Symmetry breaking for orthogonal groups and a conjecture by B. Gross and D. Prasad

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Abstract

We consider **irreducible unitary representations** A_i of $G = SO(n+1,1)$ with the same infinitesimal character as the trivial representation and representations B_j of $H = SO(n,1)$ with the same properties and discuss H-equivariant homomorphisms $\mathrm{Hom}_H(A_i, B_j)$. For tempered representations our results confirm the predictions of conjectures by B. Gross and D. Prasad.

I Introduction

A representation Π of a group G defines a representation of a subgroup G' by restriction. In general irreducibility is not preserved by the restriction. If G is compact then the restriction $\Pi|_{G'}$ is isomorphic to a direct sum of irreducible finite-dimensional representations π of G' with multiplicities $m(\Pi, \pi)$. These multiplicities are studied by using combinatorial techniques. We are interested in the case where G and G' are (noncompact) real reductive Lie groups. Then most irreducible representations Π of G are infinite-dimensional, and generically the restriction $\Pi|_{G'}$ is not a direct sum of irreducible representations [9]. So we have to consider another notion of multiplicity.

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For a continuous representation Π of G on a complete, locally convex topological vector space $\mathcal{H}$, the space $\mathcal{H}^\infty$ of C^∞-vectors of $\mathcal{H}$ is naturally endowed with a Fréchet topology, and $(\Pi, \mathcal{H})$ induces a continuous representation Π^∞ of G on $\mathcal{H}^\infty$. If Π is an admissible representation of finite length on a Banach space $\mathcal{H}$, then the Fréchet representation $(\Pi^\infty, \mathcal{H}^\infty)$ depends only on the underlying $(\mathfrak{g}, K)$-module $\mathcal{H}_K$, sometimes referred to as an admissible representation of moderate growth [17, Chap. 11]. We shall work with these representations and write simply Π for Π^∞. Given another continuous representation π of moderate growth of a reductive subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi).$$

The dimension $m(\Pi, \pi)$ of this space yields important information of the restriction of Π to G' and is called the *multiplicity* of π occurring in the restriction $\Pi|_{G'}$. In general, $m(\Pi, \pi)$ may be infinite. The criterion in [11] asserts that the multiplicity $m(\Pi, \pi)$ is finite for all irreducible representations Π of G and all irreducible representations π of G' if and only if a minimal parabolic subgroup P' of G' has an open orbit on the real flag variety G/P, and that the multiplicity is uniformly bounded with respect to Π and π if and only if a Borel subgroup of $G'_{\mathbb{C}}$ has an open orbit on the complex flag variety of $G_{\mathbb{C}}$.

We consider in this article the case

$$(G, G') = (SO(n+1, 1), SO(n, 1)), \tag{1.1}$$

and discuss symmetry breaking between irreducible unitary representations of the groups G and G' with the same infinitesimal character ρ as the trivial one-dimensional representations.

We state our results first in Langlands parameters by identifying the representations with the Langlands subquotients of principal series representations induced from finite-dimensional representations of a maximal parabolic subgroup. Since these representations also have nontrivial $(\mathfrak{g}, \mathfrak{k})$-cohomology we can parametrize them by characters of the Levi of a θ-stable parabolic subalgebras and we proceed to state the results in this language. Then we describe the representations as members of Vogan packets and restate the results in this language. In the last section we relate our results to the Gross–Prasad conjectures for tempered representations.

Detailed proofs of the results will be published elsewhere.

II Classification of symmetry breaking operators

The main result of this section is a classification of symmetry breaking operators for principal series representations induced from exterior tensor representations for the pair $(G, G') = (SO(n+1,1), SO(n,1))$. Theorem II.1 extends the scalar case [12] and the case of differential operators [10], and will be used in Section III.

II.1 Notation for $SO(n+1,1)$

We first recall the notation from the Memoir article [11].

Consider the quadratic form

$$x_0^2 + x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \tag{2.2}$$

of signature $(n+1, 1)$. We define G to be the indefinite special orthogonal group $SO(n+1,1)$ that preserves the quadratic form (2.2) and the orientation. Let G' be the stabilizer of the vector $e_n = {}^t(0, 0, \cdots, 0, 1, 0)$. Then $G' \simeq SO(n,1)$. We set

$$\begin{aligned} K &:= O(n+2) \cap G = \{\begin{pmatrix} A & \\ & \det A \end{pmatrix} : A \in O(n+1)\} \simeq O(n+1), \\ K' &:= K \cap G' \quad = \{\begin{pmatrix} B & & \\ & 1 & \\ & & \det B \end{pmatrix} : B \in O(n)\} \simeq O(n). \end{aligned} \tag{2.3}$$

Then K and K' are maximal compact subgroups of G and G', respectively.

Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $\mathfrak{g}' = \mathfrak{so}(n,1)$ be the Lie algebras of G and G', respectively. We take a hyperbolic element H as

$$H := E_{0,n+1} + E_{n+1,0} \in \mathfrak{g}', \tag{2.4}$$

and set

$$\mathfrak{a} := \mathbb{R}H \qquad \text{and } A := \exp \mathfrak{a}.$$

Then the centralizers of H in G and G' are given by MA and $M'A$, respectively, where

$$M := \left\{ \begin{pmatrix} \varepsilon & & \\ & A & \\ & & \varepsilon \end{pmatrix} : A \in SO(n),\ \varepsilon = \pm 1 \right\} \qquad \simeq SO(n) \times O(1),$$

$$M' := \left\{ \begin{pmatrix} \varepsilon & & & \\ & B & & \\ & & 1 & \\ & & & \varepsilon \end{pmatrix} : B \in SO(n-1) : \varepsilon = \pm 1 \right\} \simeq SO(n-1) \times O(1).$$

We observe that $\mathrm{ad}(H) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ has eigenvalues -1, 0, and $+1$. Let

$$\mathfrak{g} = \mathfrak{n}_- + (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n}_+$$

be the corresponding eigenspace decomposition, and P a minimal parabolic subgroup with $P = MAN_+$ its Langlands decomposition. We remark that P is also a maximal parabolic subgroup of G. Likewise, $P' = M'AN'_+$ is a compatible Langlands decomposition of a minimal (also maximal) parabolic subgroup P' of G' given by

$$\mathfrak{p}' = \mathfrak{m}' + \mathfrak{a} + \mathfrak{n}'_+ = (\mathfrak{m} \cap \mathfrak{g}') + (\mathfrak{a} \cap \mathfrak{g}') + (\mathfrak{n}_+ + \mathfrak{g}'). \tag{2.5}$$

We note that we have chosen $H \in \mathfrak{g}'$ so that $P' = P \cap G'$ and we can take a common maximally split abelian subgroup A in P' and P.

II.2 Principal series representations of $SO(n+1,1)$

The character group of $O(1)$ consists of two characters. We write $+$ for the trivial character, and $-$ for the nontrivial character. Since $M \simeq SO(n) \times O(1)$, any irreducible representation of M is the outer tensor product of a representation (σ, V) of $SO(n)$ and a character δ of $O(1)$.

