An integral formula for L^2-eigenfunctions of a fourth order Bessel-type differential operator

Toshiyuki Kobayashi*
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan.
E-mail address: `toshi@ms.u-tokyo.ac.jp`

Jan Möllers†
Institut für Mathematematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn, Germany
E-mail address: `moellers@math.uni-paderborn.de`

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Abstract

We find an explicit integral formula for the eigenfunctions of a fourth order differential operator against the kernel involving two Bessel functions. Our formula establishes the relation between K-types in two different realizations of the minimal representation of the indefinite orthogonal group, namely the L^2-model and the conformal model.

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1 Introduction and statement of the results

Let $\theta = x\frac{\mathrm{d}}{\mathrm{d}x}$ be the one-dimensional Euler operator. We consider the following representation of the Bessel differential operator

$$\frac{1}{x^2}Q_\nu(\theta) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\nu+1}{x}\frac{\mathrm{d}}{\mathrm{d}x} - 1, \qquad \nu \in \mathbb{C},$$

where Q_ν is the quadratic transform of the Weyl algebra $\mathbb{C}[x, \frac{\mathrm{d}}{\mathrm{d}x}]$ defined by

$$Q_\nu(P) = P(P+\nu) - x^2, \qquad \text{for } P \in \mathbb{C}\left[x, \frac{\mathrm{d}}{\mathrm{d}x}\right].$$

Our object of study is the L^2-eigenfunctions of the fourth order differential operator

$$\begin{aligned} D_{\mu,\nu} &:= \frac{1}{x^2}Q_\nu(\theta+\mu)Q_\nu(\theta) \\ &= \frac{1}{x^2}\left((\theta+\mu)(\theta+\mu+\nu) - x^2\right)\left(\theta(\theta+\nu) - x^2\right). \end{aligned}$$

Throughout this article we assume that the parameters μ and ν satisfy the following integrality condition:

$$\mu \geq \nu \geq -1 \text{ are integers of the same parity, not both equal to } -1. \tag{1.1}$$

We then have the following fact (see [4, Theorem A]):

Fact. *The differential operator $D_{\mu,\nu}$ extends to a self-adjoint operator on $L^2(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$ with only discrete spectrum which is given by*

$$\lambda_j^{\mu,\nu} := 4j(j+\mu+1), \qquad j = 0, 1, 2, \ldots$$

The corresponding L^2-eigenspaces are one-dimensional.

For instance, it is easily seen that the normalized K-Bessel function $\widetilde{K}_{\frac{\nu}{2}}(z) := (\frac{z}{2})^{-\frac{\nu}{2}}K_{\frac{\nu}{2}}(z)$ is an L^2-eigenfunction of $D_{\mu,\nu}$ for the eigenvalue $\lambda_0^{\mu,\nu} = 0$.

The purpose of this article is to establish the following integral formula for L^2-solutions of the differential equation

$$D_{\mu,\nu}u = \lambda_j^{\mu,\nu}u. \tag{1.2}$$

Theorem A. *Assume* (1.1) *and let u be an L^2-solution of the differential equation* (1.2)*. Then there exists a constant $A_j^{\mu,\nu}(u)$ such that for $\cos\vartheta + \cos\varphi > 0$:*

$$\begin{aligned} &\int_0^\infty u(x)\widetilde{J}_{\frac{\mu}{2}}(ax)\widetilde{J}_{\frac{\nu}{2}}(bx)x^{\mu+\nu+1}\,\mathrm{d}x \\ &\quad = A_j^{\mu,\nu}(u)\left(\frac{\cos\vartheta+\cos\varphi}{2}\right)^{\frac{\mu+\nu+2}{2}}\widetilde{C}_j^{\frac{\mu+1}{2}}(\cos\vartheta)\widetilde{C}_{j+\frac{\mu-\nu}{2}}^{\frac{\nu+1}{2}}(\cos\varphi), \end{aligned} \tag{1.3}$$

where we set $a := \frac{\sin\vartheta}{\cos\vartheta+\cos\varphi}$ and $b := \frac{\sin\varphi}{\cos\vartheta+\cos\varphi}$.

Here $\widetilde{J}_\alpha(x) = \left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x)$ denotes the normalized J-Bessel function and $\widetilde{C}_n^\lambda(x) = \Gamma(\lambda) C_n^\lambda(x)$ is the normalized Gegenbauer polynomial.

