}} \mathrm{vol}(e^{Y'} C' \cap C')^p \widetilde{D}_{\mathfrak{h}}(Y')dY'. \end{aligned}$$

$(iii) \Longleftrightarrow (iv)$ We notice that we can find constants $a_1, a_2 > 0$, such that for any $Y \in \mathfrak{a}$, the following inequality holds:

$$a_1\, e^{\rho_{\mathfrak{h}}(Y)} \leq \widetilde{D}_{\mathfrak{h}}(Y) \leq a_2\, e^{\rho_{\mathfrak{h}}(Y)}.$$

$(iv) \Longleftrightarrow (v)$ We use Proposition 3.7, that we will prove in Section 3.4, and that gives, for C large enough, constants $m, M > 0$ such that, for any $Y \in \mathfrak{a}$,

$$m\, e^{-\,\rho_V(Y)} \leq \mathrm{vol}(e^Y C \cap C) \leq M\, e^{-\,\rho_V(Y)}.$$

$(v) \Longleftrightarrow (vi)$ We recall that the function $\rho_{\mathfrak{h}} - p\rho_V$ is continuous and piecewise linear.

This proves Proposition 3.6 provided the following Proposition 3.7. $\square$

3.4 Estimate of $\mathrm{vol}(e^Y C \cap C)$

The following asymptotic estimate of $\mathrm{vol}(e^Y C \cap C)$ for the linear representation in V will become a prototype of the volume estimate for the action on G/H which we shall discuss in Section 4 (Theorem 4.4).

Proposition 3.7. *Let H be an algebraic reductive Lie group, $\tau\colon H \to SL_{\pm}(V)$ a volume preserving algebraic representation, and C be a compact neighborhood of 0 in V. Then there exist constants $m \equiv m_C > 0$, $M \equiv M_C > 0$ such that*

$$me^{-\rho_V(Y)} \leq \mathrm{vol}(e^Y C \cap C) \leq Me^{-\rho_V(Y)} \quad \textit{for any } Y \in \mathfrak{a}.$$

To see this, write $\Delta \equiv \Delta(V, \mathfrak{a}) \subset \mathfrak{a}^*$ for the set of weights of the representation $d\tau|_{\mathfrak{a}}\colon \mathfrak{a} \to \mathrm{End}(V)$, and

$$V = \bigoplus\nolimits_{\lambda \in \Delta} V_\lambda, \quad v = \sum v_\lambda \tag{3.6}$$

for the corresponding weight space decomposition.

Lemma 3.8. *For each $\lambda \in \Delta$, let B_λ be a convex neighborhood of 0 in V_λ, and let $B := \prod_\lambda B_\lambda$. Then, one has*

$$\mathrm{vol}(e^Y B \cap B) = \mathrm{vol}(B) e^{-\rho_V(Y)} \quad \textit{for any } Y \in \mathfrak{a}.$$

Proof. For any real t, one has $B_\lambda \cap e^{-t}B_\lambda = e^{-t^+}B_\lambda$ where $t^+ := \max(t,0)$. Then we get

$$B \cap e^{-Y}B = \prod_\lambda (B_\lambda \cap e^{-\lambda(Y)}B_\lambda) = \prod_\lambda e^{-\lambda(Y)^+}B_\lambda,$$

and $$\mathrm{vol}(e^Y B \cap B) = \mathrm{vol}(B \cap e^{-Y}B) = e^{-\rho_V(Y)}\,\mathrm{vol}(B).$$ □

Proof of Proposition 3.7. We take $\{B_\lambda\}$ and $\{B'_\lambda\}$ such that

$$\prod_\lambda B_\lambda \subset C \subset \prod_\lambda B'_\lambda$$

and we apply Lemma 3.8. □

4 Representations in $L^2(G/H)$

In this chapter we study the representations of an algebraic semisimple Lie group in $L^2(X)$ where X is a homogeneous space with reductive isotropy.

4.1 Criterion for temperedness of $L^2(G/H)$

Let G be an algebraic reductive Lie group and H an algebraic reductive subgroup of G. Since the homogeneous space $X = G/H$ carries a G-invariant Radon measure, there is a natural unitary representation of G on $L^2(G/H)$ as in Section 2.4. We want to study the temperedness of this representation.

Let $\mathfrak{q}$ be an H-invariant complementary subspace of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{g}$. We fix a maximal split abelian subspace $\mathfrak{a}$ of $\mathfrak{h}$ and we define $\rho_\mathfrak{q} : \mathfrak{a} \to \mathbb{R}_{\geq 0}$ for the H-module $\mathfrak{q}$ as in Section 3.1.

Here is the main result of this chapter :

Theorem 4.1. *Let G be an algebraic semisimple Lie group, H an algebraic reductive subgroup of G, and p a positive even integer. Then, one has the equivalences :*
a) $L^2(G/H)$ is tempered $\iff$ $\rho_\mathfrak{h}(Y) \leq \rho_\mathfrak{q}(Y)$ for any $Y \in \mathfrak{a}$.
b) $L^2(G/H)$ is almost L^p $\iff$ $\rho_\mathfrak{g}(Y) \leq p\,\rho_\mathfrak{q}(Y)$ for any $Y \in \mathfrak{a}$.

Remark 4.2. *Since $\rho_\mathfrak{g} = \rho_\mathfrak{h} + \rho_\mathfrak{q}$, one has the equivalence*

$$\rho_\mathfrak{h} \leq \rho_\mathfrak{q} \Leftrightarrow \rho_\mathfrak{g} \leq 2\,\rho_\mathfrak{q}.$$

The inequality $\rho_\mathfrak{g} \leq p\,\rho_\mathfrak{q}$ holds on $\mathfrak{a}$ if and only if it holds on $\mathfrak{a}_+$.

