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Harmonische Analysis und Darstellungstheorie Topologischer Gruppen

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Abstract. This was a meeting in the general area of representation theory and harmonic analysis on reductive groups.

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Introduction by the Organisers

This international conference was organized by Toshiyuki Kobayashi (University of Tokyo), Bernhard Krötz (MPIM Bonn), Erez Lapid (Hebrew University) and Charles Torrosian (Paris VI). The general theme was representation theory of real and p-adic Lie groups, especially in connection to automorphic forms. In addition there were very interesting presentations on infinite dimensional Lie groups and quantization.

It was very pleasing that new results of the highest caliber were presented by the youngest of the participants. There were many discussions at this relaxed and influential meeting. We would like to thank the MFO for having given us this wonderful opportunity. The organizers would like to thank Dmitry Gourevitch for his careful work in compiling and editing this report.

Workshop: Harmonische Analysis und Darstellungstheorie Topologischer Gruppen

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Abstracts

On inverse limits of classical groups.

Yuri Neretin

Denote by $\mathrm{U}(n)$ the group of unitary matrices of order n. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(n+m)$. Define the matrix

$$\Upsilon_n^{n+m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(n+m) := a - b(1+d)^{-1}c..$$

Theorem. a) *If $g \in \mathrm{U}(n+m)$, then $\Upsilon_n^{n+m}(g) \in \mathrm{U}(n)$.*

b) *The pushforward of the Haar measure is the Haar measure.*

c) $\Upsilon_q^p \circ \Upsilon_r^q = \Upsilon_r^p$.

More generally, fix α, $\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha+\beta) \geq -1$. Define a charge $d\mu_{\alpha,\beta}^n(g)$ on $\mathrm{U}(n)$ by

$$d\mu_{\alpha,\beta}^n(g) := \prod_{j=1}^n \frac{\Gamma(j+\alpha)\Gamma(j+\mu)}{\Gamma(j)\Gamma(j+\alpha+\beta)} \cdot d_n(g),$$

where $d_n(g)$ is the Haar measure. If $\lambda = \overline{\mu}$, then this charge is a positive measure.

Theorem. a) *The total charge of* $\mathrm{U}(n)$ *is 1.*

b) *The pushforward of $d\mu_{\alpha,\beta}^{n+1}(g)$ under Υ_n^{n+1} is $d\mu_{\alpha,\beta}^n(g)$.*

These theorems allow to define an inverse limit $\mathfrak{U}$ of the spaces $\mathrm{U}(n)$ and the two-parametric family of measures on the inverse limit of unitary groups.

The topic of the lecture was a discussion of consequences of these phenomena.

Multiplicities of holomorphic representations relatively to compact subgroups

Paul-Emile Paradan

The aim of this talk is to explain how one can “compute” geometrically the mutiplicities of a holomorphic representation of a real simple Lie group relatively to a compact subgroup. This computation follows the line of the orbit method [3] and is a non-compact example of the “quantization commutes with reduction” phenomenon [2, 6, 7, 8].

Let G be a real simple Lie group with finite center and let K be a maximal compact subgroup. We make the choice of a maximal torus T in K. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$ be the Lie algebras of G, K, T. We consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We assume that G admits holomorphic discrete series representations. It is the case if the real vector space $\mathfrak{p}$ admits a K-invariant complex structure, or equivalently, if the center $Z(K)$ of K is equal to the circle group : hence the complex structure on $\mathfrak{p}$ is defined by the adjoint action of an element z_o in the Lie algebra of $Z(K)$.

Let $\wedge_K^* \subset \mathfrak{t}^*$ be the weight lattice : $\alpha \in \wedge_K^*$ if $i\alpha$ is the differential of a character of T. Let $\mathfrak{R} \subset \wedge_K^*$ be the set of roots for the action of T on $\mathfrak{g} \otimes \mathbb{C}$. We have $\mathfrak{R} = \mathfrak{R}_c \cup \mathfrak{R}_n$ where $\mathfrak{R}_c$ and $\mathfrak{R}_n$ are respectively the set of roots for the action of T on $\mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p} \otimes \mathbb{C}$. We fix a system of positive roots $\mathfrak{R}_c^+$ in $\mathfrak{R}_c$. We have $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^\pm := \ker(\mathrm{ad}(z_o) \mp i)$. Let $\mathfrak{R}_n^+$ be the set of roots for the action of T on $\mathfrak{p}^+$. The union $\mathfrak{R}^+ := \mathfrak{R}_c^+ \cup \mathfrak{R}_n^+$ defines then a system of positive roots in $\mathfrak{R}$.