The differential equation (1.2) has a regular singularity at $x = 0$ with characteristic exponents 0, $-\nu$, $-\mu$ and $-\mu-\nu$. Accordingly, the asymptotic behaviour of a non-zero L^2-solution u of (1.2) as $x \to 0$ is of the following form (see [4, Theorem 4.2 (1)]):

$$u(x) \sim B_j^{\mu,\nu}(u) \times \begin{cases} x^{-\nu} + o(x^{-\nu}) & \text{for } \nu > 0, \\ \log(\frac{x}{2}) + o(\log(\frac{x}{2})) & \text{for } \nu = 0, \\ 1 + o(1) & \text{for } \nu = -1, \end{cases} \tag{1.4}$$

with some non-zero constant $B_j^{\mu,\nu}(u)$. The constant $A_j^{\mu,\nu}(u)$ in Theorem A is determined by $B_j^{\mu,\nu}(u)$ as follows:

Theorem B. *For any solution u of* (1.2)*:*

$$\frac{A_j^{\mu,\nu}(u)}{B_j^{\mu,\nu}(u)} = (-1)^j \frac{j! 2^{2\mu+\nu} \Gamma(\frac{\mu+2}{2}) \Gamma(\frac{\mu-|\nu|+2}{2}) \Gamma(j + \frac{\mu-\nu+2}{2})}{\Gamma(j + \frac{\mu-|\nu|+2}{2}) \pi \Gamma(j+\mu+1)} \times \begin{cases} \dfrac{2}{\Gamma\left(\frac{\nu}{2}\right)} & \text{for } \nu > 0, \\ -1 & \text{for } \nu = 0, \\ -\dfrac{2}{\Gamma\left(\frac{\nu}{2}\right)} & \text{for } \nu = -1. \end{cases}$$

The proofs of Theorems A and B will be given in Sections 2 and 3, respectively.

In Section 4 we give some applications and discuss special values of Theorem A. One particularly interesting situation arises when both μ and ν are odd integers. In this case the solutions u of (1.2) can be expressed as

$$u(x) = \text{const} \times \begin{cases} x^{-\nu} e^{-x} M_j^{\mu,\nu}(2x) & \text{for } \nu \geq 1, \\ e^{-x} M_j^{\mu,\nu}(2x) & \text{for } \nu = -1, \end{cases}$$

for some polynomial $M_j^{\mu,\nu}$ (see [5]). For $\nu = \pm 1$ these polynomials reduce to the classical Laguerre polynomials $M_j^{\mu,\pm 1}(x) = L_j^\mu(x)$. Hence, for $\nu = \pm 1$ the integral formula in Theorem A collapses to integral formulas for the Laguerre polynomials. Even these we could not trace in the literature.

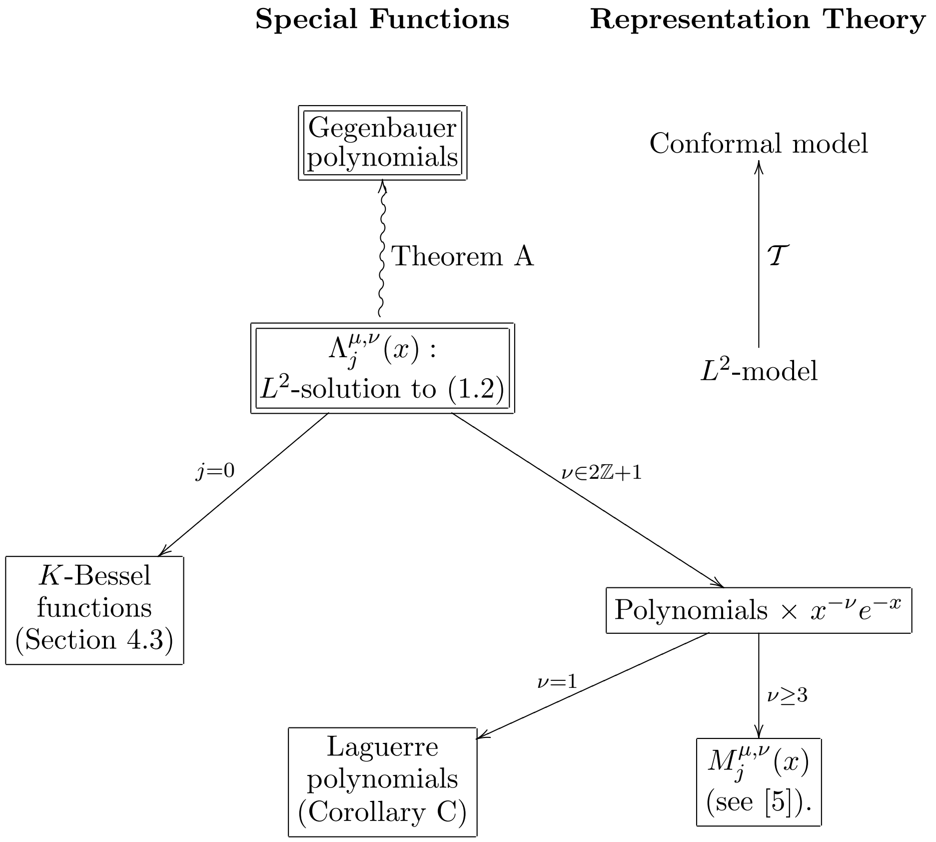
Corollary C. *Let $\mu \geq 1$ be an odd integer and $\cos\vartheta + \cos\varphi > 0$. Set $a := \frac{\sin\vartheta}{\cos\vartheta + \cos\varphi}$ and $b := \frac{\sin\varphi}{\cos\vartheta + \cos\varphi}$. Then we have*