Proof of Theorem 4.1. When the kernel of the action of G on G/H is noncompact, both sides of the equivalence are false. Hence we may assume that this kernel is compact. But then, according to Proposition 2.4 and Lemma 2.8, it is sufficient to prove the following equivalence:

$$\begin{array}{lcl} \rho_{\mathfrak{g}}(Y) \;\leq\; p\,\rho_{\mathfrak{q}}(Y) & \Longleftrightarrow & \mathrm{vol}(gC\cap C) \in L^{p+\varepsilon}(G) \textit{ for any compact} \\ \textit{for any } Y\in\mathfrak{a} & & \textit{subset } C \textit{ in } G/H \textit{ and any } \varepsilon>0. \end{array}$$

This statement is a special case of Proposition 4.3 below. $\square$

4.2 L^p-norm of $\mathrm{vol}(gC \cap C)$

We assume that the action of G on G/H has compact kernel or, equivalently, that the action of H on $\mathfrak{q}$ has compact kernel. Then, according to (3.3), one has $\rho_{\mathfrak{q}}(Y) > 0$ as soon as $Y \neq 0$. Hence the real number

$$p_{G/H} := \max_{Y\in\mathfrak{a}\setminus\{0\}} \frac{\rho_{\mathfrak{g}}(Y)}{\rho_{\mathfrak{q}}(Y)} \tag{4.1}$$

is finite.

Proposition 4.3. *Let G be an algebraic reductive Lie group and H an algebraic reductive subgroup of G such that the action of G on G/H has compact kernel. For any real $p \geq 1$, one has the equivalence :*
$p > p_{G/H} \Longleftrightarrow \mathrm{vol}(gC \cap C) \in L^p(G)$ *for any compact set C in G/H.*

In this section we will show how to deduce Proposition 4.3 from a volume estimate that we will prove in the following sections. For that we will use another equivalent definition of the constant $p_{G/H}$.

We extend $\mathfrak{a}$ to a maximal split abelian subspace $\mathfrak{a}_{\mathfrak{g}}$ of $\mathfrak{g}$ and we choose a maximal compact subgroup K of G such that $H_K := H \cap K$ is a maximal compact subgroup of H, and that $G = K(\exp\mathfrak{a}_{\mathfrak{g}})K$ and $H = H_K(\exp\mathfrak{a})H_K$ are Cartan decompositions of G and H, respectively.

Let W_G be the finite group $W_G := N_G(\mathfrak{a}_{\mathfrak{g}})/Z_G(\mathfrak{a}_{\mathfrak{g}}) \simeq N_K(\mathfrak{a}_{\mathfrak{g}})/Z_K(\mathfrak{a}_{\mathfrak{g}})$. When G is connected, W_G is the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{g}})$.

For $Y \in \mathfrak{a}$, we define a subset of W_G by

$$W(Y;\mathfrak{a}) := \{w \in W_G : wY \in \mathfrak{a}\}. \tag{4.2}$$

We notice that $W(Y;\mathfrak{a}) \ni e$ for any $Y \in \mathfrak{a}$, and $W(0;\mathfrak{a}) = W_G$. We set

$$\rho_{\mathfrak{q}}^{\min}(Y) := \min_{w \in W(Y;\mathfrak{a})} \rho_{\mathfrak{q}}(wY). \tag{4.3}$$

We can then rewrite Definition (4.1) by the equivalent formula

$$p_{G/H} = \max_{Y \in \mathfrak{a}\setminus\{0\}} \frac{\rho_{\mathfrak{g}}(Y)}{\rho_{\mathfrak{q}}^{\min}(Y)}. \tag{4.4}$$

Proof of Proposition 4.3. The Haar measure dg on G is given as

$$\int_G f(g)dg = \int_{\mathfrak{a}_{\mathfrak{g}}} f(e^Y) D_{\mathfrak{g}}(Y) dY, \tag{4.5}$$

for any K-biinvariant measurable function f on G, where $D_{\mathfrak{g}}$ is the W_G-invariant function on $\mathfrak{a}_{\mathfrak{g}}$ given by

$$D_{\mathfrak{g}}(Y) := \prod_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a}_{\mathfrak{g}})} |\sinh\langle \alpha, Y\rangle|^{\dim \mathfrak{g}_\alpha}, \quad Y \in \mathfrak{a}_{\mathfrak{g}}.$$

and where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ are the (restricted) root spaces.

We also introduce the function on $\mathfrak{a}_{\mathfrak{g}}$

$$\widetilde{D}_{\mathfrak{g}}(Y) := \int_{\|Z\| \le 1} D_{\mathfrak{g}}(Y+Z) dZ.$$

We shall prove successively the following equivalences

$$\begin{array}{rll}
 & (i) & \mathrm{vol}(gC \cap C) \in L^p(G), \text{ for any compact } C \subset X, \\
\iff & (ii) & \mathrm{vol}(e^Y C \cap C)^p \, D_{\mathfrak{g}}(Y) \in L^1(\mathfrak{a}_{\mathfrak{g}}), \text{ for any compact } C \subset X, \\
\iff & (iii) & \mathrm{vol}(e^Y C \cap C)^p \, \widetilde{D}_{\mathfrak{g}}(Y) \in L^1(\mathfrak{a}_{\mathfrak{g}}), \text{ for any compact } C \subset X, \\
\iff & (iv) & \mathrm{vol}(e^Y C \cap C)^p \, e^{\rho_{\mathfrak{g}}(Y)} \in L^1(\mathfrak{a}_{\mathfrak{g}}), \text{ for any compact } C \subset X, \\
\iff & (v) & \mathrm{vol}(e^Y C \cap C)^p \, e^{\rho_{\mathfrak{g}}(Y)} \in L^1(\mathfrak{a}), \text{ for any compact } C \subset X, \\
\iff & (vi) & e^{\rho_{\mathfrak{g}}(Y) - p\,\rho_{\mathfrak{q}}^{\min}(Y)} \in L^1(\mathfrak{a}), \\
\iff & (vii) & p\,\rho_{\mathfrak{q}}^{\min}(Y) - \rho_{\mathfrak{g}}(Y) > 0, \text{ for any } Y \in \mathfrak{a} \setminus 0.
\end{array}$$