The holomorphic discrete series representations of G is parametrized by the set

$$\mathcal{E}_2^{\mathrm{hol}} := \{\lambda \in \mathfrak{t}^* \mid (\lambda, \alpha) > 0 \ \forall \alpha \in \mathfrak{R}^+ \text{ and } \lambda - \rho \in \wedge_K^*\},$$

where ρ is half the sum of the elements of $\mathfrak{R}^+$. We have $\rho = \rho_c + \rho_n$ where ρ_c and ρ_n are respectively half the sum of the elements of $\mathfrak{R}_c^+$ and $\mathfrak{R}_n^+$.

For each $\lambda \in \mathcal{E}_2^{\mathrm{hol}}$, we denote by $\mathcal{H}_\lambda$ the corresponding holomorphic representation of G. The element λ is the *Harish-Chandra parameter* of the representation $\mathcal{H}_\lambda$, and

$$\Lambda := \lambda - \rho_c + \rho_n \in \wedge_K^*$$

is the *Blattner parameter* of the representation $\mathcal{H}_\lambda$.

We now recall how the representations $\mathcal{H}_\lambda$, $\lambda \in \mathcal{E}_2^{\mathrm{hol}}$ are constructed. We consider the coadjoint orbit $\mathcal{O}_\Lambda := G \cdot \Lambda \subset \mathfrak{g}^*$. For any $X \in \mathfrak{g}$, we denote VX the vector field on $\mathcal{O}_\Lambda$ defined by : $VX(\xi) := \frac{d}{dt} e^{-tX} \cdot \xi|_{t=0}$ for $\xi \in \mathcal{O}_\Lambda$.

We have on the coadjoint orbit $\mathcal{O}_\Lambda$ the following data :

(1) The Kirillov-Kostant-Souriau symplectic form Ω_Λ which is defined by the relation : for any $X, Y \in \mathfrak{g}$ and $\xi \in \mathcal{O}_\Lambda$ we have

$$\Omega_\Lambda(VX, VY)|_\xi = \langle \xi, [X, Y] \rangle.$$

(2) The inclusion $\Phi_G : \mathcal{O}_\Lambda \hookrightarrow \mathfrak{g}^*$ is a moment map relative to the Hamiltonian action of G on $(\mathcal{O}_\Lambda, \Omega_\Lambda)$.

(3) The line bundle $\mathcal{L}_\Lambda := G \times_{K_\Lambda} \mathbb{C}_\Lambda$. Here K_Λ is the stabilizer subgroup of Λ in G : it is contained in K since $(\Lambda, \alpha) \neq 0$ for any $\alpha \in \mathfrak{R}_n^+$. We denote $\mathbb{C}_\Lambda$ the one dimensional representation of the group K_Λ attached to the weight $\Lambda \in \wedge_K^*$.

(4) An equivariant complex structure J characterized by the following fact. The holomorphic tangent bundle $T^{1,0}\mathcal{O}_\Lambda \to \mathcal{O}_\Lambda$ is equal, above $\Lambda \in \mathcal{O}_\Lambda$, to the complex vector space

$$\sum_{\substack{\alpha \in \mathfrak{R}_c^+ \\ \langle \alpha, \Lambda \rangle \neq 0}} (\mathfrak{g} \otimes \mathbb{C})_\alpha + \sum_{\alpha \in \mathfrak{R}_n^-} (\mathfrak{g} \otimes \mathbb{C})_\alpha \ .$$

One can check that the complex structure is positive relatively to the symplectic form, e.g. $\Omega_\Lambda(-, J-)$ defines a Riemannian metric. Hence $(\mathcal{O}_\Lambda, \Omega_\Lambda, J)$ is a Kähler manifold.

Moreover the first Chern class of $\mathcal{L}_\Lambda$ is equal to $\left[\frac{\Omega_\Lambda}{2\pi}\right]$: the line bundle $\mathcal{L}_\Lambda$ is an equivariant *pre-quantum* line bundle over $(\mathcal{O}_\Lambda, \Omega_\Lambda)$ [5].

The representation $\mathcal{H}_\lambda$ is constructed as the vector space of holomorphic sections of $\mathcal{L}_\Lambda \to \mathcal{O}_\Lambda$ which are L^2 integrable [4].

Let $H \subset K$ be a compact connected Lie group with Lie algebra $\mathfrak{h}$. The H-action on $(\mathcal{O}_\Lambda, \Omega_\Lambda)$ is Hamiltonian with moment map $\Phi_H : \mathcal{O}_\Lambda \to \mathfrak{h}^*$ equal to the composition of $\Phi_G : \mathcal{O}_\Lambda \to \mathfrak{g}^*$ with the projection $\mathfrak{g}^* \to \mathfrak{h}^*$.