$$\int_0^\infty L_j^\mu(2x) \widetilde{J}_{\frac{\mu}{2}}(ax) \cos(bx) x^\mu e^{-x} \, \mathrm{d}x$$
$$= (-1)^j \frac{2^\mu}{\sqrt{\pi}} \left(\frac{\cos\vartheta + \cos\varphi}{2} \right)^{\frac{\mu+1}{2}} \cos\left(j + \frac{\mu+1}{2}\right) \varphi \; \widetilde{C}_j^{\frac{\mu+1}{2}}(\cos\vartheta)$$

and

$$\int_0^\infty L_j^\mu(2x)\widetilde{J}_{\frac{\mu}{2}}(ax)\sin(bx)x^\mu e^{-x}\,\mathrm{d}x$$
$$= (-1)^j \frac{2^\mu}{\sqrt{\pi}}\left(\frac{\cos\vartheta+\cos\varphi}{2}\right)^{\frac{\mu+1}{2}} \sin\left(j+\frac{\mu+1}{2}\right)\varphi\;\widetilde{C}_j^{\frac{\mu+1}{2}}(\cos\vartheta).$$

Note that there appear 5 parameters in the integral formula in Theorem A, namely μ, ν, ϑ, φ and j. Our scheme (specialization of parameters, relation to representation theory) is summarized in the following diagram:



Notation: $\mathbb{N}_0=\{0,1,2,\ldots\}$, $\mathbb{R}_+=\{x\in\mathbb{R}: x>0\}$.

2 Two models for the minimal representation of the indefinite orthogonal group

In the proof of Theorem A we will use representation theory, namely two different models for the minimal representation of the indefinite orthogonal group $G=O(p,q)$ where $p\geq q\geq 2$ and $p+q\geq 6$ is even. These two models were constructed by T. Kobayashi and B. Ørsted [7, 8] and investigated further by T. Kobayashi and G. Mano [6]. This unitary representation is irreducible and attains the minimum Gelfand–Kirillov dimension among all irreducible unitary representations of G. In physics the minimal representation of $O(4,2)$ appears as the bound states of the Hydrogen atom, and incidentally as the quantum Kepler problem.

2.1 The conformal model

We begin with a quick review of the conformal model for the minimal representation of $G = O(p,q)$ from [7].

We equip $M := S^{p-1} \times S^{q-1}$ with the standard indefinite Riemannian metric of signature $(p-1, q-1)$ by letting the second factor be negative definite. Then G acts on M by conformal transformations. The solution space

$$\mathcal{S}ol(\widetilde{\Delta}_M) := \left\{ f \in C^\infty(M) : \widetilde{\Delta}_M f = 0 \right\}$$

of the Yamabe operator $\widetilde{\Delta}_M = \Delta_{S^{p-1}} - \Delta_{S^{q-1}} - \left(\frac{p-2}{2}\right)^2 + \left(\frac{q-2}{2}\right)^2$ is infinite-dimensional. Further, it is invariant under the 'twisted action' ϖ of G and hence defines a representation. The minimal representation of G is realized on the Hilbert completion

$$\mathcal{H} := \overline{\mathcal{S}ol(\widetilde{\Delta}_M)}$$

of $\mathcal{S}ol(\widetilde{\Delta}_M)$ with respect to a certain G-invariant inner product.

