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Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups

III. Restriction of Harish-Chandra modules and associated varieties

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Abstract. Let $H \subset G$ be real reductive Lie groups and π an irreducible unitary representation of G. We introduce an algebraic formulation (discretely decomposable restriction) to single out the nice class of the branching problem (breaking symmetry in physics) in the sense that there is no continuous spectrum in the irreducible decomposition of the restriction $\pi|_H$. This paper offers basic algebraic properties of discretely decomposable restrictions, especially for a reductive symmetric pair (G, H) and for the Zuckerman-Vogan derived functor module $\pi = \overline{A_{\mathfrak{q}}(\lambda)}$, and proves that the sufficient condition [Invent. Math. '94] is in fact necessary. A finite multiplicity theorem is established for discretely decomposable modules which is in sharp contrast to known examples of the continuous spectrum. An application to the restriction $\pi|_H$ of discrete series π for a symmetric space G/H is also given.

0. Introduction

0.1. This paper is a continuation of the work [Ko2]. Let G be a real reductive linear Lie group and denote by $\widehat{G}$ the unitary dual of G. The object of study is the restriction of $\pi \in \widehat{G}$ with respect to a reductive subgroup H.

0.2. In general, the restriction of $\pi \in \widehat{G}$ to a subgroup H may have a wild behavior even if H is a maximal reductive subgroup of G. For instance,

(0.2.1) the tensor product of principal series representations of a complex simple Lie group G is decomposed into only continuous spectrum with infinite multiplicity except the case where G is locally isomorphic to $SL(2, \mathbb{C})$ (Gelfand-Graev, Williams, see [GG], [Wi]).

Here are well-known opposite extremal cases:

(0.2.2) The restriction of π with respect to a maximal compact subgroup K is decomposed discretely into irreducibles with finite multiplicity for any $\pi \in \widehat{G}$ (Harish-Chandra).

(0.2.3) The restriction of holomorphic discrete series representations with respect to certain symmetric pairs is discretely decomposable (Martens, Jakobsen-Vergne; see Fact 5.4). Also, the restriction of the Segal-Shale-Weil representation with respect to a reductive dual pair $H_1 \times H_2$ with H_2 compact decomposes discretely.

These are also important in their applications: (0.2.2) has enabled us to study $(\mathfrak{g}, K)$-modules by algebraic methods instead of representations of G; the decomposition in (0.2.3) gives Howe's correspondence (e.g. Howe, Kashiwara-Vergne, Adams; [Ho], [KV], [A1]) and interacts closely with the theory of θ-series.

0.3. In this paper, we focus on the discretely decomposable restriction as a generalization of the feature of (0.2.2) and (0.2.3). The concept of discretely decomposable restriction might be very interesting not only in representation theory but also in other branches of mathematics. In fact, discrete decomposability of the restriction, especially for Zuckerman's modules $A_{\mathfrak{q}}(\lambda)$, has recently found its applications to the vanishing theorem of middle Hodge components of modular symbols for the arithmetic quotient of Hermitian symmetric domains [KO], the construction of discrete series representations for certain non-symmetric spherical homogeneous manifolds ([Ko2], Sect. 5; [Ko4]), spherical harmonics on pseudo-Riemannian homogeneous spaces and minimal unipotent representations [KØ], and the finite dimensional theorem of global holomorphic solutions to the generalization (to higher order) of the Gauss-Aomoto-Gelfand hypergeometric system ([Se], Theorem 7.1).

0.4. In our previous paper, we have given a sufficient condition on (G, H, π) where $H \subset G$ and $\pi \in \widehat{G}$ in order that the restriction $\pi|_H$ is discretely decomposable. Results follow from algebraic methods in [Ko2] and from microlocal analysis in Part II of [Ko1]. The main theorem was stated in the case where (G, H) is a reductive symmetric pair and where π is Zuckerman's module $A_{\mathfrak{q}}(\lambda)$. However, these methods provided only a sufficient condition for the discrete decomposable decomposition.

In the present paper, we shall reformulate the discretely decomposable restriction in a purely algebraic way in Sect. 1, and give a necessary condition for the discretely decomposable restriction for a reductive pair (G, H) in Sect. 3. One of our main results here is that the aforementioned sufficient condition in the symmetric case is in fact a necessary condition; that is, the restriction of $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module if and only if $\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \sqrt{-1}\left(\mathfrak{t}_0^{-\sigma}\right)^* = 0$, where $\mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p} \rangle$ is a closed cone determined by a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and $\mathfrak{t}_0^{-\sigma}$ is a subspace determined by a reductive symmetric pair (G, H) (see Theorem 4.2).

Furthermore, because the necessary condition is proved here by a weaker assumption (namely, no assumption on multiplicities), our results also yield a finite multiplicity theorem on the level of Harish-Chandra modules generalizing the case $H = K$ (see (0.2.2)); *If* (G, H) is a reductive symmetric pair, then

$$\dim \operatorname{Hom}_{\mathfrak{h}, H \cap K}\big(Y, A_{\mathfrak{q}}(\lambda)\big) < \infty$$

for any irreducible $(\mathfrak{h}, H \cap K)$-module Y and for any Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$ (see Corollary 4.3). This result is striking in contrast to the example of the *infinite multiplicity* of the continuous spectrum for a symmetric pair (e.g. (0.2.1)) (here we recall that principal series representations for complex reductive groups are written in the form $A_{\mathfrak{q}}(\lambda)$). In general, $\dim \operatorname{Hom}_{\mathfrak{h}, H \cap K}\big(Y, A_{\mathfrak{q}}(\lambda)\big)$ is not uniformly bounded, but we will give a sufficient condition for the uniformly bounded multiplicities in a subsequent paper which supports the Gross-Prasad conjecture.

0.5. In Sects. 5 and 6, in the context of discretely decomposable restrictions, we try to reveal the representation theoretic principles which were hidden in the following sharp contrast of known results:

1) The restriction $\pi|_K$ decomposes discretely for any $\pi \in \widehat{G}$ (see (0.2.2)), while there is no discrete spectrum in $L^2(G/K)$ for the Riemannian symmetric space G/K (Harish-Chandra) (see Theorem 6.2 and Remark 6.6 (1)).

2) The tensor product $\pi \otimes \pi'$ is discretely decomposable if both π and π' are holomorphic discrete series representations [Ma], [JV], while it always contains the continuous spectrum in the irreducible decomposition if π is holomorphic and π' is anti-holomorphic discrete series representation [R] (cf. Theorem 5.3 and Remark 6.6 (4)).

3) The non-vanishing theorem [TW] and the vanishing theorem [KO] of modular symbols defined by arithmetic subgroups (see Remark 6.6 (3)).

0.6. I have omitted in this paper all applications to harmonic analysis. In the last two sections we illustrate the theorem only by some examples. Irreducible $(\mathfrak{g}, K)$-modules are discretely decomposable as $(\mathfrak{h}, H \cap K)$-modules in the following cases:

1) $(G, H) = (G, K)$ (Riemannian symmetric pair): Any π is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module (see (0.2.2)).

2) $(G, H) = \big(G'_{\mathbb{C}}, G'_{\mathbb{R}}\big)$ with $G'_{\mathbb{R}}$ a normal real form of $G'_{\mathbb{C}}$: No π is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module except for $\dim \pi < \infty$ (see Theorem 8.1).

3) $(G, H) = (U(2,2), Sp(1,1)) \approx (SO(4,2), SO(4,1))$: 12 series among 18 series of irreducible, infinitesimally unitary $(\mathfrak{g}, K)$-modules with regular integral infinitesimal character are discretely decomposable as $(\mathfrak{h}, H \cap K)$-modules (Proposition 7.5).

0.7. The results in Sect. 3 were announced in Part II of [Ko1]. Most of other results were obtained while the author was a guest at the Institut Mittag-Leffler supported by the Royal Swedish Academy of Sciences. He expresses his sincere gratitude to the staff of the Institute and to the organizers of the special year "Analysis on Lie Groups". Thanks are also due to R. Donley, M. Duflo, M. Flensted-Jensen, D. Vogan, T. Ohta, G. Ólafsson and B. Ørsted for helpful discussions and their encouragement.

1. Discretely decomposable modules

In this section, we introduce the notion of discretely decomposable modules and present some basic properties.

1.1. Let $\mathfrak{h}$ be a complex Lie algebra, and X an $\mathfrak{h}$-module.

Definition 1.1. *We say X is* ***discretely decomposable*** *if there is an increasing filtration $\{X_m\}$ of $\mathfrak{h}$-submodules such that*
(1.1.1) $X = \bigcup_{m=0}^{\infty} X_m$,
(1.1.2) X_m *is of finite length as an $\mathfrak{h}$-module (i.e. has finite composition series).*

We note that an $\mathfrak{h}$-module of finite length is obviously discretely decomposable.

1.2. Here is a set up that we shall use frequently in this paper:

Let G be a real reductive linear Lie group, and H a closed subgroup of G. We assume that H has at most finitely many connected components and that there exists a Cartan involution θ of G which stabilizes H. Then H is also a reductive subgroup with maximal compact subgroup $H \cap K$, where $K = G^{\theta}$ is a maximal compact subgroup of G. We write $\mathfrak{g}_0$ for the Lie algebra of G and write $\mathfrak{g} := \mathfrak{g}_0 \otimes \mathbb{C}$. In what follows analogous notation will be applied to Lie groups denoted by other Roman upper case letters without comment. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition corresponding to the Cartan involution θ, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the complexification. There is a non-degenerate symmetric $\mathrm{Ad}(G)$-invariant bilinear form, say B, on $\mathfrak{g}_0$, which is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$ such that $\mathfrak{k}_0$ is orthogonal to $\mathfrak{p}_0$.

We say X is *discretely decomposable as an* $(\mathfrak{h}, H \cap K)$*-module* if X is an $(\mathfrak{h}, H \cap K)$-module and if X is discretely decomposable as an $\mathfrak{h}$-module.

1.3. Suppose that we are in the setting of Sect. 1.2. An $(\mathfrak{h}, H \cap K)$-module X is said to be *infinitesimally unitary* if there is an inner product on X such that $\mathfrak{h}$ acts on X by skew-Hermitian operators and that $H \cap K$ does unitarily on X. In this case, the terminology "discretely decomposable" is justified by the following lemma:

Lemma 1.3. *Let X be an infinitesimally unitary $(\mathfrak{h}, H \cap K)$-module. We suppose that X has at most countable basis over $\mathbb{C}$. Then X is discretely de-*

composable if and only if X decomposes into the algebraic sum of irreducible $\mathfrak{h}$*-modules.*

Proof. If X is decomposed into the algebraic sum of irreducible $\mathfrak{h}$-modules, say, $\bigoplus_{i=0}^{\infty} Y_i$, then we put $X_m := \bigoplus_{i=0}^{m} Y_i$ $(m \in \mathbb{N})$ which obviously satisfies the conditions (1.1.1) and (1.1.2). Conversely, assume that X is discretely decomposable. We may and do assume that H is connected in order to prove that X decomposes into the algebraic sum of irreducible $\mathfrak{h}$-modules. Let $X = \bigcup X_m$ be a filtration as in Definition 1.1. It is convenient to put $X_m := \{0\}$ for a negative integer m. We note that each $\mathfrak{h}$-submodule X_m is also an $(\mathfrak{h}, H \cap K)$-submodule. Let $\overline{X_m}$ be the Hilbert space that is the completion of X_m with respect to the pre-Hilbert structure of X. A basic result due to Harish-Chandra (cf. [V1], Theorem 0.3.5) asserts that there is a lattice isomorphism between closed H-invariant subspaces of $\overline{X_m}$ and $(\mathfrak{h}, H \cap K)$-invariant subspaces of X_m because X_m is of finite length as an $(\mathfrak{h}, H \cap K)$-module. In particular, we can take the orthogonal complementary subspace of X_m in X_{m+1}, which is decomposed into the finite sum of irreducible $(\mathfrak{h}, H \cap K)$-modules. By the induction on m starting from $m = -1$, $X = \bigcup_m X_m$ is decomposed into the algebraic sum of irreducible $(\mathfrak{h}, H \cap K)$-modules. □

1.4. Here is an elementary result for producing a family of discretely decomposable modules from a given discretely decomposable module.

Lemma 1.4. *Suppose we are in the setting of Sect.* 1.2. *Assume that the* $(\mathfrak{h}, H \cap K)$*-module X is discretely decomposable.*

1) *Any submodule or quotient of X is discretely decomposable.*

2) *The tensor product* $X \otimes F$ *is discretely decomposable for any finite dimensional* $(\mathfrak{h}, H \cap K)$*-module F.*

Proof. 1) Let $X = \bigcup_{m=0}^{\infty} X_m$ be a filtration satisfying (1.1.2). Suppose that Y and Z are $(\mathfrak{h}, H \cap K)$-modules and that $\iota : Y \to X$ and $\pi : X \to Z$ are injective and surjective $(\mathfrak{h}, H \cap K)$-homomorphisms, respectively. Then $Y_m := \iota^{-1}(X_m)$ and $Z_m := \pi(X_m)$ give the desired filtration of Y and Z, respectively.