(i) $\iff$ *(ii)* We may choose C to be K-invariant. We apply then the integration formula (4.5) to the K-biinvariant function $\mathrm{vol}(gC \cap C)$.
(ii) $\iff$ *(iii)* We just replace C by a larger compact $C' := e^{\mathfrak{a}_{\mathfrak{g}}(1)}C$ where

$\mathfrak{a}_{\mathfrak{g}}(1)$ is the unit ball $\{Z \in \mathfrak{a}_{\mathfrak{g}} : \|Z\| \leq 1\}$.
$(iii) \iff (iv)$ We notice that we can find constants $a_1, a_2 > 0$, such that for any $Y \in \mathfrak{a}_{\mathfrak{g}}$, one has

$$a_1\, e^{\rho_{\mathfrak{g}}(Y)} \leq \widetilde{D}_{\mathfrak{g}}(Y) \leq a_2\, e^{\rho_{\mathfrak{g}}(Y)}.$$

$(iv) \iff (v)$ The main point of this equivalence is to replace an integration on $\mathfrak{a}_{\mathfrak{g}}$ by an integration on $\mathfrak{a}$. For that we will bound the support of the function φ_C on $\mathfrak{a}_{\mathfrak{g}}$, $\varphi_C(Y) := \mathrm{vol}(e^Y C \cap C)^p\, e^{\rho_{\mathfrak{g}}(Y)}$. We may choose C to be K-invariant so that, φ_C is W_G-invariant. We recall now the Cartan projection

$$\mu \colon G \to \mathfrak{a}_{\mathfrak{g}}/W_G, \quad k_1 e^Y k_2 \mapsto Y \bmod W_G$$

with respect to the Cartan decomposition $G = K(\exp \mathfrak{a}_{\mathfrak{g}})K$. It follows from either [3, Prop. 5.1] or [17, Th. 1.1] that, for any compact subsets $S \subset G$, there exists $\delta > 0$ such that

$$\mu(SHS^{-1}) \subset \mu(H) + \mathfrak{a}_{\mathfrak{g}}(\delta) \bmod W_G \tag{4.6}$$

where $\mathfrak{a}_{\mathfrak{g}}(\delta)$ stands for the δ-ball $\{Y \in \mathfrak{a}_{\mathfrak{g}} : \|Y\| \leq \delta\}$. If we take this compact set $S \subset G$ such that $C \subset SH/H$, then $Y \in \mathfrak{a}_{\mathfrak{g}}$ satisfies $e^Y C \cap C \neq \emptyset$ only if $e^Y \in SHS^{-1}$, and therefore, only if $Y \in \mu(SHS^{-1})$. Hence we get the bound on the support

$$\mathrm{Supp}\, \varphi_C \subset \bigcup_{w \in W_G} w(\mathfrak{a} + \mathfrak{a}_{\mathfrak{g}}(\delta)). \tag{4.7}$$

By W_G-invariance of φ_C, we only have to integrate on the δ-neighborhood of $\mathfrak{a}$. Hence the assertion (iv) is equivalent to the following assertion

(iv') $\mathrm{vol}(e^Y C \cap C)^p\, e^{\rho_{\mathfrak{g}}(Y)} \in L^1(\mathfrak{a} + \mathfrak{a}_{\mathfrak{g}}(R))$ for any compact $C \subset X$, $R > 0$.

To see that this assertion (iv') is equivalent to (v), we just have, for both implications, to replace the compact C by a larger compact $C' := e^{\mathfrak{a}_{\mathfrak{g}}(R)} C$ and to notice that the map $Y \mapsto \max\limits_{Z \in \mathfrak{a}_{\mathfrak{g}}(R)} |\rho_{\mathfrak{g}}(Y+Z) - \rho_{\mathfrak{g}}(Y)|$ is uniformly bounded on $\mathfrak{a}$.
$(v) \iff (vi)$ We use Theorem 4.4, that we will prove in the next section, and that gives, for C large enough, constants $m, M > 0$ such that

$$m\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \leq \mathrm{vol}(e^Y C \cap C) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \text{for any } Y \in \mathfrak{a}.$$

$(vi) \iff (vii)$ We recall that the function $\rho_{\mathfrak{g}} - p\rho_{\mathfrak{q}}^{\min}$ is continuous and piecewise linear.

This proves Proposition 4.3 provided the following Theorem 4.4. $\square$

4.3 Estimate of $\mathrm{vol}(e^Y C \cap C)$

Let C be a compact subset of X. We shall give both lower and upper bounds of the volume of $e^Y C \cap C$ as $Y \in \mathfrak{a}$ goes to infinity. For that we will use the function $\rho_{\mathfrak{q}}^{\min}$ defined by formula (4.3). Let $x_0 = eH \in X = G/H$ and $W_G x_0$ be the orbit of this point under the Weyl group of G.

Theorem 4.4. *Let G be an algebraic reductive Lie group, H an algebraic reductive subgroup and C a compact neighborhood of Kx_0 in $X := G/H$. Then there exist constants $m \equiv m_C > 0$ and $M \equiv M_C > 0$ such that*

$$m e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \leq \mathrm{vol}(e^Y C \cap C) \leq M e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{a}.$$

The proof of the lower bound will be given in Section 4.4.

We will give the proof of the upper bound in eight steps which will last from Section 4.4 to 4.8. Clearly, the upper bound in Theorem 4.4 is equivalent to the following statement:

For any compact sets C_1, C_2 in X, there exists $M \equiv M_{C_1,C_2} > 0$ such that

$$\mathrm{vol}(e^Y C_1 \cap C_2) \leq M e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{a}. \tag{4.8}$$

The strategy of the proof of (4.8) will be to see G/H as a closed orbit in a representation of G and to decompose C_1 and C_2 into smaller compact pieces.