The maximal compact subgroup $K = O(p) \times O(q)$ of G acts on M as isometries, and the restriction of ϖ to K is given just by rotations. To see the K-types we recall the space of spherical harmonics

$$\mathcal{H}^k(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} \varphi = -k(k+n-2)\varphi \right\},$$

or equivalently, the space of restrictions of harmonic homogeneous polynomials on $\mathbb{R}^n$ of degree k to the sphere S^{n-1}. The orthogonal group $O(n)$ acts irreducibly on $\mathcal{H}^k(\mathbb{R}^n)$ for any k by rotations in the argument. Then clearly

$$\mathcal{H}^j(\mathbb{R}^p) \otimes \mathcal{H}^k(\mathbb{R}^q) \subseteq \mathcal{S}ol(\widetilde{\Delta}_M) \text{ if and only if } k = j + \frac{p-q}{2},$$

and we put

$$V^j := \mathcal{H}^j(\mathbb{R}^p) \otimes \mathcal{H}^{j+\frac{p-q}{2}}(\mathbb{R}^q), \qquad j = 0, 1, 2, \ldots,$$

on which K acts irreducibly. The multiplicity-free sum $\oplus_{j=0}^\infty V^j$ of irreducible representations of K is dense in the Hilbert space $\mathcal{H}$.

Let K' be the isotropy group of K at $((1,0,\ldots,0),(0,\ldots,0,1)) \in S^{p-1} \times S^{q-1}$. Then $K' \cong O(p-1) \times O(q-1)$. We write $\mathcal{H}^{K'}$ for the space of K'-fixed vectors.

Lemma 2.1. *In each K-type V^j ($j \in \mathbb{N}_0$) the subspace $V^j \cap \mathcal{H}^{K'}$ is one-dimensional and spanned by the functions*

$$\psi_j : S^{p-1} \times S^{q-1} \to \mathbb{C}, (v_0, v', v'', v_{p+q-1}) \mapsto \widetilde{C}_j^{\frac{p-2}{2}}(v_0)\widetilde{C}_{j+\frac{p-q}{2}}^{\frac{q-2}{2}}(v_{p+q-1}). \tag{2.1}$$

Proof. It is well-known that any $O(n-1)$-invariant spherical harmonic is a scalar multiple of the Gegenbauer polynomial

$$S^{n-1} \ni (x_1, x') \mapsto \widetilde{C}_k^{\frac{n-2}{2}}(x_1),$$

which shows the claim. □

2.2 The L^2-model

We recall the L^2-model (Schrödinger model) of the minimal representation of G which is unitarily equivalent to ϖ (see [6, 8]). Consider the isotropic cone

$$C = \{(x', x'') \in \mathbb{R}^{p-1} \times \mathbb{R}^{q-1} : |x'| = |x''| \neq 0\} \subseteq \mathbb{R}^{p+q-2}.$$

Then the group G acts unitarily in a non-trivial way on the Hilbert space $L^2(C, \mathrm{d}\mu)$ and defines a minimal representation of G. Here $\mathrm{d}\mu$ is the $O(p-1, q-1)$-invariant measure on C which is in bipolar coordinates

$$\mathbb{R}_+ \times S^{p-2} \times S^{q-2} \xrightarrow{\sim} C, \ (r, \omega, \eta) \mapsto (r\omega, r\eta)$$

normalized by $\mathrm{d}\mu = \frac{1}{2} r^{p+q-5}\, \mathrm{d}r\, \mathrm{d}\omega\, \mathrm{d}\eta$. ($\mathrm{d}\omega$ and $\mathrm{d}\eta$ denote the Euclidean measures on S^{p-2} and S^{q-2}, respectively.) The representation of the whole group G on $L^2(C)$ does not come from the geometry C, but the action of the subgroup K' is given by rotation in the argument. Hence the K'-invariant functions only depend on the radial parameter $r \in \mathbb{R}_+$ and the space of K'-invariants in $L^2(C)$ is identified as $L^2(C)^{K'} \cong L^2(\mathbb{R}_+, \frac{1}{2} r^{p+q-5}\, \mathrm{d}r)$.

Let W^j be the V^j-isotypic component in $L^2(C)$. In this model it is more difficult to find explicit K-finite vectors. By highlighting K'-fixed vectors, the following result was proved in [4, Section 8]:

Lemma 2.2. *In each K-type W^j ($j \in \mathbb{N}_0$) the subspace $W^j \cap L^2(C)^{K'}$ is one-dimensional and given by the radial functions*

$$u(2r), \tag{2.2}$$

where u is an L^2-solution of (1.2) with $\mu = p-3$, $\nu = q-3$.