Given $(\sigma, V) \in \widehat{SO(n)}$, $\delta \in \{\pm\} \simeq \widehat{O(1)}$, and a character $e_\lambda(\exp(tH)) = e^{\lambda t}$ of A for $\lambda \in \mathbb{C}$, we define the (unnormalized) principal series representation

$$I_\delta(V, \lambda) = \mathrm{Ind}_P^G(V \otimes \delta, \lambda)$$

of $G = SO(n+1,1)$ on the Fréchet space of smooth maps $f \colon G \to V$ subject to

$$f(gmm'e^{tH}n) = \sigma(m)^{-1}\delta(m')e^{-\lambda t} f(g) \qquad \text{for all } g \in G,\ mm' \in M \simeq SO(n) \times O(1),\ t \in \mathbb{R},\ n \in N_+.$$

If V is the representation of $SO(n)$ on the exterior tensor space $\bigwedge^i(\mathbb{C}^n)$ ($2i \neq n$), we use the notation $I_\delta(i, \lambda)$ for $I_\delta(V, \lambda)$. Then the $SO(n)$-isomorphism on the exterior representations $\bigwedge^i(\mathbb{C}^n) \simeq \bigwedge^{n-i}(\mathbb{C}^n)$ leads us to the following G-isomorphism:

$$I_\delta(i, \lambda) \simeq I_\delta(n-i, \lambda).$$

If n is even and $n = 2i$, the exterior representation $\bigwedge^i(\mathbb{C}^n)$ splits into two irreducible representations of $SO(n)$:

$$\bigwedge^{\frac{n}{2}}(\mathbb{C}^n) \simeq \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_+ \oplus \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_-$$

with highest weights $(1, \cdots, 1, 1)$ and $(1, \cdots, 1, -1)$, respectively, with respect to a fixed positive system for $\mathfrak{so}(n, \mathbb{C})$. Accordingly, we have a direct sum decomposition of the induced representation:

$$\mathrm{Ind}_P^G(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n) \otimes \delta, \lambda) = I_\delta(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_+, \lambda) \oplus I_\delta(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_-, \lambda),$$

which we shall write as

$$I_\delta\left(\frac{n}{2}, \lambda\right) = I_\delta^{(+)}\left(\frac{n}{2}, \lambda\right) \oplus I_\delta^{(-)}\left(\frac{n}{2}, \lambda\right). \tag{2.6}$$

Via the Harish-Chandra isomorphism, the $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character of the trivial one-dimensional representation $\mathbf{1}$ is given by

$$\rho = \left(\frac{n}{2}, \frac{n}{2} - 1, \cdots, \frac{n}{2} - [\frac{n}{2}]\right)$$

in the standard coordinates of the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(n+2, \mathbb{C})$, whereas that of $I_\delta(i, \lambda)$ and $I_\delta^{(\pm)}(i, \lambda)$ (when $n = 2i$) is given by

$$\left(\frac{n}{2}, \frac{n}{2} - 1, \cdots, \frac{n}{2} - i + 1, \widehat{\frac{n}{2} - i}, \frac{n}{2} - i - 1, \cdots, \frac{n}{2} - [\frac{n}{2}], \lambda - \frac{n}{2}\right). \tag{2.7}$$

For the group $G' = SO(n, 1)$, we shall use the notation $J_\varepsilon(j, \nu)$ for the unnormalized parabolic induction $\mathrm{Ind}_{P'}^{G'}(\bigwedge^j(\mathbb{C}^{n-1}) \otimes \varepsilon, \nu)$ for $0 \leq j \leq n-1$, $\varepsilon \in \{\pm\}$, and $\nu \in \mathbb{C}$.

II.3 Classification of symmetry breaking operators

Let $(G, G') = (SO(n+1, 1), SO(n, 1))$ with $n \geq 3$. In this section we provide a complete classification of symmetry breaking operators from $I_\delta(i, \lambda)$ to $J_\varepsilon(j, \nu)$. The two recent articles [10] and [12] gave an explicit construction and the classification of symmetry breaking operators in the following settings.

(1) $i = j = 0$. The classification was accomplished in [12].

(2) *Differential* symmetry breaking operators for i, j general. The classification was accomplished in [10, Thm. 2.8].

The proof of our general case (Theorem II.1 below) relies partially on the results and techniques that are developed in [10, 12]. We note that the above literature treats the pair $(O(n+1,1), O(n,1))$, from which one can readily deduce the classification for the pair $(SO(n+1,1), SO(n,1))$ as we explained in [10, Chap. 2, Sec. 5].

For the admissible smooth representations $\Pi = I_\delta(i,\lambda)$ of $G = SO(n+1,1)$ and $\pi = J_\varepsilon(j,\nu)$ of $G' = SO(n,1)$, we set

$$m(i,j) \equiv m(I_\delta(i,\lambda), J_\varepsilon(j,\nu)) := \dim \operatorname{Hom}_{G'}(I_\delta(i,\lambda)|_{G'}, J_\varepsilon(j,\nu)).$$

In order to give a closed formula of $m(i,j)$ as a function of $(\lambda,\nu,\delta,\varepsilon)$, we introduce the following subsets of $\mathbb{Z}^2 \times \{\pm 1\}$:

$$\begin{aligned} L :=& \{(-i,-j,(-1)^{i+j}) : (i,j) \in \mathbb{Z}^2, 0 \le j \le i\}, \\ L' :=& \{(\lambda,\nu,\gamma) \in L : \nu \neq 0\}. \end{aligned}$$

To simplify the notation we also use will use ε, $\delta \in \{\pm\}$ and ε, $\delta \in \{\pm 1\}$.

In the theorem below, we shall see

$$\begin{aligned} m(i,j) \in& \{1,2,4\} && \text{if } j = i-1 \text{ or } i, \\ m(i,j) \in& \{0,1,2\} && \text{if } j = i-2 \text{ or } i+1, \\ m(i,j) =& 0 && \text{otherwise.} \end{aligned}$$

Here is an explicit formula of the multiplicity for the restriction of nonunitary principal series representations in this setting:

Theorem II.1. *Suppose $n \ge 3$, $0 \le i \le [\frac{n}{2}]$, $0 \le j \le [\frac{n-1}{2}]$, δ, $\varepsilon \in \{\pm\} \equiv \{\pm 1\}$, and $\lambda, \nu \in \mathbb{C}$. Let $\Pi = I_\delta(i,\lambda)$ and $\pi = J_\varepsilon(j,\nu)$ be the admissible representations of $G = SO(n+1,1)$ and $G' = SO(n,1)$, respectively, as before. Then we have the following.*

(1) *Suppose $j = i$.*

(a) *Case $i = 0$.*

$$m(0,0) = \begin{cases} 2 & \text{if } (\lambda,\nu,\delta\varepsilon) \in L, \\ 1 & \text{otherwise.} \end{cases}$$

(b) *Case* $1 \leq i < \frac{n}{2} - 1$.

$$m(i,i) = \begin{cases} 2 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(i,i,+)\}, \\ 1 & \textit{otherwise.} \end{cases}$$

(c) *Case* $i = \frac{n}{2} - 1$ *(n: even).*

$$m(\frac{n}{2}-1, \frac{n}{2}-1) = \begin{cases} 2 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(i,i,+)\} \cup \{(i,i+1,-)\}, \\ 1 & \textit{otherwise.} \end{cases}$$

(d) *Case* $i = \frac{n-1}{2}$ *(n: odd).*

$$m(\frac{n-1}{2}, \frac{n-1}{2}) = \begin{cases} 4 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(i,i,+)\}, \\ 2 & \textit{otherwise.} \end{cases}$$

(2) *Suppose* $j = i - 1$.