2) Since X_m is of finite length as an $(\mathfrak{h}, H \cap K)$-module, so is $X_m \otimes F$ (see the proof of [V1], Corollary 4.5.6). Therefore, $X_m \otimes F$ $(m = 0, 1, 2, \ldots)$ gives the desired filtration of $X \otimes F$. □

1.5. Retain the setting of Sect. 1.2. Here is another characterization of discretely decomposable $\mathfrak{h}$-modules:

Lemma 1.5. *Suppose* (π, X) *is an irreducible* $\mathfrak{g}$*-module. Then X is discretely decomposable as an* $\mathfrak{h}$*-module if and only if there exists an irreducible* $\mathfrak{h}$*-module Y such that* $\operatorname{Hom}_{\mathfrak{h}}(Y, X) \neq 0$.

Proof. Let $U(\mathfrak{g}) \supset U(\mathfrak{h})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Because $\mathfrak{h}$ is reductive in $\mathfrak{g}$, the invariant form B (see Sect. 1.2) is non-degenerate when restricted to $\mathfrak{h} \times \mathfrak{h}$. We denote by $\mathfrak{h}^{\perp}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B. Then $\mathfrak{h}^{\perp}$ is an $(\mathfrak{h}, H \cap K)$-invariant subspace such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ is a direct sum decomposition. We set

$$U'_k(\mathfrak{h}^{\perp}) := \mathbb{C}\text{-span}\langle Y_1 \cdots Y_m : Y_1, \ldots, Y_m \in \mathfrak{h}^{\perp}, m \le k \rangle \subset U(\mathfrak{g}), \tag{1.5.1}$$

$$U'(\mathfrak{h}^{\perp}) := \bigcup_{k=0}^{\infty} U'_k(\mathfrak{h}^{\perp}). \tag{1.5.2}$$

Then we have $U(\mathfrak{g}) = U'(\mathfrak{h}^{\perp})\ U(\mathfrak{h})$ by the Poincaré-Birkhoff-Witt theorem. Assume that there exists an irreducible $(\mathfrak{h}, H \cap K)$-module Y such that $\mathrm{Hom}_{\mathfrak{h}}(Y, X) \neq 0$. We may regard Y as a submodule of X. We set $X_k := U'_k(\mathfrak{h}^{\perp})\ Y\ (\subset X)$. Because X is an irreducible $\mathfrak{g}$-module, we have

$$X = U(\mathfrak{g})\ Y = U'(\mathfrak{h}^{\perp})\ U(\mathfrak{h})\ Y = \bigcup_{k=0}^{\infty} U'_k(\mathfrak{h}^{\perp})\ Y = \bigcup_{k=0}^{\infty} X_k.$$

Let $Z_k\ (\subset X)$ be the image of the following $(\mathfrak{h}, H \cap K)$-homomorphism:

$$\underbrace{\mathfrak{h}^{\perp} \otimes \cdots \otimes \mathfrak{h}^{\perp}}_{k} \otimes Y \to X, (Y_1 \otimes \cdots \otimes Y_k) \otimes w \mapsto \pi(Y_1) \cdots \pi(Y_k) w. \tag{1.5.3}$$

Then $X_k = \sum_{i=0}^{k} Z_i$ (not necessarily a direct sum). This shows that X_k is of finite length as an $(\mathfrak{h}, H \cap K)$-module, since $\mathfrak{h}^{\perp} \otimes \cdots \otimes \mathfrak{h}^{\perp}$ is a finite dimensional $(\mathfrak{h}, H \cap K)$-module. Hence the conditions (1.1.1) and (1.1.2) are satisfied. Therefore X is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

Conversely, if X is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module, then X contains an $(\mathfrak{h}, H \cap K)$-submodule X_0 of finite length. Take an irreducible $(\mathfrak{h}, H \cap K)$-submodule Y of X_0. Then we have $\mathrm{Hom}_{\mathfrak{h}, H \cap K}(Y, X) \neq 0$. □

1.6. Here is a source of discretely decomposable $(\mathfrak{h}, H \cap K)$-modules in the context of the restriction of $(\mathfrak{g}, K)$-modules. Suppose $(\pi, \mathscr{H})$ is a unitary representation of G. We say the restriction $\pi|_H$ is *H-admissible* if $\mathscr{H}$ is decomposed into a discrete Hilbert direct sum with finite multiplicity for each irreducible representation of H (see [Ko2], Sect. 1). There are known sufficient conditions for irreducible unitary representations to be $H \cap K$-admissible by algebraic methods in [Ko2] and by using the singularity spectrum (the analytic wave front set) in Part II of [Ko1]. The following property is also useful in the application to harmonic analysis on non-symmetric spaces [Ko4]:

Proposition 1.6. *Let* $(\pi, V) \in \widehat{G}$. *Assume that the restriction* $\pi|_{H \cap K}$ *is* $H \cap K$-*admissible. Then we have*:

1) *The space of* K*-finite vectors* V_K *coincides with that of* $H \cap K$*-finite vectors* $V_{H\cap K}$*. In particular,* V_K *is discretely decomposable as an* $(\mathfrak{h}, H \cap K)$*-module.*

2) $\dim \operatorname{Hom}_{\mathfrak{h},H\cap K}(\tau_{H\cap K}, V_K) = \dim \operatorname{Hom}_H(\tau, V)$*, for any irreducible unitary representation* τ *of* H*. Here* $\tau_{H\cap K}$ *denotes the underlying* $(\mathfrak{h}, H \cap K)$*-module of* τ*.*

Proof. Since $H \cap K \subset K$, the inclusion $V_K \subset V_{H\cap K}$ is obvious. For each $(\sigma, U_\sigma) \in \widehat{K}$, we write $V(K;\sigma)$ for the σ-isotypical component of V, that is, $V(K;\sigma) := \sum_\varphi \varphi(U_\sigma)$ $(\subset V_K)$ where φ runs over $\operatorname{Hom}_K(\sigma, V)$, the space of K homomorphisms $U_\sigma \to V$. Similarly, we write $V(H\cap K;\tau)$ $(\subset V_{H\cap K})$ for the τ-isotypical component of V if $\tau \in \widehat{H\cap K}$. Then we have

$$V(H\cap K;\tau) \subset \sum_{\substack{\sigma\in\widehat{K} \\ [\sigma|_{H\cap K}:\tau]\neq 0}}^{\oplus} V(K;\sigma), \tag{1.6.1}$$

where $[\sigma_{|H\cap K} : \tau] := \dim \operatorname{Hom}_{H\cap K}(\tau, \sigma)$ and the right hand side of (1.6.1) is a Hilbert direct sum. We will show that it is actually a finite sum. Because the restriction $\pi|_{H\cap K}$ is $H \cap K$-admissible, we have $\dim \operatorname{Hom}_{H\cap K}(\tau, V) < \infty$ for any $\tau \in \widehat{H\cap K}$. By Frobenius reciprocity, we have

$$\dim \operatorname{Hom}_{H\cap K}(\tau, V) = \sum_{\sigma\in\widehat{K}} [\sigma_{|H\cap K} : \tau] \dim \operatorname{Hom}_K(\sigma, V),$$

which implies that there are only finitely many $\sigma \in \widehat{K}$ satisfying both $[\sigma_{|H\cap K} : \tau] \neq 0$ and $V(K;\sigma) \neq 0$. Therefore the right side of (1.6.1) is in fact a finite sum. Hence $V(H\cap K;\tau) \subset V_K$ for any $\tau \in \widehat{H\cap K}$, which means $V_{H\cap K} \subset V_K$. Thus we have proved $V_{H\cap K} = V_K$.

Because (π, V) is $H \cap K$-admissible, it is also *H-admissible* (see [Ko2], Theorem 1.2). Namely, we have a Hilbert direct sum decomposition

$$\pi \simeq \sum_{\tau\in\widehat{H}}^{\oplus} m(\tau)\tau, \tag{1.6.2}$$

where the multiplicity $m(\tau) < \infty$ for any $\tau \in \widehat{H}$. Take $(\tau, W) \in \widehat{H}$ such that $m(\tau) > 0$. Then V contains a closed subspace, say U, which is isomorphic to the direct sum of $m(\tau)$ copies of W. Then we have

$$W_{H\cap K} \oplus \cdots \oplus W_{H\cap K} \simeq U_{H\cap K} \subset V_{H\cap K} = V_K.$$

This shows that

$$m(\tau) \leq \dim \operatorname{Hom}_{\mathfrak{h},H\cap K}(\tau_{H\cap K}, V_K). \tag{1.6.3}$$

In particular, V_K is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module from Lemma 1.4. The opposite inequality of (1.6.3) is obvious. Hence we have proved the second statement. $\square$

1.7. Here are some examples of discretely decomposable restrictions due to Proposition 1.6:

Example 1.7. A $(\mathfrak{g}, K)$-module X of finite length is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module if one of the following assumptions is satisfied:

1) H is compact.
2) X is a highest weight $(\mathfrak{g}, K)$-module and H contains the center of K.
3) $X = A_{\mathfrak{q}}(\lambda)$ and (G, G^{σ}) is a symmetric pair with $\mathbb{R}_{+}\langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \sqrt{-1}\left(\mathfrak{t}_0^{-\sigma}\right)^{*} = 0$.
4) X appears in a subquotient of the coherent family through a $(\mathfrak{g}, K)$-module Y, where Y is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

(1) is obvious, (2) is obtained in [Ma] and also in [L], Theorem 4.2 (see also [Ko2], Corollary 4.4 for a generalization), and (3) is in [Ko2], Theorem 3.2 (see Theorem 4.2 for the notation and the opposite implication). (4) follows from Lemma 1.2 (see [V1], Definition 7.2.5 for the definition of the coherent continuation).

2. Associated varieties of $U(\mathfrak{g})$-modules

In this section we make a quick review on known results on associated varieties of $U(\mathfrak{g})$-modules.

2.1. If V is a finite dimensional complex vector space, we use the following notation:

V^* : the dual vector space of V over $\mathbb{C}$,
$S(V)$: the symmetric algebra of $V \simeq$ the polynomial algebra on V^*,
$S^k(V)$: the subspace of $S(V)$ of homogeneous elements of degree k,
$S_k(V) : = \bigoplus_{j=0}^{k} S^j(V)$.

Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated $S(V)$-module. We say M is a graded $S(V)$-module if $S^i(V) M_j \subset M_{i+j}$ $(i, j \geq 0)$. We define a closed cone in V^* by

$$\mathrm{Supp}_{S(V)}(M) := \{\lambda \in V^* : f(\lambda) = 0 \text{ for any } f \in \mathrm{Ann}_{S(V)}(M)\},$$

where $\mathrm{Ann}_{S(V)}(M) := \{f \in S(V) : f \cdot m = 0 \text{ for any } m \in M\}$.

2.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. For each integer $n \geq 0$, let $U_n(\mathfrak{g})$ denote by the subspace spanned by elements of the form $Y_1 \cdots Y_k$ with $Y_1, \ldots, Y_k \in \mathfrak{g}$ and $k \leq n$. We note that $U_0(\mathfrak{g}) = \mathbb{C}$. It is convenient to put $U_{-1}(\mathfrak{g}) = 0$. Then $U(\mathfrak{g})$ is a filtered algebra in the sense that

$$U(\mathfrak{g}) = \bigcup_{k=1}^{\infty} U_k(\mathfrak{g}), \qquad U_i(\mathfrak{g}) U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}).$$

The associated graded algebra $\mathrm{gr}U(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} S^k(\mathfrak{g})$ of $\mathfrak{g}$, by the Poincaré-Birkhoff-Witt theorem.

2.3. Suppose X is a finitely generated $U(\mathfrak{g})$-module. We take a finite dimensional subspace X_0 which generates X as a $U(\mathfrak{g})$-module. We put $X_k := U_k(\mathfrak{g})X_0$ $(k \in \mathbb{N})$. It is convenient to put $X_{-1} := \{0\}$. Then we have an increasing filtration $\{X_k\}_k$ such that

$$X = \bigcup_{k=0}^{\infty} X_k, \qquad U_i(\mathfrak{g})X_j = X_{i+j} \ (i, j \geq 0).$$

Therefore, if we put $\mathrm{gr}\, X := \bigoplus_{k=0}^{\infty} \overline{X_k}$ with $\overline{X_k} := X_k/X_{k-1}$, then $\mathrm{gr}\, X$ is a finitely generated $\mathrm{gr}\ U(\mathfrak{g}) \simeq S(\mathfrak{g})$-module. Define the variety $\mathscr{V}(X)$ by

$$\mathscr{V}(X) \equiv \mathscr{V}_{\mathfrak{g}}(X) = \mathrm{Supp}_{S(\mathfrak{g})}(\mathrm{gr}\ X) \ \subset \mathfrak{g}^*.$$

Then $\mathscr{V}_{\mathfrak{g}}(X)$ is independent of the choice of the generating subspace X_0 and is called the *associated variety* of the $U(\mathfrak{g})$-module X.