2.3 The G-intertwiner

Let $\mathcal{T} : L^2(C) \xrightarrow{\sim} \mathcal{H}$ be the intertwining operator as given in [6, Section 2.2]. It is the composition of the Fourier transform $\mathcal{S}'(\mathbb{R}^{p+q-2}) \to \mathcal{S}'(\mathbb{R}^{p+q-2})$ and an operator coming from the conformal transformation from the flat indefinite Euclidean space $\mathbb{R}^{p-1,q-1}$ to M. For radial functions $f \in L^2(\mathbb{R}_+, \frac{1}{2}r^{p+q-5}\,\mathrm{d}r) \cong L^2(C)^{K'}$ this operator can be written by means of the Hankel transform (cf. [6, Lemma 3.3.1]):

$$\mathcal{T}f(v_0, v', v'', v_{p+q-1}) = \frac{1}{(v_0 + v_{p+q-1})^{\frac{p+q-4}{2}}} \int_0^\infty f(r) \\ \times \widetilde{J}_{\frac{p-3}{2}}\left(\frac{2|v'|r}{v_0 + v_{p+q-1}}\right) \widetilde{J}_{\frac{q-3}{2}}\left(\frac{2|v''|r}{v_0 + v_{p+q-1}}\right) r^{p+q-5}\,\mathrm{d}r \quad (2.3)$$

for $(v_0, v', v'', v_{p+q-1}) \in M$ with $v_0 + v_{p+q-1} > 0$. Since $\mathcal{T}$ intertwines the actions of G on both models, it clearly maps K'-invariant functions to K'-invariant functions and also preserves K-types, i.e. $\mathcal{T}(W^j) = V^j$. Hence we obtain the following diagram:

$$\begin{array}{rccc} & W^j & \xrightarrow{\sim} & V^j \\ & \cap & & \cap \\ \mathcal{T}: & L^2(C) & \xrightarrow{\sim} & \mathcal{H}^{K'} \\ & \cup & & \cup \\ & L^2(C)^{K'} & \xrightarrow{\sim} & \mathcal{H}^{K'}. \end{array}$$

Now $W^j \cap L^2(C)^{K'}$ and $V^j \cap \mathcal{H}^{K'}$ are one-dimensional and we have formulas (2.2) and (2.1) for their generators. Hence the intertwiner $\mathcal{T}$ has to map the functions (2.2) to multiples of the functions (2.1). Thus, for any L^2-solution u of (1.2) and $v_0 + v_{p+q-1} > 0$ we obtain

$$\int_0^\infty u(2r)\widetilde{J}_{\frac{p-3}{2}}\left(\frac{2|v'|r}{v_0 + v_{p+q-1}}\right) \widetilde{J}_{\frac{q-3}{2}}\left(\frac{2|v''|r}{v_0 + v_{p+q-1}}\right) r^{p+q-5}\,\mathrm{d}r \\ = \mathrm{const} \cdot (v_0 + v_{p+q-1})^{\frac{p+q-4}{2}} \widetilde{C}_j^{\frac{p-2}{2}}(v_0) \widetilde{C}_{j+\frac{p-q}{2}}^{\frac{q-2}{2}}(v_{p+q-1}) \quad (2.4)$$

Substituting $x = 2r$, $\mu = p - 3$ and $\nu = q - 3$ and putting

$$\cos\vartheta = v_0, \qquad \cos\varphi = v_{p+q-1},$$
$$\sin\vartheta = |v'|, \qquad \sin\varphi = |v''|.$$

we get (1.3) with a certain constant $A_j^{\mu,\nu}$. This finishes the proof of Theorem A.

3 A closed formula for the constants

In this section we find an explicit constant for the integral formula in Theorem A, namely we give a proof of Theorem B. Our method uses the generating function of L^2-eigenfunctions of $D_{\mu,\nu}$.

Remember that we assume the integrality condition (1.1). Let

$$G^{\mu,\nu}(t,x) = \frac{1}{(1-t)^{\frac{\mu+\nu+2}{2}}} \widetilde{I}_{\frac{\mu}{2}}\left(\frac{tx}{1-t}\right) \widetilde{K}_{\frac{\nu}{2}}\left(\frac{x}{1-t}\right), \tag{3.1}$$

where $\widetilde{I}_\alpha(z) := (\frac{z}{2})^{-\alpha} I_\alpha(z)$ and $\widetilde{K}_\alpha(z) := (\frac{z}{2})^{-\alpha} K_\alpha(z)$ denote the normalized I- and K-Bessel functions. Further, let $(\Lambda_j^{\mu,\nu}(x))_{j=0,1,2,\ldots}$ be the family of functions on $\mathbb{R}_+$ which has $G^{\mu,\nu}(t,x)$ as its generating function:

$$G^{\mu,\nu}(t,x) := \sum_{j=0}^{\infty} \Lambda_j^{\mu,\nu}(x) t^j. \tag{3.2}$$

Fact ([4, Theorem A])**.** *$\Lambda_j^{\mu,\nu}(x)$ is real analytic on $\mathbb{R}_+$ and an L^2-solution of* (1.2)*.*

We will now compute the constants $A_j^{\mu,\nu}(u)$ and $B_j^{\mu,\nu}(u)$ for $u = \Lambda_j^{\mu,\nu}$. Here we recall that $A_j^{\mu,\nu}$ and $B_j^{\mu,\nu}$ were defined in Theorem A and (1.4).