4.4 Lower bound for $\mathrm{vol}(e^Y C \cap C)$

Up to the end of this chapter we keep the setting as above : G is a connected algebraic reductive Lie group, and H an algebraic reductive subgroup.

By Chevalley theorem ([5, Th. 5.1] or [6, Section 4.2]), there exists an algebraic representation $\tau \colon G \to GL(V)$ such that the homogeneous space $X = G/H$ is realized as a closed orbit $X = Gx_0 \subset V$ where $\mathrm{Stab}_G(x_0) = H$. We can assume that $\mathrm{Ker}(d\tau) = \{0\}$. We fix such a representation (τ, V) once and for all.

Here is the first step towards both the volume upper bound and the volume lower bound in Theorem 4.4.

Lemma 4.5. *There exists a neighborhood C_{x_0} of x_0 in G/H such that for any compact neighborhood C_0 of x_0 contained in C_{x_0}, there exist constants $m, M > 0$ such that*

$$me^{-\rho_{\mathfrak{q}}(Y)} \leq \mathrm{vol}(e^Y C_0 \cap C_0) \leq M e^{-\rho_{\mathfrak{q}}(Y)} \quad \textit{for any } Y \in \mathfrak{a}. \tag{4.9}$$

Proof. Since G and H are reductive, the representation of H in $\mathfrak{q}$ is volume preserving. Hence we can apply Proposition 3.7 to the representation of H in $\mathfrak{q}$. Roughly, the strategy is then to linearize X near x_0. To make this approach precise, we need two similar but slightly different arguments for the lower bound and for the upper bound.

Lower bound. We choose a sufficiently small compact neighborhood U_0 of 0 in $\mathfrak{q}$ on which the map

$$\pi_- : \mathfrak{q} \to X \ , \ Z \mapsto e^Z x_0$$

is well-defined, injective with a Jacobian bounded away from 0. Since x_0 is H-invariant, this map π_- is H-equivariant. For any compact neighborhood $C_0 = \pi_-(C)$ of x_0 in X with $C \subset U_0$, one has, for every $Y \in \mathfrak{a}$,

$$e^Y C_0 \cap C_0 \supset \pi_-(e^Y C \cap C).$$

The lower bound in (4.9) is then a consequence of the lower bound in Proposition 3.7.

Upper bound. Since the linear tangent space $T_{x_0}X \subset V$ of X at x_0 is canonically H-isomorphic to $\mathfrak{q}$, we will also denote it by $\mathfrak{q}$. Since H is reductive, this vector subspace $\mathfrak{q} \subset V$ admits a H-invariant supplementary subspace $\mathfrak{s}$. We set $p \colon V \to \mathfrak{q}$ for the linear projector with kernel $\mathfrak{s}$. We choose a sufficiently small compact neighborhood C_{x_0} of x_0 in X on which the map

$$\pi_+ \colon X \to \mathfrak{q} \ , \ x \mapsto p(x) - p(x_0)$$

is injective with a Jacobian bounded away from 0. Since x_0 is H-invariant, this map π_+ is also H-equivariant.

For any compact subset C_0 of C_{x_0}, one has, for every $Y \in \mathfrak{a}$,

$$\pi_+(e^Y C_0 \cap C_0) \subset e^Y C \cap C$$

where $C := \pi_+(C_0)$. The upper bound in (4.9) is then a consequence of the upper bound in Proposition 3.7. $\square$

As a direct corollary we get the lower bound in Theorem 4.4 .

Corollary 4.6. *For any compact neighborhood C of Kx_0 in G/H, there exists $m > 0$ such that*

$$\mathrm{vol}(e^Y C \cap C) \geq m e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{a}.$$

Proof. Shrinking C if necessary, we can assume that $C = KC_0$ where C_0 is a compact neighborhood of x_0. According to Lemma 4.5, there exists a constant $m > 0$ such that the lower bound in (4.9) is satisfied. For each $w \in W(Y; \mathfrak{a})$ $(\subset W_G)$, we take a representative $k_w \in N_K(\mathfrak{a}_{\mathfrak{g}})$. Then one has

$$\mathrm{vol}(e^Y C \cap C) \geq \mathrm{vol}(e^Y k_w^{-1} C_0 \cap C_0) = \mathrm{vol}(e^{wY} C_0 \cap C_0) \geq m e^{-\rho_{\mathfrak{q}}(wY)}.$$

Hence one has

$$\mathrm{vol}(e^Y C \cap C) \quad \geq \quad m \max_{w \in W(Y;\mathfrak{a})} e^{-\rho_{\mathfrak{q}}(wY)} \quad = \quad m\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)}.$$

This ends the proof. □

4.5 Volume near one invariant point

Here is the second step towards the volume upper bound (4.8). It is a subtle variation of the volume upper bound given in Lemma 4.5.

For any subspace $\mathfrak{b} \subset \mathfrak{a}$, we set $X^{\mathfrak{b}} := \{x \in X : e^Y x = x, \text{ for all } Y \in \mathfrak{b}\}$.

Lemma 4.7. *For any subspace $\mathfrak{b} \subset \mathfrak{a}$ and any point $x \in X^{\mathfrak{b}}$, there exists a neighborhood C_x of x in X and $M > 0$ such that*

$$\mathrm{vol}(e^Y C_x \cap C_x) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{b}.$$