2.4. The following lemma is standard (e.g. [BB], Lemma 4.1 for (2)):

Lemma 2.4. *Let L, M and N be $\mathfrak{g}$-modules of finite length.*

1) *If $0 \to L \to M \to N \to 0$ is an exact sequence of $\mathfrak{g}$-modules, then $\mathscr{V}(M) = \mathscr{V}(N) \cup \mathscr{V}(L)$.*

2) *$\mathscr{V}(M \otimes F) = \mathscr{V}(M)$ for any finite dimensional $\mathfrak{g}$-module F.*

3) *$\mathscr{V}(M) = \{0\}$ if and only if M is finite dimensional.*

2.5. From now on, let G be a real reductive linear Lie group, K a maximal compact subgroup of G and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the Cartan decomposition as in Sect. 1.2. We define the nilpotent cone $\mathscr{N}^*$ by

$$\mathscr{N}^* \equiv \mathscr{N}^*_{\mathfrak{g}} := \left\{\lambda \in \mathfrak{g}^* \ : \ f(\lambda) = 0, \text{ for all } f \in S^+(\mathfrak{g})^G\right\}.$$

Here $S^+(\mathfrak{g}) := \bigoplus_{k=1}^{\infty} S^k(\mathfrak{g})$ is the maximal ideal of $S(\mathfrak{g})$, and $S^+(\mathfrak{g})^G$ is the ring of the G-invariant elements. Then we have

Fact 2.5 (see [V5], Corollary 5.4). *If X is a $\mathfrak{g}$-module of finite length, then the associated variety $\mathscr{V}_{\mathfrak{g}}(X)$ is contained in $\mathscr{N}^*_{\mathfrak{g}}$.*

2.6. Given an element $X \in \sqrt{-1}\mathfrak{k}_0$, we define $\mathfrak{u} \equiv \mathfrak{u}(X)$, $\mathfrak{l} \equiv \mathfrak{l}(X)$ and $\mathfrak{u}^- \equiv \mathfrak{u}^-(X)$ to be the sum of eigenspaces with positive, 0 and negative eigenvalues of $\mathrm{ad}(X) \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$, respectively. Then $\mathfrak{q} := \mathfrak{l} + \mathfrak{u}$ is said to be a θ-stable parabolic subalgebra of $\mathfrak{g}$. We note that $\mathfrak{l}$ is the complexified Lie algebra of $L := Z_G(X)$. The elliptic orbit $\mathrm{Ad}(G)X \simeq G/L$ carries a G-invariant complex structure, with the canonical line bundle $\Omega := \wedge^{\mathrm{top}} T^*(G/L) \simeq G \times_L \mathbb{C}_{2\rho(\mathfrak{u})}$. Here, $2\rho(\mathfrak{u}) := \det\left(\mathrm{Ad}_{|\mathfrak{u}}\right)$ is a character of L written in an additive way. Let $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. Given a character $\mathbb{C}_\lambda$ of L, we write $A_{\mathfrak{q}}(\lambda) \equiv A_{\mathfrak{q}}^G(\lambda)$ for the underlying $(\mathfrak{g}, K)$-module of the Dolbeault cohomology group $H^S(G/L, \Omega \otimes \mathbb{C}_\lambda)$ with coefficients in the associated holomorphic line bundle $\Omega \otimes (G \times_L \mathbb{C}_\lambda) \simeq G \times_L \mathbb{C}_{\lambda+2\rho(\mathfrak{u})}$ (cf. [V1], Chap. 6; [Wo]). We note that $A_{\mathfrak{q}}(\lambda) \simeq \mathscr{R}_{\mathfrak{q}}^S(\mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ with the notation in [V3], Definition 6.20 (cf. [VZ], Sect. 5). We take a fundamental Cartan subalgebra $\mathfrak{h}_0^c$ $(\subset \mathfrak{l}_0)$. Then $\mathfrak{h}_0^c$ contains the center $\mathfrak{z}_0$ of $\mathfrak{l}_0$ and $\mathfrak{t}_0^c := \mathfrak{h}_0^c \cap \mathfrak{k}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. $A_{\mathfrak{q}}(\lambda)$ has the $\mathscr{Z}(\mathfrak{g})$-infinitesimal character $\gamma := \lambda + \rho \in (\mathfrak{h}^c)^*$ in the Harish-Chandra parametrization, where $\rho := \rho(\mathfrak{u}) + \rho_{\mathfrak{l}}$ and $\rho_{\mathfrak{l}}$ is half the sum of positive roots of $\mathfrak{l}$. Following [V4], Definition 2.5, we say λ is *in the good range* if

$$\text{(2.6.1)(a)} \qquad \mathrm{Re}\langle \gamma, \alpha \rangle > 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c),$$

and *in the fair range* if

$$\text{(2.6.1)(b)} \qquad \mathrm{Re}\langle \gamma_{|\mathfrak{z}}, \alpha \rangle > 0 \text{ for any } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^c),$$

which is implied by (2.6.1)(a). It is *weakly good* (respectively, *weakly fair*) if the weak inequalities hold.

We recall some important results on $A_{\mathfrak{q}}(\lambda)$. See [V2] and [Wa] for the first statement; and Part III of [BB], Corollary 1.9 and Proposition 2.8; [HMSW] and [V4], Proposition 6.8 for the second. Let us identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by a non-degenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form on $\mathfrak{g}$.

Fact 2.6. *Retain the notation as above.*

1) *If $\mathbb{C}_\lambda$ is infinitesimally unitary and if λ is in the weakly fair range, then the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is infinitesimally unitary.*

2) *If λ is in the good range, then $A_{\mathfrak{q}}(\lambda)$ is non-zero and irreducible. The associated variety is given by $\mathscr{V}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda)) = \mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p})$.*

We shall write $\overline{A_{\mathfrak{q}}(\lambda)}$ for the unitary representation of G obtained as a Hilbert completion of $A_{\mathfrak{q}}(\lambda)$ with respect to a pre-Hilbert structure in Fact 2.6(1).

2.7. The Zuckerman module $A_{\mathfrak{q}}(\lambda)$ may vanish and may be reducible in the weakly fair range of parameters. Nevertheless, the associated variety of $A_{\mathfrak{q}}(\lambda)$ does not change as long as $A_{\mathfrak{q}}(\lambda)$ is non-zero. The usage of the adjointness of the translation functor was suggested by David Vogan, to whom the author is grateful.

Lemma 2.7. *If λ is in the weakly fair range and if $A_{\mathfrak{q}}(\lambda)$ is non-zero, then we have $\mathscr{V}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda)) = \mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p})$.*

Proof. It follows from [V4], Proposition 4.7 that there exists a character $\mathbb{C}_\mu$ of L in the good range such that the translation functor $\psi_{\mu+\rho}^{\lambda+\rho}$ sends $Z := A_{\mathfrak{q}}(\mu)$ to $X := A_{\mathfrak{q}}(\lambda)$. Here we define the Jantzen-Zuckerman translation functor by $\psi_{\mu+\rho}^{\lambda+\rho}(Z) = P_{\lambda+\rho}(F_{\lambda-\mu} \otimes P_{\mu+\rho}(Z))$, where P_ξ denotes the projection to $(\mathfrak{g}, K)$-modules with the generalized infinitesimal character $\xi \in (\mathfrak{h}^c)^*$ and $F_{\lambda-\mu}$ denotes the finite dimensional representation of G with extremal weight $\lambda - \mu$. In particular, we have a surjective $(\mathfrak{g}, K)$-homomorphism $F_{\mu-\rho} \otimes Z$ to X, and therefore

$$\mathscr{V}_{\mathfrak{g}}(X) \subset \mathscr{V}_{\mathfrak{g}}\big(F_{\lambda-\mu} \otimes Z\big) = \mathscr{V}_{\mathfrak{g}}(Z)$$

by Lemma 2.4. On the other hand, by the adjointness of the translation functor (e.g. [V1], Proposition 4.5.8) we have

$$\mathrm{Hom}_{\mathfrak{g},K}(X, X) \simeq \mathrm{Hom}_{\mathfrak{g},K}\left(X, \psi_{\mu+\rho}^{\lambda+\rho}(Z)\right) \simeq \mathrm{Hom}_{\mathfrak{g},K}\left(\psi_{\lambda+\rho}^{\mu+\rho}(X), Z\right).$$

If X is non-zero, then the left side contains the identity map. Hence there exists a non-zero $(\mathfrak{g}, K)$-homomorphism $\varphi : \psi_{\lambda+\rho}^{\mu+\rho}(X) \to Z$. Because Z is irreducible, φ is surjective. Using Lemma 2.4 again, we have

$$\mathscr{V}_{\mathfrak{g}}(Z) \subset \mathscr{V}_{\mathfrak{g}}\left(\psi_{\lambda+\rho}^{\mu+\rho}(X)\right) \subset \mathscr{V}_{\mathfrak{g}}\left(F_{\lambda-\mu}^{*} \otimes X\right) = \mathscr{V}_{\mathfrak{g}}(X).$$

Hence, we have $\mathscr{V}_{\mathfrak{g}}(X) = \mathscr{V}_{\mathfrak{g}}(Z)$ which coincides with $\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p})$ by Fact 2.6(2). □

3. Theorems in the general case

For discretely decomposable modules with respect to subalgebras, the algebraic approach turns out to be a powerful tool for the study of the restriction. In this section, we give a basic estimate on the associated variety of a $\mathfrak{g}$-module when restricted to a reductive subalgebra $\mathfrak{h}$. The results here will be a main tool for the study of the restriction with respect to reductive symmetric pairs in Sects. 4, 5 and 6.

3.1. Suppose that $\mathfrak{g}$ is a complex reductive Lie algebra and $\mathfrak{h}$ is a subalgebra which is reductive in $\mathfrak{g}$. Write the projection $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} : \mathfrak{g}^* \to \mathfrak{h}^*$ dual to the inclusion of complexified Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

Theorem 3.1. *Suppose X is an irreducible $\mathfrak{g}$-module and Y is an irreducible $\mathfrak{h}$-module. If* $\mathrm{Hom}_{\mathfrak{h}}(Y, X) \neq 0$ *then the associated varieties* $\mathscr{V}_{\mathfrak{g}}(X) \subset \mathfrak{g}^*$ *and* $\mathscr{V}_{\mathfrak{h}}(Y) \subset \mathfrak{h}^*$ *satisfy the following relation:*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}\big(\mathscr{V}_{\mathfrak{g}}(X)\big) \subset \mathscr{V}_{\mathfrak{h}}(Y).$$

3.2. For the proof of Theorem 3.1, we may regard Y as a submodule of X. Fix a finite dimensional vector subspace $F\,(\subset Y \subset X)$. Then $\{Y_n := U_n(\mathfrak{h})F;\ (n = 0, 1, 2, \ldots)\}$ forms an increasing filtration of Y. We denote by $\mathrm{gr}\,Y$ the graded $S(\mathfrak{h})$-module, and by $\mathrm{Ann}_{S(\mathfrak{h})}(\mathrm{gr}\,Y) \subset S(\mathfrak{h})$ the annihilator of $\mathrm{gr}\,Y$. Similarly, an increasing filtration $\{X_n := U_n(\mathfrak{g})F;\ (n = 0, 1, 2, \ldots)\}$, the graded $S(\mathfrak{g})$-module $\mathrm{gr}\,X$, and the annihilator $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\,X) \subset S(\mathfrak{g})$ are defined. We should remark that the filtration $\{X_n\}$ is different from the one in the proof of Lemma 1.5. In light of the isomorphism of graded rings

$$S(\mathfrak{h}) \otimes S(\mathfrak{h}^{\perp}) \xrightarrow{\sim} S(\mathfrak{g}), \tag{3.2.1}$$

we have:

Lemma 3.2. $\mathrm{Ann}_{S(\mathfrak{h})}(\mathrm{gr}\,Y) \otimes S(\mathfrak{h}^{\perp}) \subset \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\,X)$.