(a) *Case* $1 \leq i < \frac{n-1}{2}$.

$$m(i,i-1) = \begin{cases} 2 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(n-i,n-i,+)\}, \\ 1 & \textit{otherwise.} \end{cases}$$

(b) *Case* $i = \frac{n-1}{2}$ *(n: odd).*

$$m(\frac{n-1}{2}, \frac{n-3}{2}) = \begin{cases} 2 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(n-i,n-i,+)\} \cup \{(i,i+1,-)\}, \\ 1 & \textit{otherwise.} \end{cases}$$

(c) *Case* $i = \frac{n}{2}$ *(n: even).*

$$m(\frac{n}{2}, \frac{n}{2}-1) = \begin{cases} 4 & \textit{if } (\lambda, \nu, \delta\varepsilon) \in L' \cup \{(n-i,n-i,+)\}, \\ 2 & \textit{otherwise.} \end{cases}$$

(3) *Suppose* $j = i - 2$.

(a) *Case* $2 \leq i < \frac{n}{2}$.

$$m(i,i-2) = \begin{cases} 1 & \textit{if } (\lambda, \nu, \delta\varepsilon) = (n-i, n-i+1, -), \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *Case* $i = \frac{n}{2}$ *(n: even).*

$$m(\frac{n}{2}, \frac{n}{2} - 2) = \begin{cases} 2 & \text{if } (\lambda, \nu, \delta\varepsilon) = (\frac{n}{2}, \frac{n}{2} + 1, -), \\ 0 & \text{otherwise.} \end{cases}$$

(4) *Suppose* $j = i + 1$.

(a) *Case* $i = 0$.

$$m(0, 1) = \begin{cases} 1 & \text{if } \lambda \in -\mathbb{N}, \nu = 1, \text{ and } \delta\varepsilon = (-1)^{\lambda+1}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Case* $1 \leq i < \frac{n-3}{2}$.

$$m(i, i + 1) = \begin{cases} 1 & \text{if } (\lambda, \nu, \delta\varepsilon) = (i, i + 1, -), \\ 0 & \text{otherwise.} \end{cases}$$

(c) *Case* $i = \frac{n-3}{2}$ *(n: odd).*

$$m(\frac{n-3}{2}, \frac{n-1}{2}) = \begin{cases} 2 & \text{if } (\lambda, \nu, \delta\varepsilon) = (\frac{n-3}{2}, \frac{n-1}{2}, -), \\ 0 & \text{otherwise.} \end{cases}$$

(5) *Suppose* $j \notin \{i - 2, i - 1, i, i + 1\}$. *Then* $m(i, j) = 0$ *for all* $\lambda, \nu, \delta, \varepsilon$.

The construction of nontrivial symmetry breaking operators is proved by generalizing the techniques developed in [12] in the scalar case to the matrix-valued case for representations induced from finite-dimensional representations of M. The proof for the exhaustion of (continuous) symmetry breaking operators is built on the classification of differential symmetry breaking operators which was given in [10, Thm. 2.8].

Remark II.2 (multiplicity-one property)**.** *In [14] it is proved that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \leq 1$$

for any irreducible admissible smooth representations Π *and* π *of* $G = SO(n+1, 1)$ *and* $G' = SO(n, 1)$*, respectively. Thus Theorem II.1 fits well with their multiplicity-free results for* $\lambda, \nu \in \mathbb{C} \setminus \mathbb{Z}$*, where* $I_\delta(i, \lambda)$ *and* $J_\varepsilon(j, \nu)$ *are irreducible admissible representations of* G *and* G'*, respectively, except for the cases* $n = 2i$ *or* $n = 2j + 1$*. We note that, in addition to the subgroup* $G' = SO(n, 1)$*, the Lorentz group contains two other subgroups of index* *two, that is, $O^+(n,1)$ (containing orthochronous reflections) and $O^-(n,1)$ (containing anti-orthochronous reflections) with terminology in relativistic space-time for $n=3$. Our method gives also the multiplicity formula for such pairs, and it turns out that an analogous multiplicity-one statement fails if we replace $(G,G') = (SO(n+1,1), SO(n,1))$ by $(O^-(n+1,1), O^-(n,1))$. In fact, the multiplicity $m(\Pi,\pi)$ may equal 2 for irreducible representations Π and π of $O^-(n+1,1)$ and $O^-(n,1)$, respectively.*

III Main Results: Symmetry breaking for representations of rank one orthogonal groups

The main result in this section is a theorem about multiplicities for irreducible representations with trivial infinitesimal character ρ, namely, those representations that have the same $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character with the trivial one-dimensional representation. We first state the result using the Langlands parameters of the irreducible representations [2, 13]. In the second part we introduce θ-stable parabolic pairs $\mathfrak{q}, L$ and parametrize the representations by one-dimensional representations of L following [7, 8, 16]. We then state again the theorem in this formalism.

III.1 Irreducible representations with infinitesimal character ρ

In this section we give a description of all irreducible admissible representations of $G = SO(n+1,1)$ with trivial infinitesimal character ρ. Another description will be given in Section III.3.

By (2.7), $I_\delta(i,\lambda)$ has the $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character ρ if and only if $\lambda = i$ or $\lambda = n-i$. We identify the maximal compact subgroup K of G with $O(n+1)$ via the isomorphism (2.3). In what follows, we use the notation of [10, Chap. 2, Sect. 3] by adapting it to $SO(n+1,1)$ instead of $O(n+1,1)$. For $0 \le i \le n$, we denote by $I_\delta(i)^\flat$ and $I_\delta(i)^\sharp$ the unique irreducible subquotients of $I_\delta(i,i)$ containing the irreducible representations $\bigwedge^i(\mathbb{C}^{n+1}) \otimes \delta$ and $\bigwedge^{i+1}(\mathbb{C}^{n+1}) \otimes (-\delta)$ of $O(n+1) \simeq K$, respectively. In the case $n = 2i$, the $SO(n+1,1)$-modules $I_\delta^{(\pm)}(\frac{n}{2}, \frac{n}{2})$ are irreducible for $\delta = \pm$, and we have the following isomorphisms:

$$(3.8) \qquad I_\delta\left(\frac{n}{2}\right)^\flat \simeq I_\delta\left(\frac{n}{2}\right)^\sharp \simeq I_\delta^{(+)}\left(\frac{n}{2},\frac{n}{2}\right) \simeq I_\delta^{(-)}\left(\frac{n}{2},\frac{n}{2}\right) \qquad \text{for } \delta = \pm,$$

as representations of $SO(n+1,1)$.