From [4, Theorem 4.2] we immediately obtain

$$B_j^{\mu,\nu}(\Lambda_j^{\mu,\nu}) = \frac{\Gamma(j+\frac{\mu-|\nu|+2}{2})}{j!\Gamma(\frac{\mu+2}{2})\Gamma(\frac{\mu-|\nu|+2}{2})} \times \begin{cases} 2^{\nu-1}\Gamma\left(\frac{\nu}{2}\right) & \text{for } \nu > 0, \\ -1 & \text{for } \nu = 0, \\ \frac{1}{2}\Gamma\left(-\frac{\nu}{2}\right) & \text{for } \nu = -1. \end{cases} \tag{3.3}$$

For $A_j^{\mu,\nu}$ we have the following lemma:

Lemma 3.1. *For any $j \in \mathbb{N}_0$ we have*

$$A_j^{\mu,\nu}(\Lambda_j^{\mu,\nu}) = (-1)^j \frac{2^{2(\mu+\nu)}\Gamma(j+\frac{\mu-\nu+2}{2})}{\pi\Gamma(j+\mu+1)}. \tag{3.4}$$

Putting (3.3) and (3.4) together proves Theorem B. In the remaining part of this section, we give a proof of Lemma 3.1.

Proof of Lemma 3.1. We put $\vartheta = \varphi = 0$ in (1.3). Using the special values

$$\widetilde{J}_\alpha(0) = \frac{1}{\Gamma(\alpha+1)}, \qquad \widetilde{C}_n^\lambda(1) = \frac{\Gamma(n+2\lambda)\Gamma(\lambda)}{\Gamma(n+1)\Gamma(2\lambda)},$$

we obtain

$$A_j^{\mu,\nu}(\Lambda_j^{\mu,\nu}) = \frac{2^{\mu+\nu} j!\Gamma(j+\frac{\mu-\nu+2}{2})}{\pi\Gamma(j+\mu+1)\Gamma(j+\frac{\mu+\nu+2}{2})} \int_0^\infty \Lambda_j^{\mu,\nu}(x) x^{\mu+\nu+1}\,\mathrm{d}x.$$

Together with the next lemma this finishes the proof. $\square$

Lemma 3.2. *For every $j \in \mathbb{N}_0$ we have $\Lambda_j^{\mu,\nu} \in L^1(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$ and*

$$\int_0^\infty \Lambda_j^{\mu,\nu}(x) x^{\mu+\nu+1}\,\mathrm{d}x = (-1)^j \frac{2^{\mu+\nu}\Gamma(j+\frac{\mu+\nu+2}{2})}{j!}.$$

Proof. The fact that $\Lambda_j^{\mu,\nu} \in L^1(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$ is derived from the asymptotic behaviour of $\Lambda_j^{\mu,\nu}(x)$ (see [4, Theorem 4.2]). To calculate the integral we use the following integral formula which is valid for $\Re(\lambda+\alpha\pm\beta+1) > 0$ and $b > a > 0$ (see e.g. [3, formula 6.576 (5)]):

$$\int_0^\infty I_\alpha(ax)K_\beta(bx)x^\lambda\,\mathrm{d}x = \frac{a^\alpha\Gamma(\frac{\lambda+\alpha+\beta+1}{2})\Gamma(\frac{\lambda+\alpha-\beta+1}{2})}{2^{1-\lambda}b^{\lambda+\alpha+1}\Gamma(\alpha+1)} \times {}_2F_1\left(\frac{\lambda+\alpha+\beta+1}{2}, \frac{\lambda+\alpha-\beta+1}{2}; \alpha+1; \frac{a^2}{b^2}\right),$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ denotes the hypergeometric function. With (3.1) we obtain

$$\begin{aligned}\int_0^\infty G^{\mu,\nu}(t,x)x^{\mu+\nu+1}\,\mathrm{d}x &= 2^{\mu+\nu}\Gamma\left(\frac{\mu+\nu+2}{2}\right)(1-t)^{\frac{\mu+\nu+2}{2}} \times \\ &\qquad {}_2F_1\left(\frac{\mu+\nu+2}{2}, \frac{\mu+2}{2}; \frac{\mu+2}{2}; t^2\right) \\ &= 2^{\mu+\nu}\Gamma\left(\frac{\mu+\nu+2}{2}\right)(1+t)^{-\frac{\mu+\nu+2}{2}} \\ &= \sum_{j=0}^\infty \frac{2^{\mu+\nu}\Gamma(j+\frac{\mu+\nu+2}{2})}{j!}(-t)^j.\end{aligned}$$

Then, in view of (3.2), the claim follows by comparing coefficients of t^j. □

Hence, the proof of Theorem B is completed.