Proof. Let $U'(\mathfrak{h}^{\perp}) = \bigcup_{k=0}^{\infty} U'_k(\mathfrak{h}^{\perp})$ be as in (1.5.1) and (1.5.2). Then the Poincaré-Birkhoff-Witt theorem leads to

$$U_n(\mathfrak{g}) = \sum_{i+j \le n} U'_i(\mathfrak{h}^{\perp})\ U_j(\mathfrak{h}) \quad \text{for } n \in \mathbb{N}.$$

Thus, we have

$$X_n = U_n(\mathfrak{g})\ F = \sum_{i+j \le n} U'_i(\mathfrak{h}^{\perp})\ U_j(\mathfrak{h})\ F = \sum_{i+j \le n} U'_i(\mathfrak{h}^{\perp})\ Y_j. \tag{3.2.2}$$

We fix homogeneous elements $u \in S^a(\mathfrak{h}) \cap \mathrm{Ann}_{S(\mathfrak{h})}(\mathrm{gr}\,Y)$ and $v \in S^b(\mathfrak{h}^{\perp})$. Let $\widetilde{u} \in U_a(\mathfrak{h})$ be the symmetrization of u and let $\widetilde{v} \in U'_b(\mathfrak{h}^{\perp})$ be that of v. It follows from $u \in \mathrm{Ann}_{S(\mathfrak{h})}(\mathrm{gr}\,Y)$ that $\widetilde{u} Y_j \subset Y_{a+j-1}$ for any $j \in \mathbb{N}$. Then we have

$$\widetilde{u}\ \widetilde{v}\big(U'_i(\mathfrak{h}^{\perp}) Y_j\big) \subset \widetilde{u}\big(U'_{i+b}(\mathfrak{h}^{\perp}) Y_j\big) \subset U'_{i+b}(\mathfrak{h}^{\perp})(\widetilde{u} Y_j) \subset U'_{i+b}(\mathfrak{h}^{\perp}) Y_{a+j-1}. \tag{3.2.3}$$

Here the second inclusion follows from the fact that $\mathrm{ad}(\mathfrak{h})$ stabilizes $\mathfrak{h}^{\perp}$. By (3.2.2) and (3.2.3), we have

$$\widetilde{u}\ \widetilde{v} X_n \subset \sum_{i+j \le n} \widetilde{u}\ \widetilde{v} U'_i(\mathfrak{h}^{\perp}) Y_j \subset \sum_{i+j \le n} U'_{i+b}(\mathfrak{h}^{\perp}) Y_{a+j-1} \subset X_{a+b+n-1}.$$

Since the natural map $U_{a+b}(\mathfrak{g}) \to U_{a+b}(\mathfrak{g})/U_{a+b-1}(\mathfrak{g}) \simeq S^{a+b}(\mathfrak{g})$ sends $\widetilde{u}\ \widetilde{v}$ to $u \otimes v$, we have

$$(u \otimes v) \cdot (X_n/X_{n-1}) = (\widetilde{u}\ \widetilde{v} \cdot X_n)/X_{a+b+n-1} = 0$$

in the graded module $\mathrm{gr}\,X$ for any $n \in \mathbb{N}$. Hence we have $u \otimes v \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}\,X)$, which we wanted to prove. $\square$

3.3. *Proof of Theorem 3.1.* For $V = \mathfrak{h}$ or $\mathfrak{g}$, if $f \in S(V)$ then we regard f as a polynomial over V^* and denote the evaluation of f at $v \in V^*$ by $\langle f, v \rangle$. Then we want to prove

$$\Big\langle u, \mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\lambda) \Big\rangle = 0$$

for any $u \in \mathrm{Ann}_{S(\mathfrak{h})}(\mathrm{gr} Y)$ and for any $\lambda \in \mathscr{V}_{\mathfrak{g}}(X) \subset \mathfrak{g}^*$, where u is regarded as a polynomial over $\mathfrak{h}^*$. It follows from Lemma 3.2 that $u \otimes 1 \left(\in S(\mathfrak{h}) \otimes S(\mathfrak{h}^{\perp}) \simeq S(\mathfrak{g}) \right)$ annihilates $\mathrm{gr} X$, so we have $\langle u \otimes 1, \lambda \rangle = 0$, where $u \otimes 1$ is regarded as a polynomial over $\mathfrak{g}^*$. Now, Theorem 3.1 follows from the formula $\langle u, \mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\lambda) \rangle = \langle u \otimes 1, \lambda \rangle$. $\square$

Theorem 3.1 gives rise to a necessary condition for a $\mathfrak{g}$-module to be discretely decomposable as an $\mathfrak{h}$-module. We recall that $\mathscr{N}_{\mathfrak{h}}^* \subset \mathfrak{h}^*$ is the nilpotent cone for $\mathfrak{h}$.

Corollary 3.4. *Let X be a $\mathfrak{g}$-module of finite length. Assume that X is discretely decomposable as an $\mathfrak{h}$-module. Then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}\left(\mathscr{V}_{\mathfrak{g}}(X)\right) \subset \mathscr{N}_{\mathfrak{h}}^*.$$

Proof. We may assume that X is irreducible as a $\mathfrak{g}$-module by using Lemma 1.2 (1) and Lemma 2.4 (1). We take an irreducible $\mathfrak{h}$-submodule Y of X (see Lemma 1.5). It follows from Theorem 3.1 that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}\left(\mathscr{V}_{\mathfrak{g}}(X)\right) \subset \mathscr{V}_{\mathfrak{h}}(Y)$. By Fact 2.5 applied to $\mathfrak{h}$, we have $\mathscr{V}_{\mathfrak{h}}(Y) \subset \mathscr{N}_{\mathfrak{h}}^*$. Hence, Corollary follows. $\square$

Applying Corollary 3.4 to $X = A_{\mathfrak{q}}(\lambda)$ and using Proposition 1.6, we have:

Corollary 3.5. *Let us identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as usual. We assume that a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ satisfies*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p})) \not\subset \mathscr{N}_{\mathfrak{h}}^*.$$

If $\mathbb{C}_\lambda$ is in the weakly fair range and if $A_{\mathfrak{q}}(\lambda) \neq \{0\}$, then the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is not discretely decomposable as an $\mathfrak{h}$-module. In particular, the restriction of the unitary representation $\overline{A_{\mathfrak{q}}(\lambda)}$ to $H \cap K$ is not $H \cap K$-admissible.

Proof. The first assertion follows from Lemma 2.7 and Corollary 3.4. The latter is immediate from Proposition 1.6. $\square$

Remark 3.6. Let X be an irreducible $(\mathfrak{g}, K)$-module. If $H = K$, then X is always discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module (for example, use Lemma 1.5). In this special case, Theorem 3.1 implies a well-known result (see [V5], Corollary 5.13):

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{k}}\left(\mathscr{V}_{\mathfrak{g}}(X)\right) = \{0\}, \quad \text{namely,} \quad \mathscr{V}_{\mathfrak{g}}(X) \subset (\mathfrak{g}/\mathfrak{k})^*, \tag{3.6.1}$$

because the associated variety of a finite dimensional representation is zero and because any irreducible representation of K is finite dimensional.

3.7. The following Theorem gives a useful information on $\widehat{H}$ occurring as direct summands of the restriction.

Theorem 3.7. *Let X be an irreducible $(\mathfrak{g}, K)$-module and let Y_i be irreducible $(\mathfrak{h}, H \cap K)$-modules such that* $\mathrm{Hom}_{\mathfrak{h}, H\cap K}(Y_i, X) \neq 0$ $(i = 1, 2)$. *Then we have*

$$\mathscr{V}_{\mathfrak{h}}(Y_1) = \mathscr{V}_{\mathfrak{h}}(Y_2).$$

Proof. Using the notation of the proof of Lemma 1.5 with Y replaced by Y_1, we set $X_m := U'_m(\mathfrak{h}^\perp)Y_1 = \sum_{j=0}^m Z_j$ for $m \in \mathbb{N}$, where Z_j is the image of the map of $\underbrace{\mathfrak{h}^\perp \otimes \cdots \otimes \mathfrak{h}^\perp}_{j} \otimes Y_1 \to X$ as in (1.5.3). Hence $\mathscr{V}_{\mathfrak{h}}(Z_j) \subset \mathscr{V}_{\mathfrak{h}}(\mathfrak{h}^\perp \otimes \cdots \otimes \mathfrak{h}^\perp \otimes Y_1) = \mathscr{V}_{\mathfrak{h}}(Y_1)$ for all j because of Lemma 2.4. Again using Lemma 2.4 (1), we have

$$\mathscr{V}_{\mathfrak{h}}(X_m) \subset \bigcup_{j=0}^{m} \mathscr{V}_{\mathfrak{h}}(Z_j) \subset \mathscr{V}_{\mathfrak{h}}(Y_1) \quad \text{for any } m. \tag{3.7.1}$$

Because $X = \bigcup_{m=0}^\infty X_m$, there exists m such that $X_m \cap Y_2 \neq \{0\}$. Since Y_2 is an irreducible $\mathfrak{h}$-module, we have $Y_2 \subset X_m$. Using Lemma 2.4 (1) again, we have

$$\mathscr{V}_{\mathfrak{h}}(Y_2) \subset \mathscr{V}_{\mathfrak{h}}(X_m). \tag{3.7.2}$$

By (3.7.1) and (3.7.2), we have $\mathscr{V}_{\mathfrak{h}}(Y_2) \subset \mathscr{V}_{\mathfrak{h}}(Y_1)$. The opposite inclusion $\mathscr{V}_{\mathfrak{h}}(Y_1) \subset \mathscr{V}_{\mathfrak{h}}(Y_2)$ is similar. $\square$

3.8. In [Ko1]; [Ko2], Sect. 6, we have observed a phenomenon in the restriction formula: It can happen that the unitary representation $\overline{A_{\mathfrak{q}}(\lambda)}$ of G is decomposed into the Hilbert direct sum of *different series* of irreducible unitary representations of H such as

$$\overline{A_{\mathfrak{q}}^G(\lambda)}_{|H} = \bigoplus_{j=1}^{m} \sum_{v_i^{(j)} \in \Lambda_j}^{\oplus} \overline{\left(A_{\mathfrak{q}'_j}^H\left(v_i^{(j)}\right)\right)} \tag{3.8.1}$$

where $\mathbb{C}_{v_i^{(j)}}$ is weakly fair with respect to θ-stable parabolic subalgebras $\mathfrak{q}'_j = \mathfrak{l}'_j + \mathfrak{u}'_j (1 \leq j \leq m)$ of $\mathfrak{h}$ for any $v_i^{(j)} \in \Lambda_j$. Theorem 3.7 implies that

$$\begin{aligned}\mathrm{Ad}((H \cap K)_{\mathbb{C}})(\mathfrak{u}'_1 \cap \mathfrak{p}) &= \mathrm{Ad}((H \cap K)_{\mathbb{C}})(\mathfrak{u}'_2 \cap \mathfrak{p}) \\ &= \cdots = \mathrm{Ad}((H \cap K)_{\mathbb{C}})\left(\mathfrak{u}'_m \cap \mathfrak{p}\right).\end{aligned}$$

Example 3.8. Let $(G, H) = (SO(4,4), U(2,2))$. For representations and associated varieties for $H = U(2,2)$ we shall use the notation of Sects. 7.1 and 7.3. If π is a discrete series representation for the symmetric space

$SO(4,4)/SO(4,3)$, then the underlying $(\mathfrak{g},K)$-module π_K is discretely decomposable as an $(\mathfrak{h},H\cap K)$-module. If the infinitesimal character is sufficiently regular, the decomposition formula of $\pi|_{U(2,2)}$ (see [Ko2], Theorem 6.1) contains "different series" of representations $A_{\mathfrak{q}(Y_2)}(\lambda)$, $A_{\mathfrak{q}(X_3)}(\lambda')$, and $A_{\mathfrak{q}(Y_5)}(\lambda'')$ with each parameter in the weakly fair range (namely, $m=3$ in (3.8.1)). As we shall see in Figure 7.1.2 and Lemma 7.4, these modules have the same associated variety $\mathcal{O}_{1110}(5)$.

3.9. Theorem 3.7 is mainly intended to the case where $\dim Y_1 = \infty$. But, we mention here that Theorem 3.7 with $\dim Y_1 = 1$ implies a weaker form of Moore's theorem [Mo]. Here, we recall briefly that Moore's theorem asserts that "*a unitary representation π of G satisfying* $\mathrm{Hom}_H(\mathbf{1},\pi) \neq 0$ *must be finite dimensional, if H is a non-compact closed subgroup of a simple Lie group G*".

Corollary 3.9. *Retaining the setting of Sect. 1.2, we suppose H is semisimple without compact factors. We denote by $\mathbf{1}$ the trivial representation of H. Let π be an irreducible, infinitesimally unitary $(\mathfrak{g},K)$-module which is $H\cap K$-admissible. If* $\mathrm{Hom}_{\mathfrak{h},H\cap K}(\mathbf{1},\pi) \neq 0$ then π is finite dimensional.