One knows that the representation $\mathcal{H}_\lambda|_H$ is *admissible* if and only if the map $\Phi_H : \mathcal{O}_\Lambda \to \mathfrak{h}^*$ is *proper* [1]. It is the case for example if H contains the circle subgroup $Z(K)$ since $\mathcal{H}_\lambda|_{Z(K)}$ is admissible.

The irreducible representations of the compact Lie group H are parametrized by a set of dominant weights $\widehat{H} \subset \mathfrak{h}^*$. For any $\mu \in \widehat{H}$, we denote V_μ^H the irreducible representation of H with highest weight μ.

We suppose now that the moment map Φ_H is proper, and one wants to compute the multiplicities of $\mathcal{H}_\lambda|_H$.

If $\xi \in \mathfrak{h}^*$ is a regular value of Φ_H the Marsden-Weinstein reduction

$$(\mathcal{O}_\Lambda)_\xi := \Phi_H^{-1}(H \cdot \xi)/H$$

is a compact Kähler orbifold. If moreover ξ is integral, e.g. $\xi = \mu \in \widehat{H}$, there exists a holomorphic line orbibundle $\mathcal{L}(\mu)$ that prequantizes the symplectic orbifold $(\mathcal{O}_\Lambda)_\mu$. In this situation, one defines the integer

$$\mathcal{Q}\left((\mathcal{O}_\Lambda)_\mu\right) \in \mathbb{Z}$$

as the holomorphic Euler characteristic of $((\mathcal{O}_\Lambda)_\mu, \mathcal{L}(\mu))$.

In the general case where μ is not necessarily a regular value of Φ_H, $\mathcal{Q}\left((\mathcal{O}_\Lambda)_\mu\right) \in \mathbb{Z}$ can still be defined (see [6], [7], [8]). The integer $\mathcal{Q}\left((\mathcal{O}_\Lambda)_\mu\right)$ only depends on the data $(\mathcal{O}_\Lambda, \mathcal{L}_\Lambda, J)$ in a small neighborhood of $\Phi^{-1}(\mu)$: in particular $\mathcal{Q}\left((\mathcal{O}_\Lambda)_\mu\right)$ vanishes when $\Phi^{-1}(\mu) = \emptyset$.

Now we can state our main result.

Theorem *Let $H \subset K$ be a compact connected Lie group such that the representation $\mathcal{H}_\lambda|_H$ is admissible. Then we have*

$$\mathcal{H}_\lambda|_H := \sum_{\mu \in \widehat{H}} \mathcal{Q}\left((\mathcal{O}_\Lambda)_\mu\right) V_\mu^H.$$

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Multiplicity one Theorems

Dmitry Gourevitch
(joint work with Avraham Aizenbud, Steve Rallis, and Gerard Schiffmann)

arXiv:0709.4215 [math.RT]

Let $\mathbb{F}$ be a local field non archimedean and of characteristic 0. Let W be a vector space over $\mathbb{F}$ of finite dimension $n+1 \geqslant 1$ and let $W = V \oplus U$ be a direct sum decomposition with $\dim V = n$. Then we have an imbedding of $GL(V)$ into $GL(W)$. Our goal is to prove the following Theorem:

Theorem 1: *If π (resp. ρ) is an irreducible admissible representation of $GL(W)$ (resp. of $GL(V)$) then*

$$\dim \left(\mathrm{Hom}_{GL(V)}(\pi_{|GL(V)}\,,\,\rho)\right) \leqslant 1.$$

We choose a basis of V and a non zero vector in U thus getting a basis of W. We can identify $GL(W)$ with $GL(n+1,\mathbb{F})$ and $GL(V)$ with $GL(n,\mathbb{F})$. The transposition map is an involutive anti-automorphism of $GL(n+1,\mathbb{F})$ which leaves $GL(n,\mathbb{F})$ stable. It acts on the space of distributions on $GL(n+1,\mathbb{F})$.

Theorem 1 is a Corollary of :

Theorem 2: *A distribution on $GL(W)$ which is invariant under the adjoint action of $GL(V)$ is invariant by transposition.*

One can raise a similar question for orthogonal and unitary groups. Let $\mathbb{D}$ be either $\mathbb{F}$ or a quadratic extension of $\mathbb{F}$. If $x \in \mathbb{D}$ then $\overline{x}$ is the conjugate of x if $\mathbb{D} \neq \mathbb{F}$ and is equal to x if $\mathbb{D} = \mathbb{F}$.