Proof of Lemma 4.7. Let H' be the stabilizer of x in G and $\mathfrak{h}'$ its Lie algebra. Since x is in $X^{\mathfrak{b}}$, one has $\mathfrak{b} \subset \mathfrak{h}'$. Hence there exists a maximal split abelian subspace $\mathfrak{a}'$ of $\mathfrak{h}'$ containing $\mathfrak{b}$. Since all the maximal split abelian subspaces of $\mathfrak{h}$ are H-conjugate, one can find an element $g \in G$ such that $x = gx_0$. Then one has $H' := gHg^{-1}$ and $\mathfrak{h}' := \mathrm{Ad}(g)\mathfrak{h}$. After replacing g by a suitable element gh with h in H, we also have $\mathfrak{a}' = \mathrm{Ad}(g)\mathfrak{a}$. We set $\mathfrak{q}' := \mathrm{Ad}(g)\mathfrak{q}$ and introduce the function $\rho_{\mathfrak{q}'} \colon \mathfrak{a}' \to \mathbb{R}_{\geq 0}$ associated to the representation of H' on $\mathfrak{q}'$ as in Section 3.1. By definition, we have the following identity:

$$\rho_{\mathfrak{q}'}(\mathrm{Ad}(g)Z) = \rho_{\mathfrak{q}}(Z) \quad \text{for any } Z \in \mathfrak{a}. \tag{4.10}$$

Applying Lemma 4.5 to the homogeneous space G/H', we see that there exist a compact neighborhood C_x of x in X and a constant $M > 0$ such that

$$\mathrm{vol}(e^Y C_x \cap C_x) \leq M e^{-\rho_{\mathfrak{q}'}(Y)} \text{ for any } Y \in \mathfrak{a}'. \tag{4.11}$$

Now, for $Y \in \mathfrak{b}$, we set $Z = \mathrm{Ad}(g^{-1})Y$. This element Z belongs also to $\mathfrak{a}$. Since the Cartan subspace $\mathfrak{a}_{\mathfrak{g}}$ contains $\mathfrak{a}$ and since two elements of $\mathfrak{a}_{\mathfrak{g}}$ which are G-conjugate are always W_G-conjugate, there exists $w \in W_G$ such that $Z = wY$. Using (4.10), we get

$$\rho_{\mathfrak{q}'}(Y) = \rho_{\mathfrak{q}}(Z) = \rho_{\mathfrak{q}}(wY) \geq \rho_{\mathfrak{q}}^{\min}(Y).$$

Hence, Lemma 4.7 follows from (4.11). □

4.6 Volume near two invariant points

Here is the third step towards the volume upper bound (4.8).

Lemma 4.8. *For any vector subspace $\mathfrak{b} \subset \mathfrak{a}$ and any points x_1, x_2 in $X^{\mathfrak{b}}$, there exist compact neighborhoods C_1 of x_1 and C_2 of x_2 in X, and $M > 0$ such that*

$$\mathrm{vol}(e^Y C_1 \cap C_2) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{b}. \tag{4.12}$$

We set

$$V^{\mathfrak{b}} := \{v \in V : \mathfrak{b}v = 0\} \tag{4.13}$$

so that $X^{\mathfrak{b}} = X \cap V^{\mathfrak{b}}$ and we set $\pi^{\mathfrak{b}} \colon V \to V^{\mathfrak{b}}$ to be the $\mathfrak{b}$-equivariant projection.

Proof. When $x_1 = x_2$ this is Lemma 4.7. When $x_1 \neq x_2$, we choose C_1 and C_2 with disjoint projections $\pi^{\mathfrak{b}}(C_1) \cap \pi^{\mathfrak{b}}(C_2) = \emptyset$ so that, for any Y in $\mathfrak{b}$, the intersection $e^Y C_1 \cap C_2$ is also empty. □

Here is the fourth step towards the volume upper bound (4.8).

Lemma 4.9. *For any vector subspace $\mathfrak{b} \subset \mathfrak{a}$ and any compact subsets S_1, S_2 included in $X^{\mathfrak{b}}$, there exist $M > 0$ and compact neighborhoods C_{S_1} of S_1 and C_{S_2} of S_2 in X such that*

$$\mathrm{vol}(e^Y C_{S_1} \cap C_{S_2}) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{b}. \tag{4.14}$$

Proof. This is a consequence of Lemma 4.8 by a standard compactness argument. Let $x_1 \in S_1$. For any $x_2 \in S_2$, there exist compact neighborhoods $C_1(x_1, x_2)$ of x_1 and $C_2(x_1, x_2)$ of x_2 satisfying (4.12).

First we fix x_1 in S_1. By compactness of C_2, one can find a finite set $F_2 \equiv F_2(x_1) \subset S_2$ for which the union $C_2(x_1, S_2) := \cup_{x_2 \in F_2} C_2(x_1, x_2)$ is a compact neighborhood of S_2. The intersection $C_1(x_1, S_2) := \cap_{x_2 \in F_2} C_1(x_1, x_2)$ is still a compact neighborhood of x_1.

By compactness of C_1, one can find a finite set $F_1 \subset S_1$ for which the union $C_{S_1} := \cup_{x_1 \in F_1} C_1(x_1, S_2)$ is a compact neighborhood of S_1. The intersection $C_{S_2} := \cap_{x_1 \in F_1} C_2(x_1, S_2)$ is still a compact neighborhood of S_2.

Since only finitely many constants M are involved in this process, the compact neighborhoods C_{S_1} and C_{S_2} satisfy (4.14) $\square$

4.7 Facets

In this section, we shall introduce a decomposition of $\mathfrak{a}$ in convex pieces F called facets by using the representation $d\tau|_{\mathfrak{a}} \colon \mathfrak{a} \to \mathrm{End}(V)$.