Proof. By Lemma 1.5 and Lemma 1.3, π is decomposed into an algebraic direct sum of irreducible $(\mathfrak{h},H\cap K)$-modules:

$$\pi \simeq \sum_{\tau} m(\tau)\tau. \tag{3.9.1}$$

Here, the multiplicity $m(\tau) < \infty$ for each irreducible $(\mathfrak{h},H\cap K)$-module τ, because π is $H\cap K$-admissible (see Proposition 1.6). If an irreducible $(\mathfrak{h},H\cap K)$-module τ occurs in (3.9.1), we have $\mathcal{V}_{\mathfrak{h}}(\tau) = \mathcal{V}_{\mathfrak{h}}(\mathbf{1}) = \{0\}$ by Theorem 3.7. Hence τ is finite dimensional by Lemma 2.4 (3). Because there are only finitely many equivalence classes of finite dimensional unitary representations of H, π must be finite dimensional by (3.9.1).

4. Restriction with respect to a symmetric pair

4.1. Let σ be an involutive automorphism of G, and H an open subgroup of the fixed point subgroup $G^\sigma := \{g\in G : \sigma g = g\}$. Then (G,H) is called a *reductive symmetric pair*. There exists a Cartan involution θ of G which commutes with σ. Let $K = G^\theta$ be a maximal compact subgroup of G and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the Cartan decomposition. The differential of σ or its complexification will be also denoted by σ. We set $\mathfrak{g}_0^{-\sigma} := \{X\in\mathfrak{g}_0 : \sigma X = -X\}$ and $\mathfrak{k}_0^{-\sigma} := \mathfrak{k}_0 \cap \mathfrak{g}_0^{-\sigma}$. We fix a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$ such that $\mathfrak{t}_0^{-\sigma} := \mathfrak{t}_0^c \cap \mathfrak{k}_0^{-\sigma}$ is a maximal abelian subspace of $\mathfrak{k}_0^{-\sigma}$. We fix compatible positive systems $\Delta^+(\mathfrak{k},\mathfrak{t}^c)$ and $\Sigma^+(\mathfrak{k},\mathfrak{t}^{-\sigma})$, namely, $\{\alpha_{|\mathfrak{t}^{-\sigma}} : \alpha\in\Delta^+(\mathfrak{k},\mathfrak{t}^c)\}\setminus\{0\}$ equals $\Sigma^+(\mathfrak{k},\mathfrak{t}^{-\sigma})$. Without loss of generality, we may assume that a θ-stable parabolic subalgebra $\mathfrak{q}\equiv\mathfrak{q}(X)$ is defined by a dominant element $X\in\sqrt{-1}\mathfrak{t}_0^c$ with respect to $\Delta^+(\mathfrak{k},\mathfrak{t}^c)$ (see Sect. 2.6). Define a closed cone in $\sqrt{-1}(\mathfrak{t}_0^c)^*$ by

$$(4.1.1)\qquad \mathbb{R}_+\langle \mathfrak{u}\cap\mathfrak{p}\rangle := \left\{ \sum_{\beta\in\Delta(\mathfrak{u}\cap\mathfrak{p},\mathfrak{t}^c)} n_\beta\,\beta : \ n_\beta \geq 0 \right\}.$$

Theorem 4.2. *Suppose that (G,H) is a reductive symmetric pair and that $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ is a θ-stable parabolic subalgebra defined by a dominant element of $\sqrt{-1}(\mathfrak{t}_0^c)^*$. In what follows, $\mathbb{C}_\lambda$ is a unitary character of L and $\overline{A_\mathfrak{q}(\lambda)}$ is a unitary representation of G obtained by the Hilbert completion of a Vogan-Zuckerman $(\mathfrak{g},K)$-module $A_\mathfrak{q}(\lambda)$ (see Sect. 2.6). Then the following eight conditions on $(G,H,\mathfrak{q})$ are equivalent:*

1) *$\overline{A_\mathfrak{q}(\lambda)}|_{H\cap K}$ is $H\cap K$-admissible for any $\mathbb{C}_\lambda$ in the weakly fair range.*
2) *$\overline{A_\mathfrak{q}(\lambda)}|_{H\cap K}$ is $H\cap K$-admissible for some $\mathbb{C}_\lambda$ in the weakly fair range such that $A_\mathfrak{q}(\lambda)\neq\{0\}$.*
3) *$A_\mathfrak{q}(\lambda)$ is discretely decomposable as an $(\mathfrak{h},H\cap K)$-module for any $\mathbb{C}_\lambda$ in the weakly fair range.*
4) *$A_\mathfrak{q}(\lambda)$ is discretely decomposable as an $(\mathfrak{h},H\cap K)$-module for some $\mathbb{C}_\lambda$ in the weakly fair range such that $A_\mathfrak{q}(\lambda)\neq\{0\}$.*
5) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathfrak{u}\cap\mathfrak{p})\subset\mathcal{N}_\mathfrak{h}^*$.
6) $\sigma\mathfrak{u}\cap\mathfrak{u}^-\subset\mathfrak{k}$.
7) $\sigma(\Delta(\mathfrak{u}\cap\mathfrak{p},\mathfrak{t}^c))\cap\Delta(\mathfrak{u}^-\cap\mathfrak{p},\mathfrak{t}^c)=\emptyset$.
8) $\mathbb{R}_+\langle\mathfrak{u}\cap\mathfrak{p}\rangle\cap\sqrt{-1}\left(\mathfrak{t}_0^{-\sigma}\right)^*=0$.

Here, we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$ in (5) and have regarded $\left(\mathfrak{t}_0^{-\sigma}\right)^*$ as a subspace of $\mathfrak{t}_0^*$ by using a non-degenerate symmetric $\mathrm{Ad}(G)$-invariant bilinear form on $\mathfrak{g}$.

It follows from [Ko2], Corollary 1.3 that the above equivalent eight conditions imply that the unitary representation $\overline{A_\mathfrak{q}(\lambda)}$ of G decomposes discretely as a representation of H:

$$\overline{A_\mathfrak{q}(\lambda)}|_H \simeq \sum_{\tau\in\widehat{H}}^{\oplus} m(\tau)\tau, \qquad m(\tau)<\infty.$$

Examples of the multiplicity free decomposition, namely, $m(\tau)\leq 1$ for any $\tau\in\widehat{H}$, are given in [JV], Sect. 4; [Ko1], Theorem 3.4, Theorem 3.5 and Theorem 3.6; [Ko2], Theorem 6.1, Theorem 6.4; and [GW], Sect. 6.

4.3. Each K-type occurring in any irreducible unitary representation of G is of finite multiplicity. On the other hand, it often occurs that the multiplicity of the continuous spectrum in the irreducible decomposition of $\pi|_H$ is infinite where (G,H) is a symmetric pair and $\pi\in\widehat{G}$. The following Corollary gives the finite multiplicity theorem for the discrete part on the level of Harish-Chandra modules, generalizing the case with $H=K$.

Corollary 4.3. *Suppose (G,H) is a reductive symmetric pair. Then*

$$\dim \operatorname{Hom}_{\mathfrak{h}, H\cap K}\big(Y, A_{\mathfrak{q}}(\lambda)\big) < \infty$$

for any irreducible $(\mathfrak{h}, H\cap K)$*-module* Y *and for any Zuckerman's derived functor module* $A_{\mathfrak{q}}(\lambda)$ *with* $\mathbb{C}_\lambda$ *a unitary character of* L *in the weakly fair range.*

Proof of Corollary 4.3. If $\operatorname{Hom}_{\mathfrak{h}, H\cap K}\big(Y, A_{\mathfrak{q}}(\lambda)\big) = \{0\}$ for any irreducible $(\mathfrak{h}, H\cap K)$-module Y, then there is nothing to prove. If $\operatorname{Hom}_{\mathfrak{h}, H\cap K}\big(Y, A_{\mathfrak{q}}(\lambda)\big) \neq \{0\}$ for some irreducible $(\mathfrak{h}, H\cap K)$-module Y, then $A_{\mathfrak{q}}(\lambda)$ is discretely decomposable as an $(\mathfrak{h}, H\cap K)$-module by Lemma 1.5. By Theorem 4.2, the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_{H\cap K}$ is $H\cap K$-admissible. By Theorem 1.2 of [Ko2], the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_H$ is H-admissible; namely, the multiplicity of each irreducible representation of H occurring in the irreducible decomposition of the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_H$ is finite. □

4.4. *Strategy of Proof of Theorem 4.2.* The implication (8) $\Rightarrow$ (1) is proved in [Ko2], Theorem 3.2. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious. The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are proved in Proposition 1.6. The implication (4) $\Rightarrow$ (5) follows from Corollary 3.5 because the condition (4) leads to

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathfrak{u}^- \cap \mathfrak{p}) \subset \mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathrm{Ad}(K_{\mathbb{C}})(\mathfrak{u}^- \cap \mathfrak{p})) \subset \mathcal{N}_{\mathfrak{h}}^*.$$

Here, we note that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathfrak{u}^- \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{h}}^*$ if and only if $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathfrak{u} \cap \mathfrak{p}) \subset \mathcal{N}_{\mathfrak{h}}^*$.

For the rest of this section, we shall give a proof of the implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8).

4.5. (5) $\Rightarrow$ (6): For $W \in \mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, we denote by $W \mapsto \overline{W}$ the complex conjugation with respect to $\mathfrak{g}_0$. Let $X \in (\mathfrak{u} \cap \mathfrak{p}) \cap \sigma(\mathfrak{u}^- \cap \mathfrak{p})$. Assuming (5), we will show $X = 0$. We first note that $\overline{X} \in \mathfrak{u}^- \cap \mathfrak{p}$, $\sigma X \in \mathfrak{u}^- \cap \mathfrak{p}$ and $\sigma\overline{X} = \overline{\sigma X} \in \mathfrak{u} \cap \mathfrak{p}$. We put $Y := X + \sigma\overline{X} \in \mathfrak{u} \cap \mathfrak{p}$. Then $Y + \sigma Y = (X + \overline{X}) + \sigma(X + \overline{X}) \in \mathfrak{p} \cap \mathfrak{g}_0 = \mathfrak{p}_0$. By using the assumption (5), we have

$$Y + \sigma Y \in \mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathfrak{u} \cap \mathfrak{p}) \cap \mathfrak{p}_0 \subset \mathcal{N}_{\mathfrak{h}}^* \cap \mathfrak{p}_0.$$

Here, we note that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(Y) = \frac{1}{2}(Y + \sigma Y)$ under the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ and of $\mathfrak{h}$ with $\mathfrak{h}^*$. Because any element of $\mathfrak{p}_0$ is semisimple, $\mathcal{N}_{\mathfrak{h}}^* \cap \mathfrak{p}_0 = \{0\}$. Since $Y \in \mathfrak{u}$ and $\sigma Y \in \mathfrak{u}^-$, $Y + \sigma Y = 0$ implies $Y = 0$. Similarly, if we put $Z := X - \sigma\overline{X}$, then $Z + \sigma Z = (X - \overline{X}) + \sigma(X - \overline{X}) \in \sqrt{-1}\mathfrak{p}_0$ is also a semisimple element. Hence we have $Z = -\sigma Z \in \mathfrak{u} \cap \mathfrak{u}^- = \{0\}$. Therefore $X = \frac{1}{2}(Y + Z) = 0$. Hence we have proved the implication (5) $\Rightarrow$ (6).

4.6. (6) $\Rightarrow$ (7): This is easy. In fact, if there is an element $\alpha \in \sigma(\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)) \cap \Delta(\mathfrak{u}^- \cap \mathfrak{p}, \mathfrak{t}^c)$, then we take a non-zero root vector X_α. Then $X_\alpha \in \mathfrak{u}^- \cap \mathfrak{p}$ and $\sigma X_\alpha \in \mathfrak{u} \cap \mathfrak{p}$. Therefore, $\sigma(\mathfrak{u} \cap \mathfrak{p}) \cap (\mathfrak{u}^- \cap \mathfrak{p}) \neq \{0\}$. Thus we have proved (6) $\Rightarrow$ (7).

4.7. (7) $\Rightarrow$ (8): Suppose $A \in \sqrt{-1}\mathfrak{t}_0^c$ defines the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. This means that $\alpha(A) > 0$ for any $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t}^c)$ and that $\alpha(A) = 0$ for any $\alpha \in \Delta(\mathfrak{l}, \mathfrak{t}^c)$. We assume (7), equivalently,

$$\sigma(\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)) \subset \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c) \cup \Delta(\mathfrak{l} \cap \mathfrak{p}, \mathfrak{t}^c). \tag{4.7.1.}$$

Let $\beta := \sum_{\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)} a_\alpha \alpha \in \mathbb{R}_+\langle \mathfrak{u} \cap \mathfrak{p} \rangle$ with $a_\alpha \geq 0$ for all α. If β belongs to $\sqrt{-1}(\mathfrak{t}_0^{-\sigma})^*$, then we have $\sigma\beta + \beta = 0$. By (4.7.1), we have $(\sigma\beta)(A) \geq 0$. Therefore $\beta(A) \leq 0$, which occurs only if $a_\alpha = 0$ for all α, namely, $\beta = 0$. Therefore, we have proved (7) $\Rightarrow$ (8).