Then we have G-isomorphisms:

$$I_\delta(i)^\sharp \simeq I_{-\delta}(i+1)^\flat \quad \text{for } 0 \le i \le n \text{ and } \delta \in \{\pm\}. \tag{3.9}$$

For $0 \le \ell \le n+1$ and $\delta \in \{\pm\}$, we set

$$\Pi_{\ell,\delta} := \begin{cases} I_\delta(\ell)^\flat & (0 \le \ell \le n), \\ I_{-\delta}(\ell-1)^\sharp & (1 \le \ell \le n+1). \end{cases} \tag{3.10}$$

In view of (3.9), $\Pi_{\ell,\delta}$ is well-defined.

For $0 \le i \le n$ with $n \ne 2i$ and $\delta \in \{\pm\}$, we have a nonsplitting exact sequence of G-modules:

$$0 \to \Pi_{i,\delta} \to I_\delta(i,i) \to \Pi_{i+1,-\delta} \to 0.$$

As we mentioned in (2.6), $I_\delta(\frac{n}{2},\frac{n}{2}) = \Pi_{\frac{n}{2},\delta} \oplus \Pi_{\frac{n}{2}+1,-\delta}$ when $n = 2i$.

The properties of irreducible representations $\Pi_{\ell,\delta}$ ($0 \le \ell \le n+1$, $\delta = \pm$) can be summarized as follows [2, 10].

Proposition III.1. *Let $G := SO(n+1,1)$ with $n \ge 1$.*

(1) $\Pi_{\ell,\delta} \simeq \Pi_{n+1-\ell,-\delta}$ *as G-modules for all $0 \le \ell \le n+1$ and $\delta = \pm$.*

(2) *Irreducible admissible representations of moderate growth with $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character ρ are classified as*

$$\begin{aligned} &\{\Pi_{\ell,\delta} : 0 \le \ell \le \frac{n-1}{2}, \delta = \pm\} \cup \{\Pi_{\frac{n+1}{2},+}\} && \textit{if } n \textit{ is odd,} \\ &\{\Pi_{\ell,\delta} : 0 \le \ell \le \frac{n}{2}, \delta = \pm\} && \textit{if } n \textit{ is even.} \end{aligned}$$

(3) *Every $\Pi_{\ell,\delta}$ is unitarizable.*

By abuse of notation, we use the same symbol $\Pi_{l,\delta}$ to denote the unitarization.

(4) *For n odd, $\Pi_{\frac{n+1}{2},+}$ is a discrete series. For n even, $\Pi_{\frac{n}{2},\pm}$ are tempered representations. All the other representations in the list (2) are nontempered.*

(5) *For n even, the center of G acts nontrivially on $\Pi_{\ell,\delta}$ if and only if $\delta = (-1)^{\ell+1}$. For n odd, the center of G is trivial, and thus acts trivially on $\Pi_{\ell,\delta}$ for any ℓ and δ.*

For the subgroup $G' = SO(n,1)$, we shall use similar notation $\pi_{j,\varepsilon}$ for the subrepresentations of $J_\varepsilon(j,j)$ (or the quotients of $J_\varepsilon(j-1,j-1)$).

In view of Proposition III.1, in particular, the G-isomorphism $\Pi_{\frac{n+1}{2},+} \simeq \Pi_{\frac{n+1}{2},-}$ for n odd and the G'-isomorphism $\pi_{\frac{n}{2},+} \simeq \pi_{\frac{n}{2},-}$ for n even, we shall use the following convention:

1. if $n+1 = 2i$; we identify $\delta = +$ and $\delta = -$
2. if $n = 2j$ we identify $\varepsilon = +$ and $\varepsilon = -$

in statements and theorems about representations $\Pi_{i,\delta}$ and $\pi_{j,\varepsilon}$ with indices $(0 \le i \le [\frac{n+1}{2}])$ and $(0 \le j \le [\frac{n}{2}])$.

III.2 Formulation I of the main theorem

As we saw in Proposition III.1, all the representations $\Pi_{i,\delta}$ are unitarizable, but the restriction of $\Pi_{i,\delta}$ to the subgroup G' does not decompose into a direct sum of irreducible representations [9]. Hence to obtain information about the restriction we consider G'-intertwining operators (*symmetry breaking operators*) for smooth admissible representations:

$$\mathrm{Hom}_{G'}(\Pi_{i,\delta}|_{G'}, \pi_{j,\varepsilon}). \tag{3.11}$$

By using the classification of all symmetry breaking operators for principal series representations (Theorem II.1) and by analyzing their restrictions to the subquotients of principal series representations, we can determine the dimension of the space (3.11), and, in particular, we obtain a necessary and sufficient condition for this space to be nonzero. Here is a statement.

Theorem III.2. *Let* $(G,G') = (SO(n+1,1), SO(n,1))$*. Suppose* $0 \le i \le [\frac{n+1}{2}]$, $0 \le j \le [\frac{n}{2}]$*, and* δ, $\varepsilon = \pm$ *with the convention* (III.1)*. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi_{i,\delta}|_{G'}, \pi_{j,\varepsilon}) = \begin{cases} 1 & \text{if } \delta = \varepsilon \text{ and } j \in \{i-1, i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem III.2 can be rephrased as follows.

Theorem III.3. *Suppose* $0 \le i \le [\frac{n+1}{2}]$, $0 \le j \le [\frac{n}{2}]$*, and* $\delta, \varepsilon = \pm$*. Then the following three conditions on the quadruple* (i,j,δ,ε) *are equivalent.*

(i)
$$\mathrm{Hom}_{G'}(\Pi_{i,\delta}|_{G'}, \pi_{j,\varepsilon}) \neq \{0\}.$$

(ii)
$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\Pi_{i,\delta}|_{G'}, \pi_{j,\varepsilon}) = 1.$$

(iii) *There is an arrow connecting the representations in the following tables with $\delta = \varepsilon$. (For simplicity, we omit the subscripts δ and ε in the tables below.)*

In (iii), the convention (III.1) *is applied to the cases $j = m$ when $n = 2m$ (see Table 1) and $i = m+1$ when $n = 2m+1$ (see Table 2), where $\pi_{m,+} \simeq \pi_{m,-}$ and $\Pi_{m+1,+} \simeq \Pi_{m+1,-}$ hold, respectively.*

Table 1: Symmetry breaking for $(SO(2m+1,1), SO(2m,1))$

$$\begin{array}{ccccccccccc}
\Pi_0 & & \Pi_1 & & \cdots & \cdots & \cdots & \Pi_{m-1} & & \Pi_m \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & & & \downarrow & \swarrow & \downarrow \\
\pi_0 & & \pi_1 & & \cdots & \cdots & \cdots & \pi_{m-1} & & \pi_m
\end{array}$$

Table 2: Symmetry breaking for $(SO(2m+2,1), SO(2m+1,1))$

$$\begin{array}{cccccccccccc}
\Pi_0 & & \Pi_1 & & \cdots & \cdots & \cdots & \Pi_{m-1} & & \Pi_m & \Pi_{m+1} \\
\downarrow & \swarrow & \downarrow & \swarrow & \cdots & & & \downarrow & \swarrow & \downarrow & \swarrow \\
\pi_0 & & \pi_1 & & \cdots & \cdots & \cdots & \pi_{m-1} & & \pi_m &
\end{array}$$

We note that the equivalence (i) $\Leftrightarrow$ (ii) in Theorem III.3 could be derived also from the general theory [14] because $\Pi_{i,\delta}$ and $\pi_{j,\varepsilon}$ are irreducible.