4 Applications and special values

We conclude this article with some applications of Theorem A and discuss on special values of the integral formula.

4.1 The L^2-norm of $\Lambda_j^{\mu,\nu}$

As a first application of Theorem A we can give a closed formula for the L^2-norms of the orthogonal basis $(\Lambda_j^{\mu,\nu}(x))_{j\in\mathbb{N}_0}$ in $L^2(\mathbb{R}_+, x^{\mu+\nu+1}\,\mathrm{d}x)$. The same result was obtained in [4, Theorem B] by different methods.

Corollary 4.1. *The L^2-norm of the eigenfunction $\Lambda_j^{\mu,\nu}$ is given by*

$$\|\Lambda_{2,j}^{\mu,\nu}\|^2_{L^2(\mathbb{R}_+,x^{\mu+\nu+1}\,\mathrm{d}x)} = \frac{2^{\mu+\nu-1}\Gamma(j+\frac{\mu+\nu+2}{2})\Gamma(j+\frac{\mu-\nu+2}{2})}{j!(2j+\mu+1)\Gamma(j+\mu+1)}.$$

Proof. Let $p=\mu+3$, $q=\nu+3$. We define functions $\varphi_j \in L^2(C)$ in bipolar coordinates by

$$\varphi_j(r,\omega,\eta) := \Lambda_j^{\mu,\nu}(2r).$$

Then it is immediate that

$$\|\Lambda_j^{\mu,\nu}\|^2_{L^2(\mathbb{R}_+,x^{\mu+\nu+1}\,\mathrm{d}x)} = \frac{2^{\mu+\nu+3}}{\mathrm{vol}(S^{p-2})\mathrm{vol}(S^{q-2})}\|\varphi_j\|^2_{L^2(C)}.$$

Now, the intertwining operator $\mathcal{T}: L^2(C)\to\mathcal{H}$ is unitary up to a constant, namely (see [6, Remark 2.2.2]):

$$\|\mathcal{T}u\|^2_{\mathcal{H}} = \frac{1}{2}\|u\|^2_{L^2(C)}.$$

Using formula (2.3) for the intertwiner $\mathcal{T}$ and Theorem A one also obtains easily that

$$\mathcal{T}\varphi_j = 2^{-\frac{3(\mu+\nu+2)}{2}} A_j^{\mu,\nu}(\Lambda_j^{\mu,\nu})\psi_j$$

with ψ_j as in (2.1). By [6, Fact 2.1.1 (4)] the G-invariant norm on $\mathcal{H}$ is given by

$$\|u\|^2_{\mathcal{H}} = \left(j+\frac{p-2}{2}\right)\|u\|^2_{L^2(M)}, \qquad \text{for } u\in V^j.$$

By using the formula (see [3, 7.313 (2)])

$$\int_0^\pi \left[\widetilde{C}_n^\lambda(\cos\vartheta)\right]^2 \sin^{2\lambda}\vartheta\,\mathrm{d}\vartheta = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)},$$

we get

$$\begin{aligned}
\|\psi_j\|^2_{L^2(M)} &= \int_{S^{p-1}\times S^{q-1}} \left|\widetilde{C}_j^{\frac{p-2}{2}}(v_0)\widetilde{C}^{\frac{q-2}{2}}_{j+\frac{p-q}{2}}(v_{p+q-1})\right|^2 \mathrm{d}v\\
&= \mathrm{vol}(S^{p-2})\mathrm{vol}(S^{q-2})\int_0^\pi\left[\widetilde{C}_j^{\frac{p-2}{2}}(\cos\vartheta)\right]^2\sin^{p-2}\vartheta\,\mathrm{d}\vartheta\\
&\qquad\times\int_0^\pi\left[\widetilde{C}^{\frac{q-2}{2}}_{j+\frac{p-q}{2}}(\cos\varphi)\right]^2\sin^{q-2}\varphi\,\mathrm{d}\varphi\\
&= \frac{\pi^2\Gamma(j+\mu+1)\Gamma(j+\frac{\mu+\nu+2}{2})}{j!2^{\mu+\nu}(j+\frac{\mu+1}{2})^2\Gamma(j+\frac{\mu-\nu+2}{2})}\mathrm{vol}(S^{p-2})\mathrm{vol}(S^{q-2}).
\end{aligned}$$