This completes the proof of Theorem 4.2.

5. Restriction of holomorphic discrete series representations

5.1. In this section, we assume that G is a simple linear Lie group such that G/K is Hermitian, namely, the center $\mathfrak{c}(\mathfrak{k}_0)$ of $\mathfrak{k}_0$ is not trivial. It is known that $\mathfrak{c}(\mathfrak{k}_0)$ is one dimensional and that there exists $Z \in \mathfrak{c}(\mathfrak{k}_0)$ so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ are 0, $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\operatorname{ad}(Z)$. A $(\mathfrak{g}, K)$-module is said to be a *highest weight module* (resp. *lowest weight module*) if there exists a non-zero vector annihilated by $\mathfrak{p}^+$(resp. $\mathfrak{p}^-$). It will be convenient to allow 'highest weight module' to refer also to an irreducible unitary representation of G whose underlying $(\mathfrak{g}, K)$-module is a highest weight module. A discrete series representation for G is said to be a *holomorphic discrete series representation* (resp. *anti-holomorphic discrete representation*) if it is a highest weight module (resp. a lowest weight module).

5.2. Suppose σ is an involutive automorphism of G commuting with the Cartan involution θ. Since $\sigma\mathfrak{c}(\mathfrak{k}_0) = \mathfrak{c}(\mathfrak{k}_0)$, there are two exclusive possibilities:

$$\sigma Z = Z, \tag{5.2.1}$$

$$\sigma Z = -Z. \tag{5.2.2}$$

In this section, we suppose $H = G^\sigma$, the subgroup of G consisting of fixed points by σ.

The geometric meaning of (5.2.1) and (5.2.2) is the following: $H/H \cap K \subset G/K$ is a complex submanifold in the case (5.2.1) and is a totally real submanifold in the case (5.2.2)

Theorem 5.3. *Suppose G is a non-compact simple Lie group. Let π be a holomorphic discrete series representation of G. If σ satisfies* (5.2.2), *then the underlying $(\mathfrak{g}, K)$-module π_K is not discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module. In particular, the restriction $\pi|_{H \cap K}$ is not $H \cap K$-admissible.*

5.4. The above theorem is in sharp contrast with the following fact (cf. [Ma], [JV], [Ko4]):

Fact 5.4. *Let $\pi \in \widehat{G}$ be an irreducible unitary highest weight module. If σ satisfies (5.2.1), then the restriction $\pi|_{H\cap K}$ is $H \cap K$*-admissible. In particular, the underlying $(\mathfrak{g}, K)$*-module π_K is discretely decomposable as an* $(\mathfrak{h}, H \cap K)$*-module.*

5.5. *Proof of Theorem 5.3.* Any holomorphic discrete series representation π is of the form $\overline{A_{\mathfrak{q}}(\lambda)}$ such that $\mathbb{C}_\lambda$ is in the good range with respect to a θ-stable parabolic subalgebra (actually, Borel subalgebra) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ with $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{p}_-$. On the other hand, the assumption (5.2.2) implies $\sigma\mathfrak{p}_- = \mathfrak{p}_+$. Therefore, if the condition (6) of Theorem 4.2 holds, then we have $\{0\} = \sigma\mathfrak{p}_- \cap \mathfrak{p}_+$. Hence $\mathfrak{p}_+ = \{0\}$. Then $\mathfrak{p} = \mathfrak{p}_- + \mathfrak{p}_+ = \sigma\mathfrak{p}_+ + \mathfrak{p}_+ = \{0\}$, which implies that G is compact, yielding a contradiction. Therefore, the condition (6) of Theorem 4.2 fails. Thus, the underlying $(\mathfrak{g}, K)$-module of π is not discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module. □

Remark 5.6. In view of Fact 5.4, one might expect that no unitary highest weight module of infinite dimension is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module if $H = G^\sigma$ with σ satisfying (5.2.2). However, this is not always the case. For instance, with the notation of Sect. 7.3 where $(G, H) = (U(2,2), Sp(1,1))$, $A_{\mathfrak{q}(Z_i)}$ $(1 \le i \le 4)$ are unitary highest weight modules due to [A2], while they are discretely decomposable as an $(\mathfrak{h}, H \cap K)$-modules (see Sect. 7).

6. Discrete series for semisimple symmetric spaces and restrictions

In this section we give another application of our main theorem in the case of discrete series representations for semisimple symmetric spaces G/H and mention some related topics in our context.

6.1. Let (G, H) be a reductive symmetric pair defined by an involution σ of G. We shall use the notation in Sect. 4.1. There is a G-invariant measure on a reductive symmetric space G/H. Then we have a natural unitary representation (a regular representation) on the Hilbert space of square integrable functions $L^2(G/H)$. An irreducible unitary representation π is said to be a *discrete series representation* for G/H if there is a non-zero unitary G-homomorphism from π into $L^2(G/H)$. We denote by $\mathrm{Disc}(G/H) \subset \widehat{G}$ the set of discrete series representations for G/H.

6.2. Here is a main theorem in this section.

Theorem 6.2. *Suppose that G/H is a non-compact reductive symmetric space. Then the underlying $(\mathfrak{g}, K)$-module of any discrete series representation for G/H is not discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module. In particular, no discrete series representation for G/H is $H \cap K$-admissible.*

6.3. With the notation in Sect. 4.1, we review the following result due to Flensted-Jensen, Matsuki-Oshima and Vogan.

Fact 6.3 ([FJ1]; [MO]; [FJ2], Ch.VIII, Sect. 2; [V4], Sect. 4). *Let G/H be a reductive symmetric space. Then* $\mathrm{Disc}(G/H) \neq \emptyset$ *if and only if* $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. *Furthermore, if the rank condition is satisfied, then any discrete series* $\pi \in \mathrm{Disc}(G/H)$ *is of the form* $\overline{A_{\mathfrak{q}}(\lambda)}$, *where* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *is defined by a generic element in* $\sqrt{-1}\mathfrak{t}_0^{-\sigma}$ *and* $\mathbb{C}_\lambda$ *is a unitary character of* $L = Z_G(\mathfrak{t}_0^{-\sigma})$ *in the fair range satisfying some integral conditions determined by* (G, H).

6.4. Before proving Theorem 6.2, we prepare the following lemma:

Lemma 6.4. *Retain the notation in Fact 6.3. Suppose that G/H is a reductive symmetric space with* $\operatorname{rank} G/H = \operatorname{rank} K/H \cap K$. *If* $\mathfrak{p} \subset \mathfrak{l}$, *then G/H is compact.*

Proof. The rank condition implies that $\mathfrak{t}^{-\sigma}$ is a maximal abelian subspace in $\mathfrak{g}^{-\sigma}$, and so we have $\mathfrak{l} \cap \mathfrak{g}^{-\sigma} = \mathfrak{t}^{-\sigma}$. Assume $\mathfrak{p} \subset \mathfrak{l}$. Then $\mathfrak{p} \cap \mathfrak{g}^{-\sigma} \subset \mathfrak{l} \cap \mathfrak{g}^{-\sigma} = \mathfrak{t}^{-\sigma} \subset \mathfrak{k}$, which implies $\mathfrak{p} \cap \mathfrak{g}^{-\sigma} = \{0\}$. Hence we have $\mathfrak{p}_0 \cap \mathfrak{g}_0^{-\sigma} = \{0\}$. Since G/H is diffeomorphic to a vector bundle over a compact manifold $K/H \cap K$ with typical fiber $\mathfrak{p}_0 \cap \mathfrak{g}_0^{-\sigma}$, it follows that G/H is compact. □

6.5. *Proof of Theorem 6.2.* Suppose there exists $\pi \in \mathrm{Disc}(G/H)$ such that the underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module. Then π_K is of the form $A_{\mathfrak{q}}(\lambda)$, with $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is defined by a generic element of $\sqrt{-1}\mathfrak{t}_0^{-\sigma}$ as in Fact 6.3. In particular, we have $\sigma\mathfrak{u} = \mathfrak{u}^-$. Then the equivalent condition (6) in Theorem 4.2 yields $\mathfrak{u} \cap \mathfrak{p} = \{0\}$. In view of the direct sum decomposition

$$\mathfrak{p} = (\mathfrak{u} \cap \mathfrak{p}) + (\mathfrak{l} \cap \mathfrak{p}) + \sigma(\mathfrak{u} \cap \mathfrak{p})$$

we have $\mathfrak{p} = \mathfrak{l} \cap \mathfrak{p}$, that is, $\mathfrak{p} \subset \mathfrak{l}$. By Lemma 6.4, G/H must be compact, which contradicts to our assumption. Hence we have proved Theorem. □

Remark 6.6. Related topics are in order:

1) Let K be a maximal compact subgroup of G as usual. Any irreducible unitary representation is K-admissible (Harish-Chandra). Therefore, Theorem 6.2 in the special case where $H = K$ is equivalent to the well-known fact that there is no discrete series representation for the non-compact Riemannian symmetric space G/K. This example does not give a new proof of this classical result, but rather a different perspective.

2) There are a number of examples of $\pi \in \mathrm{Disc}(G/H)$ such that $\pi|_L$ is $L \cap K$-admissible where σ and τ are involutive automorphisms of G and $H = G^\sigma$ and $L = G^\tau$ (see [Ko2], Sect. 5 and [Ko4] for the application to L^p-analysis). For instance, this is the case if σ satisfies (5.2.2) and if τ satisfies (5.2.1) in the setting of Sect. 5.2. On the other hand, Theorem 6.2 asserts that such examples can be found only if $L \neq H$, namely, $\sigma \neq \tau$ in the non-compact case.

3) Suppose Γ is a torsion free discrete subgroup of G such that both $\Gamma\backslash G$ and $(H \cap \Gamma)\backslash H$ are compact. Then the image of the natural map $(H \cap \Gamma)\backslash H/(H \cap K) \to \Gamma\backslash G/K$ defines a cycle which is called a *modular symbol*. The Poincaré dual of modular symbols are studied in the context of the Matsushima–Murakami formula ([MM], see also [BW]): $H^j_{\text{de Rham}}(\Gamma\backslash G/K; \mathbb{C}) \simeq \oplus_{\pi \in \widehat{G}} \operatorname{Hom}_G\big(\pi, L^2(\Gamma\backslash G)\big) \otimes H^j(\mathfrak{g}, K; \pi_K)$. Y. Tong and S. Wang proved the *non-vanishing* theorem of the modular symbols for the component of $\pi \in \operatorname{Disc}(G/H)$ ([TW]), while T. Oda and the author recently proved the *vanishing* theorem for the component of $\pi \in \widehat{G}$ whose restriction to H is H-admissible. Thus, one could expect that no $\pi \in \widehat{G}$ satisfies both assumptions. This was a motivation of Theorem 6.2. We note that it requires some elaboration to deduce Theorem 6.2 from [TW] and [KO] because the proof of the non-vanishing theorem of [TW] is not valid for the constant sheaf, whose vanishing theorem is given in [KO].

4) Let us consider the group manifold case $G/H \simeq G' \times G'/\operatorname{diag}(G')$. Then any discrete series representation for G/H is of the form $\pi \boxtimes \pi^*$ where $\pi \in \operatorname{Disc}(G')$ and π^* is the dual of π. Its restriction to $H = \operatorname{diag}(G')$ is nothing but the tensor product $\pi \mathbin{\widehat{\otimes}} \pi^*$. Theorem 6.2 means that $\pi_K \otimes \pi_K^*$ is not discretely decomposable as a $(\mathfrak{g}, K)$-module. This fact was known more explicitly in the special setting where π is a holomorphic discrete series representation. That is, the tensor product of holomorphic and anti-holomorphic discrete series representations is unitarily equivalent to the quasi-regular representation on the space of L^2-sections of a certain vector bundle over a Riemannian symmetric space by a result of Repka [R] (see also [Ko3], Sect. 6.2 for some more references), of which the irreducible decomposition involves continuous spectra and at most finitely many discrete series representations.

5) Generalizing the result of Repka, Ólafsson and Ørsted announced the irreducible decomposition of the restriction $\pi|_H$ under the assumption that G/H satisfies (5.2.2) and that π is a holomorphic discrete series representation for G/H having a one-dimensional minimal K-type [OØ]. Only continuous spectra appear in the irreducible decomposition of $\pi|_H$ in this case. We note that the decomposition formula for $\pi|_H$ is not known for a non-holomorphic discrete series representation π in general.