III.3 Formulation II of the main theorem

We have described the irreducible representations $\Pi_{\ell,\delta}$ as subquotients of principal series representations in Section III.1. The next proposition provides another characterization of the same representations $\Pi_{\ell,\delta}$.

Proposition III.4. *The irreducible representations $\Pi_{\ell,\delta}$ in Proposition III.1 (3) are the Casselman–Wallach globalization of the irreducible, unitarizable $(\mathfrak{g}, K)$-modules with nonzero $(\mathfrak{g}, \mathfrak{k})$-cohomologies.*

These $(\mathfrak{g}, K)$-modules can be described by using the Zuckerman derived functor modules. For this, let us introduce some notation. For $0 \leq i \leq [\frac{n+1}{2}]$, we consider a θ-stable parabolic subalgebra $\mathfrak{q}_i$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(n+1,1) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{so}(n+2, \mathbb{C})$ with the (real) Levi subgroup

$$L_i := N_G(\mathfrak{q}_i) \simeq SO(2)^i \times SO(n+1-2i, 1).$$

We note that L_i meets all the connected components of $G = SO(n+1,1)$. For the trivial one-dimensional representation $\mathbf{1}$ of the first factor $SO(2)^i$ and a one-dimensional representation χ of the last factor $SO(n+1-2i,1)$, we define a $(\mathfrak{g}, K)$-module

$$A_{\mathfrak{q}_i}(\chi) := \mathcal{R}_{\mathfrak{q}_i}^{S_i}(\mathbf{1} \boxtimes \chi)$$

as the cohomological parabolic induction from the one-dimensional representation $\mathbf{1} \boxtimes \chi$ of L. We adopt a 'ρ-shift' of the cohomological parabolic induction in a way that $A_{\mathfrak{q}_i}(\chi)$ has the infinitesimal character ρ if $d\chi = 0$. (The $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}_i}(0)$ in the notation of Vogan–Zuckerman [11] corresponds to $A_{\mathfrak{q}_i}(\mathbf{1})$ in our notation.) There are two characters χ of $SO(k,1)$ $(k \geq 1)$ such that $d\chi = 0$. We write $\chi_{k,1}^{+}$ for the trivial one, and $\chi_{k,1}^{-}$ for the nontrivial one. Then we have

Proposition III.5 ([7, 8])**.** *Suppose* $0 \leq i \leq [\frac{n+1}{2}]$*. For* $\varepsilon = \pm$*,*

$$(\Pi_{i,\varepsilon})_K \simeq A_{\mathfrak{q}_i}(\chi_{n+1-2i,1}^{\varepsilon}).$$

Remark III.6. We may regard $\chi_{0,1}^{-} \simeq \chi_{0,1}^{+}$ for the representation of $SO(0,1) = \{1\}$. When n is odd and $i = \frac{n+1}{2}$, $L \simeq SO(2)^{\frac{n+1}{2}} \times SO(0,1)$. This matches the $(\mathfrak{g}, K)$-isomorphism: $\Pi_{\frac{n+1}{2},+} \simeq \Pi_{\frac{n+1}{2},-}$ (see Proposition III.1 (1)).

Example III.7 (see Proposition III.1)**.**

(1) $\Pi_{0,\varepsilon}$ is one-dimensional.

(2) If $n = 2m$ then $\Pi_{m,+}$ and $\Pi_{m,-}$ are the inequivalent tempered principal series representations of $SO(2m+1,1)$ with infinitesimal character ρ.

(3) If $n = 2m-1$ then $\Pi_{m,+} \simeq \Pi_{m,-}$ is the unique discrete series representation of $SO(2m,1)$ with infinitesimal character ρ.

III.4 θ-stable parameter of $\Pi_{i,\delta}$

Suppose $0 \leq i \leq [\frac{n+1}{2}]$. Let Σ_i^+ be the set of positive roots corresponding to the nilpotent radical of the θ-stable parabolic subalgebra $\mathfrak{q}_i$ and define

$$\rho_i = \frac{1}{2} \sum_{\alpha \in \Sigma_i^+} \alpha.$$

Via the standard basis of the fundamental Cartan subalgebra, we have

$$\begin{aligned}
\rho_i &= (m, m-1, \dots, m-i+1, 0 \dots 0) && \text{if } G = SO(2m+1,1), \\
\rho_i &= (m-\frac{1}{2}, m-\frac{3}{2}, \dots, m-i-\frac{1}{2}, 0, \dots 0) && \text{if } G = SO(2m,1).
\end{aligned}$$

To make our notation consistent with the Harish-Chandra parameter for discrete series representations for $SO(2m,1)$ we define the *θ-stable parameters* of the cohomologically induced representation $(\Pi_{i,\delta})_K \simeq A_{\mathfrak{q}_i}(\chi^\delta_{n-2i+1,1})$ as follows.

Definition III.8. *Suppose $0 \leq i \leq m$ and $\delta \in \{\pm\}$.*

(1) *The θ-stable parameter of the irreducible representation $\Pi_{i,\delta}$ of $SO(2m+1,1)$ is*

$$(m, m-1, \dots, m-i+1 \mid\mid \chi^\delta_{2m-2i+1,1}),$$

where $\chi^\delta_{2m-2i+1,1}$ is the one-dimensional representation of $SO(2m-2i+1,1)$.

(2) *The θ-stable parameter of the irreducible representation $\Pi_{i,\delta}$ of $SO(2m,1)$ is*

$$(m-\frac{1}{2}, m-\frac{3}{2}, \dots, m-i+\frac{1}{2} \mid\mid \chi^\delta_{2m-2i,1}),$$

where $\chi^\delta_{2m-2i,1}$ is the one-dimensional representation of $SO(2m-2i,1)$.

We use the same convention for the representations $\pi_{j,\varepsilon}$ of G'.

Theorem III.3 can now be restated in a formulation resembling the classical branching law for finite-dimensional representations. We connect the parameter by an arrow $\Downarrow$ pointing towards the parameter of the representation of the smaller group.