We need to introduce more notations. Let $\Delta \equiv \Delta(V, \mathfrak{a})$ be the set of weights of $\mathfrak{a}$ in V. For v in V we write $v = \sum_{\lambda \in \Delta} v_\lambda$ according to the weight space decomposition $V = \bigoplus_{\lambda \in \Delta} V_\lambda$. We fix a norm $\| \ \|$ on each weight space V_λ, and define a norm on V by

$$\|v\| := \max_{\lambda \in \Delta} \|v_\lambda\|. \tag{4.15}$$

For any subset $F \subset \mathfrak{a}$, we set

$$\begin{aligned}
\Delta_F^+ &:= \{\lambda \in \Delta : \lambda(Y) > 0 \text{ for any } Y \in F\} \\
\Delta_F^0 &:= \{\lambda \in \Delta : \lambda(Y) = 0 \text{ for any } Y \in F\} \\
\Delta_F^- &:= \{\lambda \in \Delta : \lambda(Y) < 0 \text{ for any } Y \in F\}
\end{aligned}$$

We say that F is a *facet* if

$$\Delta = \Delta_F^+ \amalg \Delta_F^0 \amalg \Delta_F^- \qquad \text{and}$$

$$\begin{aligned}
F = \{Y \in \mathfrak{a} : \quad & \lambda(Y) > 0 \text{ for any } \lambda \in \Delta_F^+, \\
& \lambda(Y) = 0 \text{ for any } \lambda \in \Delta_F^0, \\
& \lambda(Y) < 0 \text{ for any } \lambda \in \Delta_F^-\}.
\end{aligned}$$

Let $\mathcal{F}$ be the totality of facets. Then we have

$$\mathfrak{a} = \bigsqcup_{F \in \mathcal{F}} F \quad \text{(disjoint union)}.$$

For any facet F we denote by $\mathfrak{a}_F$ its support, i.e. its linear span:

$$\mathfrak{a}_F := \{Y \in \mathfrak{a} : \lambda(Y) = 0 \ \text{ for any } \lambda \in \Delta_F^0\}.$$

We set

$$V_F^\varepsilon := \bigoplus_{\lambda \in \Delta_F^\varepsilon} V_\lambda \quad \text{for } \varepsilon = +, 0, -.$$

Notice that, using Notation (4.13), we have $V_F^0 = V^{\mathfrak{a}_F}$. We have a direct sum decomposition:

$$V = V_F^+ \oplus V_F^0 \oplus V_F^-. \tag{4.16}$$

Here is the fifth step towards the volume upper bound (4.8).

Lemma 4.10. *Let F be a facet, S_1 be a compact subset of $X \cap (V_F^0 \oplus V_F^-)$, and S_2 be a compact subset of $X \cap (V_F^0 \oplus V_F^+)$. Then there exist compact neighborhoods C_{S_1} of S_1 and C_{S_2} of S_2 in X, and $M > 0$ such that*

$$\mathrm{vol}(e^Y C_{S_1} \cap C_{S_2}) \le M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in \mathfrak{a}_F. \tag{4.17}$$

Proof. We recall that $\pi^{\mathfrak{a}_F}$ is the projection on $V_F^0 = V^{\mathfrak{a}_F}$. Since X is closed and is invariant under the group $e^{\mathfrak{a}_F}$, one has the inclusions

$$\pi^{\mathfrak{a}_F}(X \cap (V_F^0 \oplus V_F^-)) \subset X^{\mathfrak{a}_F} \quad \text{and} \quad \pi^{\mathfrak{a}_F}(X \cap (V_F^0 \oplus V_F^+)) \subset X^{\mathfrak{a}_F}.$$

Let $T_1 := \pi^{\mathfrak{a}_F}(S_1)$ and $T_2 := \pi^{\mathfrak{a}_F}(S_2)$ be the images of S_1 and S_2 by the projection $\pi^{\mathfrak{a}_F}$. Since

$$S_1 \subset X \cap (V_F^0 \oplus V_F^-) \quad \text{and} \quad S_2 \subset X \cap (V_F^0 \oplus V_F^+), \tag{4.18}$$

these images T_1 and T_2 are compact subsets of $X^{\mathfrak{a}_F}$. According to Lemma 4.9 with $\mathfrak{b} = \mathfrak{a}_F$, there exist $M > 0$ and compact neighborhoods C_{T_1} of T_1 and C_{T_2} of T_2 in X such that

$$\mathrm{vol}(e^Y C_{T_1} \cap C_{T_2}) \le M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \text{for any } Y \in \mathfrak{a}_F. \tag{4.19}$$

Using again (4.18), one can then find an element $Y_0 \in F$ such that

$$e^{Y_0} S_1 \subset \text{interior of } C_{T_1} \quad \text{and} \quad e^{-Y_0} S_2 \subset \text{interior of } C_{T_2}.$$

We choose then the neighborhoods

$$C_{S_1} := e^{-Y_0} C_{T_1} \quad \text{and} \quad C_{S_2} := e^{Y_0} C_{T_2}.$$

of S_1 and S_2 respectively. According to (4.19), one has, for any $Y \in \mathfrak{a}_F$,

$$\mathrm{vol}(e^Y C_{S_1} \cap C_{S_2}) = \mathrm{vol}(e^{Y-2Y_0} C_{T_1} \cap C_{T_2}) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y-2Y_0)}.$$

Since the function $Y \mapsto |\rho_{\mathfrak{q}}^{\min}(Y-2Y_0) - \rho_{\mathfrak{q}}^{\min}(Y)|$ is uniformly bounded on $\mathfrak{a}$, this gives the volume upper bound (4.17). $\square$

Here is the sixth step towards the volume upper bound (4.8).

Lemma 4.11. *Let F be a facet and C_1, C_2 compact subsets of G/H. Suppose $C_1 \cap (V_F^0 \oplus V_F^-) = \emptyset$ or $C_2 \cap (V_F^0 \oplus V_F^+) = \emptyset$. Then there exists $Y_0 \in F$ such that*

$$e^Y C_1 \cap C_2 = \emptyset \quad \textit{for any } Y \in Y_0 + F.$$

Proof. For a compact subset C of X and $\lambda \in \Delta$, we set

$$m_\lambda(C) := \min_{v \in C} \|v_\lambda\| \quad \text{and} \quad M_\lambda(C) := \max_{v \in C} \|v_\lambda\|,$$

and for $\varepsilon = \pm$, we set

$$m_F^\varepsilon(C) := \max_{\lambda \in \Delta_F^\varepsilon} m_\lambda(C) \quad \text{and} \quad M_F^\varepsilon(C) := \max_{\lambda \in \Delta_F^\varepsilon} M_\lambda(C).$$