7. Example 1 : the restriction from $U(2,2)$ to $Sp(1,1)$

In this section, we illustrate our results by an example of a symmetric pair $(G, H) = (U(2,2), Sp(1,1)) \approx (SO(4,2), SO(4,1))$. We shall give a classification of all irreducible unitary representations of G with regular integral infinitesimal character, whose underlying $(\mathfrak{g}, K)$-modules are discretely decomposable as an $(\mathfrak{h}, H \cap K)$-modules (Proposition 7.5). Among 18 series of irreducible infinitesimally unitary $(\mathfrak{g}, K)$-modules, 12 will be proved to be discretely decomposable as $(\mathfrak{h}, H \cap K)$-modules. Proposition 7.5 itself follows from Theorem 4.2 immediately by computation of the root system, however, we also present some elementary computations of associated varieties to illustrate the ingredients of Sect. 3 (cf. Example 3.8 and Remark 5.6).

7.1. Let $G = U(2,2)$ be the indefinite unitary group realized as $\{g \in GL(4,\mathbb{C}) : {}^t\bar{g}I_{2,2}g = I_{2,2}\}$, where $I_{2,2} := \operatorname{diag}(1,1,-1,-1)$. We take a maximal compact group $K = G \cap U(4) \simeq U(2) \times U(2)$ and then $K_{\mathbb{C}} \simeq GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$. We begin with the description of $\operatorname{Ad}(K_{\mathbb{C}})$-orbits on $\mathcal{N}_{\mathfrak{g}}^* \cap (\mathfrak{g}/\mathfrak{k})^*$. We identify

$$(7.1.1)\quad (\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p} \simeq \left\{ \begin{pmatrix} O & A \\ B & O \end{pmatrix} : A, B \in M(2,\mathbb{C}) \right\} \simeq M(2,\mathbb{C}) \oplus M(2,\mathbb{C}),$$

on which $K_{\mathbb{C}}$ acts by

$$(A,B) \mapsto \left(g_1 A g_2^{-1}, g_2 B g_1^{-1}\right), \quad (g_1, g_2) \in K_{\mathbb{C}} \simeq GL(2,\mathbb{C}) \times GL(2,\mathbb{C}).$$

Then the nilpotent cone $\mathcal{N}_{\mathfrak{p}}^* := \mathcal{N}_{\mathfrak{g}}^* \cap (\mathfrak{g}/\mathfrak{k})^*$ is given by

$$\{(A,B) \in M(2,\mathbb{C}) \oplus M(2,\mathbb{C}) : AB \text{ and } BA \text{ are nilpotent matrices}\}.$$

The variety $\mathcal{N}_{\mathfrak{p}}^*$ splits into 10 $\operatorname{Ad}(K_{\mathbb{C}})$-orbits:

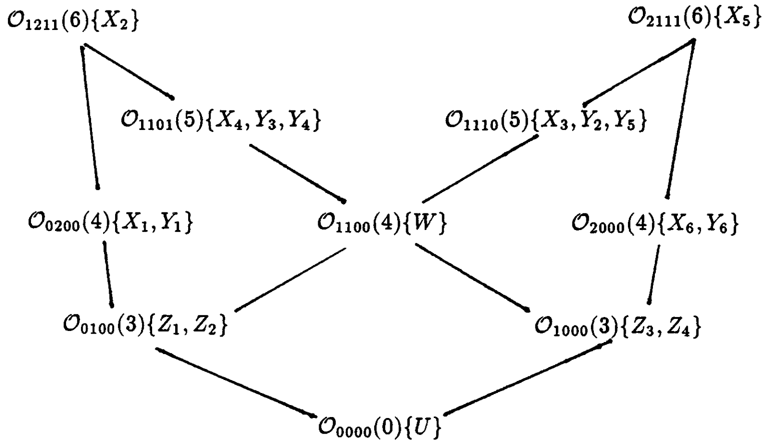

Fig. 7.1.2

Here, we have put

$$\mathcal{O}_{ijkl} := \{(A,B) \in \mathcal{N}_{\mathfrak{p}}^* : \operatorname{rank} A = i, \operatorname{rank} B = j, \operatorname{rank} AB = k, \operatorname{rank} BA = l\}.$$

Since rank$A$, rank$B$, rankAB and rankBA are invariants of the $\operatorname{Ad}(K_{\mathbb{C}})$-action, $\mathcal{O}_{ijkl}$ are $K_{\mathbb{C}}$-invariant sets. In fact, each $\mathcal{O}_{ijkl}$ in Figure 7.1.2. is a single $\operatorname{Ad}(K_{\mathbb{C}})$-orbit. For the reader's convenience, $\mathcal{O}_{ijkl}$ is also written as

$\mathcal{O}_{ijkl}(N)$ if $\dim_{\mathbb{C}} \mathcal{O}_{ijkl} = N$. The segment denotes the closure relation. The brace followed by $\mathcal{O}_{ijkl}(N)$ (e.g. $\{X_2\}$ followed by $\mathcal{O}_{1211}(6)$) describes the associated variety which we will explain in Sect. 7.3. Figure 7.1.2 is obtained as follows. For instance, if $\operatorname{rank} A = 1$ then we can assume $A = E_{12}$, after conjugating by an element of $K_{\mathbb{C}}$, where $E_{ij} \in M(2, \mathbb{C})$ stands for the matrix unit. Because AB and BA are nilpotent, the $(2,1)$-component of B must be 0. Then B is conjugate to one of $E_{11} + E_{22}$, E_{11}, E_{22}, E_{12} or O by the subgroup of $K_{\mathbb{C}}$ stabilizing A. The corresponding orbits are $\mathcal{O}_{1211}(6)$, $\mathcal{O}_{1101}(5)$, $\mathcal{O}_{1110}(5)$, $\mathcal{O}_{1100}(4)$ or $\mathcal{O}_{1000}(3)$, respectively. Other cases are similar or trivial.

7.2. Let $H = Sp(1,1)$. Corresponding to the embedding $H \hookrightarrow G$, we have the projection $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} : \mathfrak{p}^* \to (\mathfrak{p} \cap \mathfrak{h})^*$ which respects the action of $(H \cap K)_{\mathbb{C}} \hookrightarrow K_{\mathbb{C}}$. The projection is given in the matrix spaces by

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} := \mathfrak{p}^* \simeq M(2,\mathbb{C}) \oplus M(2,\mathbb{C}) \to (\mathfrak{p} \cap \mathfrak{h})^* \simeq M(2,\mathbb{C}),\ (A,B) \mapsto A + \tau B,$$

where $\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$ and the action of $(H \cap K)_{\mathbb{C}} \simeq SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \ni (g_1, g_2)$ on $(\mathfrak{p} \cap \mathfrak{h})^*$ is given by $Z \mapsto g_1 Z g_2^{-1}$. The nilpotent variety $\mathcal{N}_{\mathfrak{h}}^* \cap (\mathfrak{p} \cap \mathfrak{h})^*$ is identified with $\{X \in M(2,\mathbb{C}) : \operatorname{rank} X \le 1\}$.

Lemma 7.2. *The following conditions on a* $K_{\mathbb{C}}$-orbit $\mathcal{O}$ *in* $\mathcal{N}_{\mathfrak{p}}^*$ *are equivalent*:
1) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathcal{O}) \not\subset \mathcal{N}_{\mathfrak{h}}^*$.
2) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\overline{\mathcal{O}}) \not\subset \mathcal{N}_{\mathfrak{h}}^*$. *{Here,* $\overline{\mathcal{O}}$ *denotes the closure of* $\mathcal{O}$.
3) $\mathcal{O}$ *is one of* $\mathcal{O}_{2111}(6)$, $\mathcal{O}_{1211}(6)$, $\mathcal{O}_{2000}(4)$ or $\mathcal{O}_{0200}(4)$.

Proof. By the closure relation described in Figure 7.1.2, it suffices to show (1) $\Leftrightarrow$ (3). Since $\mathcal{O}_{2111}(6)$ contains $(A,B) := (E_{11} + E_{22}, E_{12})$ (see Sect. 7.1), $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(A,B) = A + \tau B \notin \mathcal{N}_{\mathfrak{h}}^*$ because $\operatorname{rank}(A + \tau B) = 2 > 1$. Hence $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} (\mathcal{O}_{2111}(6)) \not\subset \mathcal{N}_{\mathfrak{h}}^*$. Similarly, $\mathcal{O}_{2000}(4)$ contains $(E_{11} + E_{22}, O)$ and thus $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathcal{O}_{2000}(4)) \not\subset \mathcal{N}_{\mathfrak{h}}^*$. Changing the role of A and B for other cases, we have (3) $\Rightarrow$ (1). To prove (1) $\Rightarrow$ (3), we note that

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathrm{Ad}(K_{\mathbb{C}})(A,B)) = \mathrm{Ad}((H \cap K)_{\mathbb{C}})\{(aA + b\tau B) : a, b \in \mathbb{C}^\times\}.$$

For example, $\mathcal{O}_{1101}(5)$ contains (E_{12}, E_{11}) and $\det(aE_{12} + b\tau E_{11}) = 0$ for any $a, b \in \mathbb{C}^\times$. Hence we have $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}}(\mathcal{O}_{1101}(5)) \subset \mathcal{N}_{\mathfrak{h}}^*$. Other cases are similar. $\square$

7.3. It follows from a result of S. Salamanca Riba (see [Sa], Theorem 1.2) that any irreducible unitary representation of $G = U(2,2)$ whose underlying $(\mathfrak{g}, K)$-module has regular integral infinitesimal character is of the form $A_{\mathfrak{q}}(\lambda)$ where $\mathfrak{q}$ is a θ-stable parabolic subalgebra and λ is in the good range. Because the associated variety and the discrete decomposability are preserved by coherent continuation in the good range of parameters, we shall assume $\lambda = 0$ without loss of generality. Let $\mathfrak{t}_0^c$ be a Cartan subalgebra of $\mathfrak{k}_0 \simeq \mathfrak{u}(2) + \mathfrak{u}(2)$. We choose a coordinate in $\sqrt{-1}\mathfrak{t}_0^c$ so that $\Delta^+(\mathfrak{k}, \mathfrak{t}^c) = \{e_1 - e_2, e_3 - e_4\}$. With this basis, we put $X_1 = (4,3,2,1), X_2 = (4,2,3,1)$,

$X_3 = (4,1,3,2)$, $X_4 = (3,2,4,1)$, $X_5 = (3,1,4,2)$, $X_6 = (2,1,4,3)$, $Y_1 = (2,1,1,0)$, $Y_2 = (2,0,1,0)$, $Y_3 = (2,1,2,0)$, $Y_4 = (1,0,2,0)$, $Y_5 = (2,0,2,1)$, $Y_6 = (1,0,2,1)$, $Z_1 = (1,0,0,0)$, $Z_2 = (1,1,1,0)$, $Z_3 = (0,0,1,0)$, $Z_4 = (1,0,1,1)$, $W = (1,0,1,0)$, $U = (0,0,0,0) \in \sqrt{-1}\mathfrak{t}^c_{0-}$. Then the set of $(\mathfrak{g},K)$ -modules

$$\{A_\mathfrak{q} : \mathfrak{q} = \mathfrak{q}(X_i),\ \mathfrak{q}(Y_i)(1 \le i \le 6),\ \mathfrak{q}(Z_i)(1 \le i \le 4),\ \mathfrak{q}(W),\ \mathfrak{q}(U)\}$$

is the totality of irreducible, infinitesimally unitarizable $(\mathfrak{g},K)$-modules with trivial infinitesimal character. We note that $\overline{A_{\mathfrak{q}(X_i)}}$ $(1 \le i \le 6)$ is Harish-Chandra's discrete series for a group manifold G and $A_{\mathfrak{q}(U)} = \mathbf{C}$. We refer to the figure in [Ko2], Example 3.7 for the geometric description of these modules in the context of the Beilinson-Bernstein correspondence between irreducible Harish-Chandra modules and irreducible K-equivariant sheaves of $\mathscr{D}$-modules on the flag variety of $G_\mathbf{C}$.

7.4. Let us compute the associated variety $\mathscr{V}(A_\mathfrak{q}) = \mathrm{Ad}(K_\mathbf{C})(\mathfrak{u}^- \cap \mathfrak{p})$ for each θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$.