Finally, putting all the steps together shows the claim. $\square$

4.2 The Poisson kernel of the Gegenbauer Polynomials

As another application of Theorem A we get an integral formula for the generating function $G^{\mu,\nu}(t,x)$ of L^2-eigenfunctions, which is closely related to the Poisson kernel of the Gegenbauer polynomials. For this we put

$$I_{\mu,\nu}(t,\vartheta,\varphi) \;:=\; \int_0^\infty \widetilde{I}_{\frac{\mu}{2}}\left(\frac{tx}{1-t}\right)\widetilde{K}_{\frac{\nu}{2}}\left(\frac{x}{1-t}\right)\widetilde{J}_{\frac{\mu}{2}}(ax)\widetilde{J}_{\frac{\nu}{2}}(bx)x^{\mu+\nu+1}\,\mathrm{d}x,$$

where $a := \frac{\sin\vartheta}{\cos\vartheta+\cos\varphi}$ and $b := \frac{\sin\varphi}{\cos\vartheta+\cos\varphi}$.

Corollary 4.2. *For $\cos\vartheta+\cos\varphi>0$ and $-1<t<1$ we have the following formula:*

$$\begin{aligned}(1-t)^{-\frac{\mu+\nu+2}{2}}&\left(\frac{\cos\vartheta+\cos\varphi}{2}\right)^{-\frac{\mu+\nu+2}{2}} I_{\mu,\nu}(t,\vartheta,\varphi)\\ &= \frac{2^{2(\mu+\nu)}}{\pi}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+\frac{\mu-\nu+2}{2})}{\Gamma(j+\mu+1)}\widetilde{C}_j^{\frac{\mu+1}{2}}(\cos\vartheta)\widetilde{C}_{j+\frac{\mu-\nu}{2}}^{\frac{\nu+1}{2}}(\cos\varphi)t^j.\end{aligned}$$

For $\mu=\nu=2\lambda-1$ this yields a formula for the Poisson kernel of the Gegenbauer polynomials. Recall that the Poisson kernel $P_\lambda(t,\vartheta,\varphi)$ of the Gegenbauer polynomials is defined by (see [1, formula (6.4.5)])

$$P_\lambda(t,\vartheta,\varphi) := \sum_{n=0}^{\infty}\frac{n!(n+\lambda)}{\Gamma(n+2\lambda)}\widetilde{C}_n^\lambda(\cos\vartheta)\widetilde{C}_n^\lambda(\cos\varphi)t^n.$$

(For explicit formulas for the Poisson kernel of the Gegenbauer polynomials see e.g. [1, formula (7.5.6)].) Now, Corollary 4.2 yields a new expression for $P_\lambda(t,\vartheta,\varphi)$ ($2\lambda\in\mathbb{Z}$, $\lambda>0$):

$$\begin{aligned}P_\lambda(t,\vartheta,\varphi) = \frac{\pi}{2^{8\lambda-4}}&\left(\frac{\cos\vartheta+\cos\varphi}{2}\right)^{-2\lambda}\\ &\times(\theta_t+\lambda)\left[(1+t)^{-2\lambda}I_{2\lambda-1,2\lambda-1}(-t,\vartheta,\varphi)\right],\end{aligned}$$

where $\theta_t = t\frac{\partial}{\partial t}$.

4.3 The bottom layer

As remarked in the introduction, for $j=0$ the K-Bessel function $u(x)=\widetilde{K}_{\frac{\nu}{2}}(x)$ is a solution of the differential equation (1.2). In this case the integral formula in Theorem A can be written as

$$\begin{aligned}\int_0^\infty K_{\frac{\nu}{2}}(x)J_{\frac{\mu}{2}}&\left(\frac{x\sin\vartheta}{\cos\vartheta+\cos\varphi}\right)J_{\frac{\nu}{2}}\left(\frac{x\sin\varphi}{\cos\vartheta+\cos\varphi}\right)x^{\frac{\mu+2}{2}}\,\mathrm{d}x\\ &= \frac{2^{\frac{\nu-2}{2}}\Gamma(\frac{\mu-\nu+2}{2})}{\sqrt{\pi}}(\cos\vartheta+\cos\varphi)\sin^{\frac{\mu}{2}}\vartheta\sin^{\frac{\nu}{2}}\varphi\;\widetilde{C}_{\frac{\mu-\nu}{2}}^{\frac{\nu+1}{2}}(\cos\varphi).\end{aligned}$$

This special case was already proved in [6, Lemma 7.8.1]. Another expression for this integral can be found in [2, formula 8.13 (14)].

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