Theorem III.9. *Suppose that* $(G, G') = (SO(n+1,1), SO(n,1))$*. Let* Π *and* π *be irreducible admissible representations of moderate growth of* G *and* G' *with* $\mathfrak{Z}(\mathfrak{g})$*-infinitesimal character* ρ*, respectively.*

(1) *Suppose* $n = 2m$*. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\Pi|_{G'}, \pi) = 1$$

if and only if the θ*-stable parameters of* Π *and* π *satisfy one of the following conditions:*
$(\Pi, \pi) = (\Pi_{i,\delta}, \pi_{i,\varepsilon})$ *for* $0 \leq i \leq m$ *with* $\delta = \varepsilon \in \{\pm\}$ *(the convention* (III.1) *is applied to* ε *when* $i = m$*):*

$$(m, m-1, \ldots m+1-i \,||\, \chi^{\delta}_{2m+1-2i,1})$$
$$\Downarrow$$
$$(m - \tfrac{1}{2}, m - \tfrac{3}{2}, \ldots, m + \tfrac{1}{2} - i \,||\, \chi^{\varepsilon}_{2m-2i,1})$$

or $(\Pi, \pi) = (\Pi_{i,\delta}, \pi_{i-1,\varepsilon})$ *for* $0 < i \leq m$ *with* $\delta = \varepsilon \in \{\pm\}$*:*

$$(m, m-1, \ldots m+1-i \,||\, \chi^{\delta}_{2m+1-2i,1})$$
$$\Downarrow$$
$$(m - \tfrac{1}{2}, m - \tfrac{3}{2}, \ldots, m + \tfrac{3}{2} - i \,||\, \chi^{\varepsilon}_{2m+2-2i,1}).$$

(2) *Suppose* $n = 2m+1$*. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\Pi|_{G'}, \pi) = 1$$

if and only if the θ*-stable parameters of* Π *and* π *satisfy one of the following conditions:*
$(\Pi, \pi) = (\Pi_{i,\delta}, \pi_{i,\varepsilon})$ *for* $0 \leq i < m+1$ *with* δ*,* $\varepsilon \in \{\pm\}$*:*

$$(m + \tfrac{1}{2}, m - \tfrac{1}{2}, \ldots, m + \tfrac{3}{2} - i \,||\, \chi^{\delta}_{2m+2-2i,1})$$
$$\Downarrow$$
$$(m, m-1, \ldots, m+1-i \,||\, \chi^{\varepsilon}_{2m+1-2i,1})$$

or $(\Pi, \pi) = (\Pi_{i,\delta}, \pi_{i-1,\varepsilon})$ *for* $0 < i \leq m+1$ *with* $\delta = \varepsilon \in \{\pm\}$ *(the convention* (III.1) *is applied to* δ *when* $i = m+1$*):*

$$(m + \tfrac{1}{2}, m - \tfrac{1}{2}, \ldots, m + \tfrac{3}{2} - i \,||\, \chi^{\delta}_{2m+2-2i,1})$$
$$\Downarrow$$
$$(m, m-1, \ldots, m+2-i \,||\, \chi^{\varepsilon}_{2m+3-2i,1}).$$

Remark III.10. *The first case represents the vertical arrows and the second case represents the slanted arrow in Theorem III.3 (iii).*

IV Symmetry breaking and the Gross–Prasad conjectures

In 2000 B. Gross and N. Wallach [6] showed that the restriction of *small* discrete series representations of $G = SO(2p+1, 2q)$ to $G' = SO(2p, 2q)$ satisfies the Gross–Prasad conjectures [5]. In that case, both the groups G and G' admit discrete series representations. On the other hand, for the pair $(G, G') = (SO(n+1, 1), SO(n, 1))$, only one of G or G' admits discrete series representations. We sketch here a proof that our theorem III.2 confirms the Gross–Prasad conjectures also for **tempered** representations with infinitesimal character ρ.

In our formulation and the exposition we rely on the original article by B. Gross and D. Prasad [5] and also on [3].

The following diagram recalls our results in the previous sections about symmetry breaking operators for tempered representations with infinitesimal character ρ of the groups $SO(n+1, 1)$ for $n = 2m$, $2m$, and $2m-1$. We denote the corresponding representations by Π, π and ϖ, respectively, using the subscripts defined in Section III. For $n = 2m$, we simply write π_m for $\pi_{m,\varepsilon}$ because $\pi_{m,+} \simeq \pi_{m,-}$ as $SO(2m, 1)$-modules.

Table 3: Symmetry breaking for $SO(2m+1, 1) \supset SO(2m, 1) \supset SO(2m-1, 1)$

$$
\begin{array}{ccccc}
\Pi_{m,-} & & & & \varpi_{m-1,-} \\
 & \searrow & & \nearrow & \\
 & & \pi_m & & \\
 & \nearrow & & \searrow & \\
\Pi_{m,+} & & & & \varpi_{m-1,+}
\end{array}
$$

We will in the following only consider representations which are nontrivial on the center (see Proposition III.1 (5)) and thus are genuine representations of the orthogonal groups. So we are considering in our discussion of the Gross–Prasad conjectures only this part of the diagram.

$$\Pi_{m,(-1)^{m+1}} \quad \to \quad \pi_m \quad \to \quad \varpi_{m-1,(-1)^m}$$

The other remaining cases can be handled by using the same ideas.

We first sketch the results about Vogan packets for special orthogonal groups. The Vogan L-packet is the disjoint union of Langlands L-packet over *pure* inner forms. We refer to [1] and [15] for general information about Vogan packets and to [3] for details for special orthogonal groups.

Consider the complexification $SO(n+1,\mathbb{C})$ of a special orthogonal group $SO(n,1)$ and let $T_{\mathbb{C}} \subset SO(n+1,\mathbb{C})$ be the complexification of a fundamental Cartan subgroup T of $SO(n+1,1)$.

IV.1 Vogan packets of discrete series representations with infinitesimal character ρ of odd special orthogonal groups

We begin with the case $n = 2m-1$. In this case $SO(n+1,1) = SO(2m,1)$ has discrete series representations. We fix a set of positive roots $\Delta^+ \subset \mathfrak{t}_{\mathbb{C}}^*$ for the root system $\Delta(\mathfrak{so}(2m+1,\mathbb{C}),\mathfrak{t}_{\mathbb{C}})$, and denote by ρ half the sum of positive roots as before. For $l+k = 2m+1$, we call a real form $SO(l,k)$ *pure* if l is even. The Vogan packet containing the discrete series representation π_m is the disjoint union of discrete series representations with infinitesimal character ρ of the pure inner forms. The cardinality of this packet is

$$2^m = \sum_{\substack{0 \leq l \leq 2m \\ l\text{:even}}} \binom{m}{\frac{l}{2}}.$$

There exists a finite group $\mathcal{A}_2 \simeq (\mathbb{Z}_2)^m$ whose characters parametrize the representations in the Vogan packet. For the discrete series representation with parameter $\chi \in \widehat{\mathcal{A}_2}$ we write $\pi(\chi)$. For more details see [4] or [15]. We write $VP(\pi_m)$ for the Vogan packet containing π_m.

Example IV.1. (1) The trivial representation $\mathbf{1}$ of $SO(0,2m+1)$ is in $VP(\pi_m)$.

(2) We can define similarly a Vogan packet containing $(SO(1,2m),\pi_m)$.

By abuse of notation we may also consider π_m as a discrete series representation of $SO(1,2m)$, but the pairs $(SO(1,2m),\pi_m)$ and $(SO(2m,1),\pi_m)$ are not in the same Vogan packet.

Remark IV.2. Analogous results hold for the infinitesimal character $\lambda+\rho$ where λ is the highest weight of a finite-dimensional representation.