If $C_1 \cap (V_F^0 \oplus V_F^-) = \emptyset$, one has $m_F^+(C_1) > 0$ and we choose $Y_0 \in F$ such that, for all $\lambda \in \Delta_F^+$,

$$e^{\lambda(Y_0)} > \frac{M_F^+(C_2)}{m_F^+(C_1)}.$$

Let $Y \in Y_0 + F$. By definition of $m_F^+(C_1)$, one can find $\lambda \in \Delta_F^+$ such that, for any v in C_1, one has $\|v_\lambda\| \geq m_F^+(C_1)$. One has then

$$\|(e^Y v)_\lambda\| = e^{\lambda(Y)} \|v_\lambda\| \geq e^{\lambda(Y_0)} m_F^+(C_1) > M_F^+(C_2).$$

Hence $e^Y v$ does not belong to C_2. This proves that $e^Y C_1 \cap C_2 = \emptyset$.

Likewise, if $C_2 \cap (V_F^+ \oplus V_F^0) = \emptyset$, one has $m_F^-(C_2) > 0$, and we choose $Y_0 \in F$ such that, for all $\lambda \in \Delta_F^-$,

$$e^{-\lambda(Y_0)} > \frac{M_F^-(C_1)}{m_F^-(C_2)}.$$

$\square$

4.8 Upper bound for $\mathrm{vol}(e^Y C \cap C)$

Here is the seventh step towards the volume upper bound (4.8). For any facet F, any $Y_0 \in F$, and any $R \geq 0$, we introduce the R-neighborhood of the Y_0-translate of the facet F :

$$F(Y_0, R) := Y_0 + F + \mathfrak{a}(R), \tag{4.20}$$

where $\mathfrak{a}(R)$ is the ball $\{Y \in \mathfrak{a} : \|Y\| \leq R\}$.

Lemma 4.12. *Let F be a facet, $R \geq 0$, and C_1, C_2 compact subsets of G/H. Then there exist $Y_{F,R} \in F$ and $M > 0$ such that*

$$\mathrm{vol}(e^Y C_1 \cap C_2) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)} \quad \textit{for any } Y \in F(Y_{F,R}, R). \tag{4.21}$$

Proof. We first assume that $R = 0$. We will deduce Lemma 4.12 from the two previous steps, namely Lemmas 4.10 and 4.11. Let

$$S_1 := C_1 \cap (V_F^0 \oplus V_F^-) \quad \text{and} \quad S_2 := C_2 \cap (V_F^0 \oplus V_F^+).$$

According to Lemma 4.10 we can write

$$C_1 := C_{S_1} \cup C_1' \quad \text{and} \quad C_2 := C_{S_2} \cup C_2'$$

where C_{S_1} and C_{S_2} are respectively compact neighborhoods of S_1 in C_1 and of S_2 in C_2 satisfying the volume upper bound (4.17) for some constant $M > 0$, and where C_1' and C_2' are compact subsets of X such that

$$C_1' \cap (V_F^0 \oplus V_F^-) = \emptyset \quad \text{and} \quad C_2' \cap (V_F^0 \oplus V_F^+) = \emptyset.$$

Hence according to Lemma 4.11, there exists an element $Y_F \in F$ such that, for any $Y \in Y_F + F$, one has,

$$e^Y C_1' \cap C_2' = e^Y C_{S_1} \cap C_2' = e^Y C_1' \cap C_{S_2} = \emptyset\ .$$

Hence, one has the desired volume upper bound, for any $Y \in Y_F + F$,

$$\mathrm{vol}(e^Y C_1 \cap C_2) = \mathrm{vol}(e^Y C_{S_1} \cap C_{S_2}) \leq M\, e^{-\rho_{\mathfrak{q}}^{\min}(Y)}.$$

When R is not zero, we apply the first case to the compact sets $e^{\mathfrak{a}(R)}C_1$ and C_2 and notice that the function $Y \mapsto \max\limits_{Z \in \mathfrak{a}(R)} |\rho_{\mathfrak{q}}^{\min}(Y+Z) - \rho_{\mathfrak{q}}^{\min}(Y)|$ is uniformly bounded on $\mathfrak{a}$. $\square$

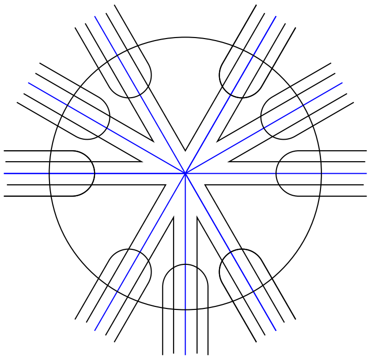

Figure 1: Cover of $\mathfrak{a}$

Proof of Theorem 4.4. Here is the eighth and last step towards the volume upper bound (4.8). Fix two compact sets C_1, C_2 of G/H. According to Lemma 4.12, *given any facet $F \in \mathcal{F}$ and any $R > 0$ there exist $Y_{F,R} \in F$ and $M > 0$ such that* (4.8) *holds for any $Y \in F(Y_{F,R}, R)$.* The following Lemma 4.13 tells us that (4.8) holds for any Y in $\mathfrak{a}$. This ends the proof of the volume upper bound (4.8) and of Theorem 4.4, $\square$

Lemma 4.13. *Assume that, for any facet F and any $R \geq 0$, we are given an element $Y_{F,R} \in F$. Then, one can choose for every facet F a constant $R_F \geq 0$ such that, using notations* (4.20), *one has*

$$\mathfrak{a} = \bigcup_{F \in \mathcal{F}} F(Y_{F,R_F}, R_F). \tag{4.22}$$

Proof. We will choose inductively on $\ell = 0, 1, \ldots, \dim \mathfrak{a}$, simultaneously the constants R_F for all the facets of codimension ℓ (see Figure 1).

We first choose $R_F = 0$ for all the open facets F.