Lemma 7.4. *The correspondence $X \mapsto A_{\mathfrak{q}(X)} \mapsto \mathscr{V}_\mathfrak{g}(A_{\mathfrak{q}(X)}) =: \overline{\mathcal{O}}$ is given in the Figure 7.1.2 described as $\mathcal{O}\{X\}$. Here $\overline{\mathcal{O}}$ stands for the closure of some* $\mathrm{Ad}(K_\mathbf{C})$*-orbit $\mathcal{O} = \mathcal{O}_{ijkl}(N)$, and X stands for one of X_m, Y_m, Z_m, W or U.*

For example, the above lemma means that $\mathscr{V}_\mathfrak{g}(A_{\mathfrak{q}(X_1)}) = \overline{\mathcal{O}_{0200}(4)}$, $\mathscr{V}_\mathfrak{g}(A_{\mathfrak{q}(X_2)}) = \overline{\mathcal{O}_{1211}(6)}$, $\mathscr{V}_\mathfrak{g}(A_{\mathfrak{q}(X_5)}) = \overline{\mathcal{O}_{2111}(6)}$, and so on. Let us sketch the computation for $\mathscr{V}_\mathfrak{g}(A_{\mathfrak{q}(X_5)}) = \overline{\mathcal{O}_{2111}}$. Corresponding to $\Delta(\mathfrak{u}^-(X_5) \cap \mathfrak{p}) = \{e_1 - e_3, -e_1 + e_4, e_2 - e_3, e_2 - e_4\}$, we have

$$\mathfrak{u}^-(X_5) \cap \mathfrak{p} \simeq \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : a,b,c,d \in \mathbf{C} \right\} \subset M(2,\mathbf{C}) \oplus M(2,\mathbf{C}).$$

Hence, $\mathrm{Ad}(K_\mathbf{C})(\mathfrak{u}^-(X_5) \cap \mathfrak{p}) = \{(A,B) : \mathrm{rank} A \le 2, \mathrm{rank} B \le 1\} = \overline{\mathcal{O}_{2111}(6)}$.

7.5. Here is a classification of discretely decomposable modules.

Proposition 7.5. *Let $G = U(2,2) \supset H = Sp(1,1)$. Let π be the irreducible unitary representation of G with regular integral infinitesimal character. Then the following three conditions are equivalent:*

1) *The restriction $\pi|_{H \cap K}$ is $H \cap K$-admissible.*

2) *The underlying $(\mathfrak{g},K)$-module π_K is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.*

3) *π is isomorphic to $\overline{A_\mathfrak{q}(\lambda)}$ where λ is in the good range of parameter and where $\mathfrak{q}$ is one of $\mathfrak{q}(X_3)$, $\mathfrak{q}(X_4)$, $\mathfrak{q}(Y_2)$, $\mathfrak{q}(Y_3)$, $\mathfrak{q}(Y_4)$, $\mathfrak{q}(Y_5)$, $\mathfrak{q}(Z_i)$ $(1 \le i \le 4)$, $\mathfrak{q}(W)$, $\mathfrak{q}(U)$.*

Proof. (3) $\Rightarrow$ (1) is given in Theorem 3.2 and Example 3.7 of [Ko2]. (1) $\Rightarrow$ (2) is proved in Proposition 1.6. (2) $\Rightarrow$ (3) follows from Corollary 3.5 and the above computations. $\square$

Remark 7.6. The above proof of (3) $\Rightarrow$ (1) is divided into four steps:

Step 1. Computation of the $K_{\mathbb{C}}$-orbits in the nilpotent cone $\mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$.
Step 2. Computation of the projection $\mathrm{pr}_{\widetilde{\mathfrak{g}} \to \mathfrak{h}}$ of the orbits in Step 1.
Step 3. Description of $\widehat{G}$ with regular integral infinitesimal character.
Step 4. Description of the associated varieties for the representations in Step 3.

More general combinatorial results may be found in the literature for some of the above steps: the description of $K_{\mathbb{C}}$-orbits on $\mathcal{N}^*_{\mathfrak{p}}$ in terms of *ab*-diagrams due to H. Kraft-C. Procesi, T. Ohta and J. Schwartz for Step 1; the description of $K_{\mathbb{C}}$-orbit Q on the flag variety F by T. Matsuki – T. Oshima for a part of Step 3; and the description of the image of the moment map from the conormal bundle $T^*_Q F$ to $\mathfrak{g}^*$ (e.g. $G = U(p,q)$) by a recent work of A. Yamamoto for Step 4. As we mentioned in Sect. 7.1, Proposition 7.5 itself can be easily proved by Theorem 4.2 instead of Corollary 3.5 because Theorem 4.2 enables us to avoid a direct computation of associated varieties. The classification for the triplet $(G, H, \mathfrak{q})$ with (G, H) a classical symmetric pair which satisfies the equivalent conditions in Theorem 4.2 will be reported elsewhere.

8. Example 2 : the restriction to normal real forms

8.1. It is well-known that π is discretely decomposable as a $(\mathfrak{g}, K)$-module for any irreducible $(\widetilde{\mathfrak{g}}, \widetilde{K})$-module π if $(\widetilde{G}, G)$ is a Riemannian symmetric pair, namely if $G \simeq \widetilde{K}$. In this section we present an opposite extremal case:

Theorem 8.1. *Let $\widetilde{G}$ be a complex reductive linear Lie group, and G a normal real form of $\widetilde{G}$. Then, no irreducible infinite dimensional $(\widetilde{\mathfrak{g}}, \widetilde{K})$-module is discretely decomposable as a $(\mathfrak{g}, K)$-module.*

We recall that a closed subgroup G of a complex reductive linear Lie group $\widetilde{G}$ is a normal real form if $\mathfrak{g}_0$ is a real form of $\widetilde{\mathfrak{g}}_0$ satisfying $\mathbb{R}\text{-rank}G = \text{rank}G$. For example, $(\widetilde{G}, G) = (GL(n, \mathbb{C}), GL(n, \mathbb{R}))$, $(SO(2n+1, \mathbb{C}), SO(n+1, n))$, $(Sp(n, \mathbb{C}), Sp(n, \mathbb{R}))$, $(SO(2n, \mathbb{C}), SO(n, n))$ are the cases. Exceptional cases are given by $(\widetilde{\mathfrak{g}}_0, \mathfrak{g}_0) = (\mathfrak{e}_n, \mathfrak{e}_{n(n)})$ $(n = 6, 7, 8)$, $(\mathfrak{f}_4, \mathfrak{f}_{4(4)})$ and $(\mathfrak{g}_2, \mathfrak{g}_{2(2)})$ (see [He], Chapter X for notation).

The rest of this section will be devoted to the proof of Theorem 8.1.

8.2. Suppose we are in the setting of Theorem 8.1. Let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{k}} + \widetilde{\mathfrak{p}}$ be a complexified Cartan decomposition of $\widetilde{\mathfrak{g}}$ compatible with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. If X is a $(\widetilde{\mathfrak{g}}, \widetilde{K})$-module of finite length, then the associated variety $\mathcal{V}_{\widetilde{\mathfrak{g}}}(X)$ is a union of $\mathrm{Ad}(\widetilde{K_{\mathbb{C}}})$-orbits in $\mathcal{N}^*_{\widetilde{\mathfrak{g}}} \cap (\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{k}})^*$ (see Remark 3.6). Furthermore, if

$\dim X = \infty$ then $\mathscr{V}_{\tilde{\mathfrak{g}}}(X) \neq \{0\}$. Since $\mathscr{N}^*_{\tilde{\mathfrak{g}}}$ consists of nilpotent elements, Theorem 8.1 follows from Corollary 3.4 if one proves the following lemma:

Lemma 8.2. *If* $\mathcal{O}$ *is a non-zero* $\mathrm{Ad}(\widetilde{K_{\mathbb{C}}})$*-orbit on* $\mathscr{N}^*_{\tilde{\mathfrak{g}}} \cap (\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^*$ *then* $\mathrm{pr}_{\tilde{\mathfrak{g}}\to\mathfrak{g}}(\mathcal{O})$ $(\subset \mathfrak{g}^*)$ *contains a non-zero semisimple element.*

8.3. We will rewrite Lemma 8.2 by using the fact that $\widetilde{G}$ is a complex Lie group. In view of the isomorphism of complex vector spaces $\tilde{\mathfrak{g}} = Lie(\widetilde{G}) \otimes_{\mathbb{R}} \mathbb{C} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}} \simeq \mathfrak{g} + \mathfrak{g}$, we can identify

$$\begin{aligned} (\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^* &\simeq \tilde{\mathfrak{p}}^* \simeq \tilde{\mathfrak{p}} \simeq \mathfrak{g}, \\ (\mathfrak{g}/\mathfrak{k})^* &\simeq \mathfrak{p}^* \simeq \mathfrak{p}. \end{aligned}$$

Then, the projection $\mathrm{pr}_{\tilde{\mathfrak{g}}\to\mathfrak{g}} : \tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ restricted to $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^* \to (\mathfrak{g}/\mathfrak{k})^*$ is identified with

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p}, \qquad X \mapsto \frac{1}{2}(X - \theta X),$$

where θ is a Cartan involution of $\mathfrak{g}_0$ extended to a $\mathbb{C}$-linear endomorphism of $\mathfrak{g}$ as usual. We note that the above isomorphism induces a bijection between nilpotent orbits of the identity component of $\widetilde{K_{\mathbb{C}}}$ on $\mathscr{N}^*_{\tilde{\mathfrak{g}}} \cap (\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^*$ and nilpotent orbits of $\mathrm{Int}(\mathfrak{g})$ on $\mathscr{N}^*_{\mathfrak{g}}$. Hence, Lemma 8.2 is equivalent to the following lemma:

Lemma 8.3. *Suppose that* G *is a real reductive linear Lie group with* $\mathbb{R}\text{-rank}\, G = \mathrm{rank}\, G$. *If* $\mathcal{O}$ *is a non-zero nilpotent orbit of* $\mathrm{Int}(\mathfrak{g})$ *in* $\mathfrak{g}$, *then* $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}}(\mathcal{O})$ *contains a non-zero semisimple element.*

8.4. Before proving Lemma 8.3, we need:

Lemma 8.4. *Suppose that* G *is a real reductive linear Lie group with* $\mathbb{R}\text{-rank}\, G = \mathrm{rank}\, G$. *Then any nilpotent orbit* $\mathcal{O}$ *of* $\mathrm{Int}(\mathfrak{g})$ *in* $\mathfrak{g}$ *meets* $\mathfrak{g}_0$.

Proof. First, we recall the proof of the Dynkin-Kostant classification of complex nilpotent orbits. By the Jacobson-Morozov theorem, there exists an $\mathfrak{sl}_2$ triple $\{H, X, Y\}$ $(\subset \mathfrak{g})$ such that $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$ and that $\mathrm{Int}(\mathfrak{g}) \cdot X = \mathcal{O}$. We put

$$\begin{aligned} &\mathfrak{g}_0(H; \lambda) := \{Z \in \mathfrak{g}_0 : [H, Z] = \lambda Z\} \quad (\lambda \in \mathbb{C}), \\ &\mathfrak{g}(H; \lambda) := \{Z \in \mathfrak{g} : [H, Z] = \lambda Z\} \quad (\lambda \in \mathbb{C}), \\ &\mathscr{P} := \{Z \in \mathfrak{g}(H; 2) : \mathrm{Ker}(\mathrm{ad}\, Z : \mathfrak{g}(H; -2) \to \mathfrak{g}(H; 0)) = \{0\}\}, \\ &L := \text{ the identity component of } \{g \in \mathrm{Int}(\mathfrak{g}) : \mathrm{Ad}(g) H = H\}. \end{aligned}$$

We note that $\mathfrak{g}(H; \lambda) \neq \{0\}$ only if $\lambda \in \mathbb{Z}$ by the representation theory of $\mathfrak{sl}_2$. Because $\mathfrak{g}_0$ is a normal real form of $\mathfrak{g}$, we have

$$\mathfrak{g}(H;\lambda) = \mathfrak{g}_0(H;\lambda) \otimes_{\mathbb{R}} \mathbb{C}\ (\lambda \in \mathbb{C}). \tag{8.4.1}$$

A theorem of Malcev says that $\mathscr{P} = \mathrm{Ad}(L) \cdot X$ and that $\mathscr{P}$ is a Zariski open set of $\mathfrak{g}(H;2)$. It follows from (8.4.1) that $\mathscr{P} \cap \mathfrak{g}_0(H;2) \neq \emptyset$. Therefore, we find $X' \in \mathscr{P} \cap \mathfrak{g}_0(H;2)\ (\subset \mathfrak{g}_0)$ and $l \in L$ such that $\mathrm{Ad}(l)X = X'$. Thus, $\mathcal{O} \cap \mathfrak{g}_0 \neq \emptyset$. □

8.5. Now, let us complete the prove of Theorem 8.1.

All what we need is to show Lemma 8.3. In the setting of Lemma 8.3, we can take a non-zero element $X \in \mathcal{O} \cap \mathfrak{g}_0$ owing to Lemma 8.4. Then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}}(X) = \frac{1}{2}(X - \theta X) \in \mathfrak{p}_0$, which shows $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}}(X)$ is a semisimple element. On the other hand, $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}}(X) \neq 0$ because any element of $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathrm{Ker}(\mathrm{pr}_{\mathfrak{g}\to\mathfrak{p}} : \mathfrak{g} \to \mathfrak{p})$ is semisimple. This proves Lemma 8.3.

Hence, we have completed the proof of Theorem 8.1.

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