IV.2 A Vogan packet of tempered induced representations with infinitesimal character ρ

Next we consider the case $n = 2m$. Then $SO(n+1, 1) = SO(2m+1, 1)$ has no discrete series representation and we consider instead a Vogan packet of tempered representations with infinitesimal character ρ which contains the pair $(SO(2m + 1, 1), \Pi_{m,\delta})$ with $\delta = (-1)^{m+1}$.

To simplify the notation we assume in this subsection that $\delta = (-1)^{m+1}$. Recall that $\Pi_{m,\delta}$ denotes the irreducible representation $I_\delta(\bigwedge^m(\mathbb{C}^{2m})_+, m) \simeq I_\delta(\bigwedge^m(\mathbb{C}^{2m})_-, m)$ which is the smooth representation of a unitarily induced principal series representation from the maximal parabolic subgroup. Its Langlands/Vogan parameter factors through the Levi subgroup of a maximal parabolic subgroup of the Langlands dual group ${}^LG$ [13]. This parabolic subgroup corresponds to a maximal parabolic subgroup of $SO(2m + 1, 1)$ whose Levi subgroup L is a real form of $SO(2m, \mathbb{C}) \times SO(2, \mathbb{C})$ and thus is isomorphic to $SO(2m, 0) \times SO(1, 1)$. Note that $SO(1, 1)$ is a disconnected group.

By [3, p. 35] there are 2^m representations in the Vogan packet containing $\Pi_{m,\delta}$ and they are parametrized by the characters of a finite group $\mathcal{A}_1 \simeq (\mathbb{Z}_2)^m$. We write $VP(\Pi_{m,\delta})$ for this Vogan packet.

We can describe the representations in the Vogan packet $VP(\Pi_{m,\delta})$ as follows: for $l+k = 2m+2$, a real form $SO(l, k)$ is called *pure* if l is odd. The Levi subgroups of parabolic subgroups in the same Vogan packet are isomorphic to $L = SO(2m - 2p, 2p) \times SO(1, 1)$. Principal series representations, which are induced from the outer tensor product of a discrete series representation with the same infinitesimal character as $\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_+$ (or $\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_-$) and a one-dimensional representation of $SO(1, 1)$, are irreducible. These induced representations are in $VP(\Pi_{m,\delta})$ if they have the same central character as $\Pi_{m,\delta}$ [15].

Remark IV.3. (1) *The Vogan packet containing* $(SO(1 + 2m, 1), \Pi_{m,\delta})$*,* $\delta = (-1)^{m+1}$ *does contain neither the pair* $(SO(2m + 2, 0)$*, finite-dimensional representation) nor* $(SO(0, 2m + 2)$*, finite-dimensional representation).*

(2) *If* $(SO(1 + 2m, 1), \Pi)$ *is in* $VP(\Pi_{m,\delta})$*, then* $\Pi = \Pi_{m,\delta}$*.*

(3) *By abuse of notation we say that the Vogan packet containing* $(SO(1+2m, 1), \Pi_{m,\delta})$ *also contains* $(SO(2m + 1, 1), \Pi_{m,\delta})$*.*

(4) *Using the same considerations for* $m - 1$ *and* $\delta = (-1)^m$ *we obtain a Vogan packet* $VP(\varpi_{m-1,\delta})$ *which contains the pair* $(SO(2m -$

$1,1), \varpi_{m-1,(-1)^m})$.

IV.3 Gross–Prasad conjecture I: Symmetry breaking from $\Pi_{m,-}$ to the discrete series representation π_m

We consider the Vogan packet of tempered representations of $SO(2m+1,1)\times SO(2m,1)$ which contains the pair $(SO(2m+1,1)\times SO(2m,1), \Pi_{m,\delta}\boxtimes\pi_m)$, or the Vogan packet which contains the pair $(SO(1,1+2m)\times SO(1,2m), \Pi_{m,\delta}\boxtimes \pi_m)$. The representations in these packets are parametrized by characters of

$$\mathcal{A}_1 \times \mathcal{A}_2 \simeq (\mathbb{Z}_2)^m \times (\mathbb{Z}_2)^m \simeq (\mathbb{Z}_2)^{2m}.$$

B. Gross and D. Prasad propose an algorithm which determines a pair $\chi_1 \in \widehat{\mathcal{A}_1}, \chi_2 \in \widehat{\mathcal{A}_2}$ hence representations

$$(\Pi(\chi_1), \pi(\chi_2)) \in VP(\Pi_{m,\delta}) \times VP(\pi_m)$$

so that

$$\operatorname{Hom}_{G(\chi_2)}(\Pi(\chi_1)|_{G(\chi_2)}, \pi(\chi_2)) \neq \{0\},$$

where $G(\chi_2)$ is the pure inner form determined by χ_2.

Let $T_{\mathbb{C}}$ be a torus in $SO(2m+2,\mathbb{C}) \times SO(2m+1,\mathbb{C})$, and $X^*(T_{\mathbb{C}})$ the character group. Fix basis

$$X^*(T_{\mathbb{C}}) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_{m+1} \oplus \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \cdots \oplus \mathbb{Z}f_m$$

such that the standard root basis Δ_0 is given by

$$e_1-e_2, e_2-e_3, \ldots, e_m - e_{m+1}, e_m + e_{m+1}, f_1-f_2, f_2-f_3, \ldots, f_{m-1}-f_m, f_m$$

if $m \geq 1$.

We fix as before $\delta = (-1)^{m+1}$. We can identify the Langlands parameter of the Vogan packet containing

$$(SO(2m+1,1) \times SO(2m,1), \Pi_{m,\delta} \boxtimes \pi_m)$$

with

$$me_1 + (m-1)e_2 + \cdots + e_m + 0e_{m+1} + (m-\frac{1}{2})f_1 + (m-\frac{3}{2})f_2 + \cdots + \frac{1}{2}f_m.$$

Let δ_i be the element which is -1 in the ith factor of $\mathcal{A}_1$ and equal to 1 everywhere else and ε_j the element which is -1 in the jth factor of $\mathcal{A}_2$ and 1 everywhere else.

Then the algorithm [5, p. 993] determines $\chi_1 \in \widehat{\mathcal{A}_1}$ and $\chi_2 \in \widehat{\mathcal{A}_2}$ by

$$\chi_1(\delta_i) = (-1)^{\#m-i+1>} \quad \text{ and } \chi_2(\varepsilon_j) = (-1)^{\#m-j+\frac{1}{2}<},$$

where $\#m-i+1 >$ is the cardinality of the set $\{j : m-i+1 > \text{the coefficients of } f_j\}$, and $\#m-j+\frac{1}{2} <$ is the cardinality of the set $\{i : m-j+\frac{1}{2} < \text{the coefficients of } e_i\}$.