Let W be a vector space over $\mathbb{D}$ of finite dimension $n+1 \geqslant 1$. Let $\langle .,. \rangle$ be a non degenerate hermitian form on W. This form is bi-additive and

$$\langle dw, d'w' \rangle = d\,\overline{d'}\langle w, w' \rangle, \quad \langle w', w \rangle = \overline{\langle w, w' \rangle}.$$

Given a $\mathbb{D}-$linear map u from W into itself, its adjoint u^* is defined by the usual formula

$$\langle u(w), w' \rangle = \langle w, u^*(w') \rangle.$$

Choose a vector e in W such that $\langle e, e\rangle \neq 0$; let $U = \mathbb{D}e$ and $V = U^{\perp}$ the orthogonal complement. Then V has dimension n and the restriction of the hermitian form to V is non degenerate.

Let M be the unitary group of W, that is to say the group of all $\mathbb{D}-$linear maps m of W into itself which preserve the hermitian form or equivalently such that $mm^* = 1$. Let G be the unitary group of V. With the p-adic topology both groups are l-groups(see [**B-Z**]).

The group G is naturally imbedded into M.

Theorem 1′: *If π (resp ρ) is an irreducible admissible representation of M (resp of G) then*

$$\dim\big(\mathrm{Hom}_G(\pi_{|G}, \rho)\big) \leq 1.$$

Choose a basis $e_1, \dots e_n$ of V such that $\langle e_i, e_j\rangle \in \mathbb{F}$. For

$$w = x_0 e + \sum_1^n x_i e_i$$

put

$$\overline{w} = \overline{x}_0 e + \sum_1^n \overline{x_i}\, e_i.$$

If u is a $\mathbb{D}-$linear map from W into itself, let $\overline{u}$ be defined by

$$\overline{u}(w) = \overline{u(\overline{w})}.$$

Let σ be the anti-involution $\sigma(m) = \overline{m}^{-1}$ of M; Theorem 1' is a consequence of

Theorem 2′: *A distribution on M which is invariant under the adjoint action of G is invariant under σ.*

Let us describe briefly our proof for $\mathrm{GL}(n)$. The proof is by induction on n; the case $n = 0$ is trivial. In general we first linearize the problem by replacing the action of G on $\mathrm{GL}(W)$ by the action on the Lie algebra of $\mathrm{GL}(W)$. As a $G-$module this Lie algebra is isomorphic to a direct sum $\mathfrak{g} \oplus V \oplus V^* \oplus \mathbb{F}$ with $\mathfrak{g}$ the Lie algebra of G, V^* the dual space of V. The group $G = \mathrm{GL}(V)$ acts trivially on $\mathbb{F}$, by the adjoint action on its Lie algebra and the natural actions on V and V^*. The component $\mathbb{F}$ plays no role. Let u be a linear bijection of V onto V^* which transforms some basis of V into its dual basis. The involution may be taken as

$$(X, v, v^*) \mapsto (u^{-1}\,{}^t X\, u, u^{-1}(v^*), u(v)).$$

We have to show that a distribution T on $\mathfrak{g} \oplus V \oplus V^*$ which is invariant under G and skew relative to the involution is 0.

First we prove that such a distribution must have "singular support". On the $\mathfrak{g}$ side, using Harish-Chandra descent we get that the support of T must be contained in $\mathfrak{z} \times \mathcal{N} \times (V \oplus V^*)$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. On the $V \oplus V^*$ side we show that the support must be contained in $\mathfrak{g} \times \Gamma$ where Γ is the cone $\langle v, v^* \rangle = 0$ in $V \oplus V^*$. On $\mathfrak{z}$ the action is trivial so we are reduced to the case of a distribution on $\mathcal{N} \times \Gamma$.

The end of the proof is based on two remarks. First, viewing the distribution as a distribution on $\mathcal{N} \times (V \oplus V^*)$ its partial Fourier transform relative to $V \oplus V^*$ has the same invariance properties and hence must also be supported by $\mathcal{N} \times \Gamma$. This implies in particular a homogeneity condition on $V \oplus V^*$. The idea of using Fourier transform in this kind of situation goes back at least to Harish-Chandra ([H]) and is conveniently expressed using a particular case of the Weil or oscillator representation.