We assume that R_F has been chosen for the facets of codimension strictly less than ℓ and we consider the set

$$\mathfrak{a}_\ell = \bigcup_{\substack{F \in \mathcal{F} \\ \operatorname{codim} F < \ell}} F(Y_{F,R_F}, R_F).$$

We assume, by induction, that there exists a constant $\delta_\ell > 0$ such that the complementary set $\mathfrak{a} \setminus \mathfrak{a}_\ell$ is included in a δ_ℓ-neighborhood of the union of the facets of codimension ℓ. We choose $R_F = \delta_\ell$ for all the facets of codimension

ℓ. This gives a new set $\mathfrak{a}_{\ell+1}$. The complementary set $\mathfrak{a} \setminus \mathfrak{a}_{\ell+1}$ is then included in a $\delta_{\ell+1}$-neighborhood of the union of the facets of codimension $\ell+1$, for some constant $\delta_{\ell+1} > 0$. And we go on by induction. $\square$

5 Application

The criterion given in Theorem 4.1 is easy to apply: it is easy to detect for a given homogeneous space G/H whether the unitary representation of G in $L^2(G/H)$ is tempered or not. We collect in this chapter a few corollaries of this criterion, omitting the details of the computational verifications that will be published elsewhere together with a complete classification of homogeneous spaces G/H for which $L^2(G/H)$ is non-tempered.

5.1 Abelian or amenable generic stabilizer

For general real reductive homogeneous spaces, we deduce the following facts :

Proposition 5.1. *Let $p \geq 2$ be an even integer. Let G be a semisimple algebraic Lie group, and $H_1 \supset H_2$ two unimodular subgroups.*
a) If $L^2(G/H_1)$ is almost L^p then $L^2(G/H_2)$ is almost L^p.
b) The converse is true when H_2 is normal in H_1 and H_1/H_2 is amenable (for instance finite, or compact, or abelian).

Proposition 5.2. *Let $p \geq 2$ be an even integer. Let G be an algebraic semisimple Lie group, and H an algebraic reductive subgroup.*
a) If the representation of $G_{\mathbb{C}}$ in $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is almost L^p, then the representation of G in $L^2(G/H)$ is almost L^p.
b) The converse is true when H is a split group.

Theorem 5.3. *Let G be an algebraic semisimple real Lie group, and H an algebraic reductive subgroup.*
a) If the representation of G in $L^2(G/H)$ is tempered, then the set of points in G/H with amenable stabilizer in H is dense.
b) If the set of points in G/H with abelian stabilizer in $\mathfrak{h}$ is dense, then the representation of G in $L^2(G/H)$ is tempered.

The proof of Theorem 5.3 leads us to the list of all the spaces G/H for which the representation of G in $L^2(G/H)$ is non-tempered.

5.2 Complex homogeneous spaces

We assume in this section that G and H are complex Lie groups. Since complex amenable reductive Lie groups are abelian, the following result is a particular case of Theorem 5.3.

Theorem 5.4. *Suppose G is a complex algebraic semisimple group and H a complex reductive subgroup. Then $L^2(G/H)$ is tempered if and only if the set of points in G/H with abelian stabilizer in $\mathfrak{h}$ is dense.*

Example 5.5. *$L^2(SL(n,\mathbb{C})/SO(n,\mathbb{C}))$ is always tempered.*
$L^2(SL(2m,\mathbb{C})/Sp(m,\mathbb{C}))$ is never tempered.
$L^2(SO(7,\mathbb{C})/G_2)$ is not tempered.

Example 5.6. *Let $n = n_1 + \cdots + n_r$ with $n_1 \geq \cdots \geq n_r \geq 1$, $r \geq 2$.*
$L^2(SL(n,\mathbb{C})/\prod SL(n_i,\mathbb{C}))$ is tempered iff $2n_1 \leq n+1$.
$L^2(SO(n,\mathbb{C})/\prod SO(n_i,\mathbb{C}))$ is tempered iff $2n_1 \leq n+2$.
$L^2(Sp(n,\mathbb{C})/\prod Sp(n_i,\mathbb{C}))$ is tempered iff $r \geq 3$ and $2n_1 \leq n$.

5.3 Real homogeneous spaces

Here are a few examples of application of our criterion.

Example 5.7. *Let G_1 be a real algebraic semisimple Lie group and K_1 a maximal compact subgroup.*
$L^2(G_1 \times G_1/\Delta(G_1))$ is always tempered.
$L^2(G_{1,\mathbb{C}}/G_1)$ is always tempered.
$L^2(G_{1,\mathbb{C}}/K_{1,\mathbb{C}})$ is tempered iff G_1 is quasisplit.

Example 5.8. *Let G/H be a symmetric space i.e. G is a real algebraic semisimple Lie group and H is the set of fixed points of an involution of G. Write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the H-invariant decomposition of $\mathfrak{g}$. Let G' be an algebraic semisimple Lie group with Lie algebra $\mathfrak{g}' = \mathfrak{h} + \sqrt{-1}\mathfrak{q}$.*

Then $L^2(G/H)$ is almost L^p iff $L^2(G'/H)$ is almost L^p.

Example 5.9. *$L^2(SL(p+q,\mathbb{R})/SO(p,q))$ is always tempered.*
$L^2(SL(2m,\mathbb{R})/Sp(m,\mathbb{R}))$ is never tempered.
$L^2(SL(m+n,\mathbb{R})/SL(m,\mathbb{R}) \times SL(n,\mathbb{R}))$ is tempered iff $|m-n| \leq 1$.

Example 5.10. *Let $p_1 + \cdots + p_r \leq p$ and $q_1 + \cdots + q_r \leq q$.*
$L^2(SO(p,q)/\prod SO(p_i,q_i))$ is tempered iff $2 \max\limits_{p_i q_i \neq 0}(p_i + q_i) \leq p + q + 2$.

The homogeneous spaces in Examples 5.6 and 5.10 are not symmetric spaces when $r \geq 3$.

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