We normalize the quasi-split form by

$$\begin{aligned} G^o &= SO(m+1, m+1) \times SO(m, m+1) \quad &&\text{if } m \text{ is even,} \\ G^o &= SO(m+2, m) \times SO(m+1, m) &&\text{if } m \text{ is odd.} \end{aligned}$$

Applying the formulæ in [5, (12.21)] we define the integers p and q with $0 \leq p \leq m$ and $0 \leq q \leq m$ by

$$p = \#\{i : \chi_1(\delta_i) = (-1)^i\} \quad \text{ and } \quad q = \#\{j : \chi_2(\varepsilon_j) = (-1)^{m+j}\}$$

and we get the pure forms

$$G = SO(2m-2p+1, 2p+1) \times SO(2q, 2m-2q+1) \quad \text{if } m \text{ is even,}$$

$$G = SO(2p+1, 2m-2p+1) \times SO(2m-2q, 2q+1) \quad \text{if } m \text{ is odd.}$$

In our setting, we get the pair of integers $(p, q) = (0, m)$ for m even; $(p, q) = (m, 0)$ for m odd. Applying [5, (12.22)] with correction by changing n by m *loc.cit.*, we deduce that this character defines the pure inner form

- $G = SO(2m+1, 1) \times SO(2m, 1)$ if m is even,
- $G = SO(2m+1, 1) \times SO(2m, 1)$ if m is odd.

The only representation in $VP(\Pi_{m,\delta}) \times VP(\pi_m)$ with this pair of pure inner forms is $\Pi_{m,\delta} \times \pi_m$. Hence Theorem III.3 implies the following.

Conclusion: *The result*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{SO(2m,1)}(\Pi_{m,(-1)^{m+1}}|_{SO(2m,1)}, \pi_m) = 1$$

confirms the conjectures by B. Gross and D. Prasad [5].

IV.4 Gross–Prasad conjecture II: Symmetry breaking from the discrete series representation π_m to $\varpi_{m-1,(-1)^m}$

We consider the Vogan packet of tempered representations containing the pair $(SO(2m,1)\times SO(2m-1,1), \pi_m \boxtimes \varpi_{m-1,(-1)^m})$, *i.e.*, the Vogan packet

$$VP(\pi_m \boxtimes \varpi_{m-1,(-1)^m}) \subset VP(\pi_m) \times VP(\varpi_{m-1,(-1)^m}).$$

The packet $VP(\pi_m)\times VP(\varpi_{m,(-1)^m})$ is parametrized by characters of the finite group

$$\mathcal{A}_2 \times \mathcal{A}_3 \simeq (\mathbb{Z}_2)^m \times (\mathbb{Z}_2)^{m-1} \simeq (\mathbb{Z}_2)^{2m-1}.$$

As in Section IV.3 we use the algorithm by B. Gross and D. Prasad to determine a pair $\chi_2 \in \widehat{\mathcal{A}_2}, \chi_3 \in \widehat{\mathcal{A}_3}$ and hence representations

$$(\pi(\chi_2), \varpi(\chi_3)) \in VP(\pi_m) \times VP(\varpi_{m-1,(-1)^m})$$

so that

$$\mathrm{Hom}_{G(\chi_3)}(\pi(\chi_2)|_{G(\chi_3)}, \varpi(\chi_3)) \neq \{0\},$$

where $G(\chi_3)$ is the pure inner form determined by χ_3.

Let $T_{\mathbb{C}}$ be a torus in $SO(2m+1,\mathbb{C})\times SO(2m,\mathbb{C})$ and $X^*(T_{\mathbb{C}})$ the character group. Fix basis

$$X^*(T_{\mathbb{C}}) = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2 \oplus \cdots \oplus \mathbb{Z}f_m \oplus \mathbb{Z}g_1 \oplus \mathbb{Z}g_2 \oplus \cdots \oplus \mathbb{Z}g_m$$

such that the standard root basis Δ_0 is given by

$$f_1-f_2, f_2-f_3, \ldots, f_{m-1}-f_m, f_m, g_1-g_2, g_2-g_3, \ldots, g_{m-1}-g_m, g_{m-1}+g_m$$

for $m \geq 2$.

Fix as before $\epsilon = (-1)^m$. We identify the Langlands parameter of the Vogan packet

$$VP(\pi_m) \times VP(\varpi_{m,\epsilon})$$

with

$$(m-\frac{1}{2})f_1+(m-\frac{3}{2})f_2+\cdots+\frac{1}{2}f_m+(m-1)g_1+(m-2)g_2+\cdots+g_{m-1}+0g_m.$$

Again applying [5, Prop. 12.18] we define characters $\chi_2 \in \widehat{\mathcal{A}_2}$, $\chi_3 \in \widehat{\mathcal{A}_3}$ as follows: Let $\varepsilon_j \in \mathcal{A}_2 \simeq (\mathbb{Z}_2)^m$ be the element which is -1 in the jth factor and equal to 1 everywhere else as in Section IV.3; $\gamma_k \in \mathcal{A}_3 \simeq (\mathbb{Z}_2)^{m-1}$ the element which is -1 in the kth factor and 1 everywhere else. Then $\chi_2 \in \widehat{\mathcal{A}_2}$ and $\chi_3 \in \widehat{\mathcal{A}_3}$ are determined by

$$\chi_2(\varepsilon_j) = (-1)^{\#m-j+1/2<} \quad \text{and } \chi_3(\gamma_k) = (-1)^{\#m-k>}$$

where $\#m - j + \frac{1}{2} <$ is the cardinality of the set $\{k : m - j + \frac{1}{2} <$ the coefficients of $g_k\}$, and $\#m-k >$ is the cardinality of the set $\{j : m-k >$ the coefficients of $f_j\}$.

We normalize the quasi-split form by

$$\begin{aligned} G^o &= SO(m+1, m) \times SO(m+1, m-1) && \text{if } m \text{ is even,} \\ G^o &= SO(m, m+1) \times SO(m, m) && \text{if } m \text{ is odd.} \end{aligned}$$

Applying the formulæ in [5, (12.21)] we define the integers p and q with $0 \le p \le m$ and $0 \le q \le m-1$ by

$$p = \#\{j : \chi_2(\varepsilon_j) = (-1)^j\} \quad \text{and} \quad q = \#\{k : \chi_3(\gamma_k) = (-1)^{m+k}\},$$

and we get

$$G = SO(2m-2p+1, 2p) \times SO(2q+1, 2m-2q-1) \quad \text{if } m \text{ is even,}$$

$$G = SO(2p+1, 2m-2p) \times SO(2m-2q-1, 2q+1) \quad \text{if } m \text{ is odd.}$$

In our setting, the pair of integers (p, q) is given by $(p, q) = (m, 0)$ for m even; $(p, q) = (0, m-1)$ for m odd. Applying [5, (12.22)] we deduce that this character defines the pure inner form

- $G^0 = SO(1, 2m) \times SO(1, 2m-1)$ if m is even,
- $G^0 = SO(1, 2m) \times SO(1, 2m-1)$ if m is odd.

The only representation in $VP(\pi_m) \times VP(\varpi_{m-1,\epsilon})$ with this pair of pure inner forms is $\pi_m \times \varpi_{m-1,\epsilon}$.

Conclusion: *The result*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{SO(1,2m-1)}(\pi_m|_{SO(1,2m-1)}, \varpi_{m-1,(-1)^m}) = 1$$

confirms the conjectures by B. Gross and D. Prasad [5].

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