For $(v, v^*) \in \Gamma$, let X_{v,v^*} be the map $x \mapsto \langle x, v^* \rangle v$ of V into itself. The second remark is that the one parameter group of transformations

$$(X, v, v^*) \mapsto (X + \lambda X_{v,v^*}, v, v^*)$$

is a group of (non linear) homeomorphisms of $[\mathfrak{g}, \mathfrak{g}] \times \Gamma$ which commute with G and the involution. It follows that the image of the support of our distribution must also be singular. Precisely this allows us to replace the condition $\langle v, v^* \rangle = 0$ by the stricter condition $X_{v,v^*} \in \operatorname{Im} \operatorname{ad} X$.

Using the stratification of $\mathcal{N}$ we proceed one nilpotent orbit at a time, transferring the problem to $V \oplus V^*$ and a fixed nilpotent matrix X. The support condition turns out to be compatible with direct sum so that it is enough to consider the case of a principal nilpotent element. In this last situation the key is the homogeneity condition coupled with an easy induction.

The orthogonal and unitary cases are proved roughly in the same way. Here the main difference is that we use Harish-Chandra descent directly on the group. Note that some Levi subgroups have components of type GL so that theorem 2 has to be assumed.

We systematically use two classical results from [**Ber**]: Bernstein's localization principle and a variant of Frobenius reciprocity.

Similar theorems should be true in the archimedean case. A partial result is given by [**A-G-S**].

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Periods and global Gross-Prasad conjecture

Tamotsu Ikeda
(joint work with Atsushi Ichino)

This is a joint work with Atsushi Ichino.

Let k be a global field with $\mathrm{char}(k) \neq 2$. Let $(V_0, Q_0) \subset (V_1, Q_1)$ be quadratic forms over k with rank n and $n+1$, respectively. We assume that $n \geq 2$ and that (V_0, Q_0) is not isomorphic to the hyperbolic plane. We regard $G_0 = \mathrm{SO}_{Q_0}$ as a subgroup of $G_1 = \mathrm{SO}_{Q_1}$. Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ be irreducible tempered cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. Put

$$\begin{aligned}
\langle \varphi_1, \varphi_1 \rangle &= \int_{G_1(k)\backslash G_1(\mathbb{A})} |\varphi_1(g_1)|^2 \, dg_1, \\
\langle \varphi_0, \varphi_0 \rangle &= \int_{G_0(k)\backslash G_0(\mathbb{A})} |\varphi_0(g_0)|^2 \, dg_0, \\
\langle \varphi_1|_{G_0}, \varphi_0 \rangle &= \int_{G_0(k)\backslash G_0(\mathbb{A})} \varphi_1(g_0)\overline{\varphi_0(g_0)} dg_0
\end{aligned}$$

for $\varphi_1 \in \pi_1$ and $\varphi_0 \in \pi_0$. Here dg_1 (resp. dg_0) is the Tamagawa measure on $G_1(\mathbb{A})$ (resp. $G_0(\mathbb{A})$). For each v, we choose a Haar measure $dg_{0,v}$ of $G_{0,v}$ and put $dg_0 = C_0 \prod_v dg_{0,v}$. Since $\pi_i \simeq \otimes_v \pi_{i,v}$ is unitary, there exists an inner product $\langle \; , \; \rangle_v$ on $\pi_{i,v}$.

We are going to formulate a conjecture, which expresses the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ in terms of L-values. Put

$$\Delta_{G_1} = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(2l) & \text{if } \dim V_1 = 2l+1, \\ \zeta(2)\zeta(4)\cdots\zeta(2l-2)\cdot L(l, \chi_{Q_1}) & \text{if } \dim V_1 = 2l, \end{cases}$$

where χ_{Q_1} is the quadratic Hecke character associated with the discriminant of Q_1.

Following [3], we put

$$\mathcal{P}_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \times \pi_0)}{L(s+(1/2), \pi_1, \mathrm{Ad})L(s+(1/2), \pi_0, \mathrm{Ad})},$$

where $L(s, \pi_1 \times \pi_0)$ is the rensor product L-function, and $L(s, \pi_1, \mathrm{Ad})$ and $L(s, \pi_0, \mathrm{Ad})$ are the adjoint L-function of π_1 and that of π_0, respectively. We assume that the L-functions $L(s, \pi_1 \times \pi_0)$, $L(s, \pi_1, \mathrm{Ad})$, and $L(s, \pi_0, \mathrm{Ad})$ have meromorphic continuation. For a sufficiently large finite set of bad places S, we denote the partial Euler products for $\mathcal{P}_{\pi_1,\pi_0}(s)$ and Δ_{G_1} by $\mathcal{P}^S_{\pi_1,\pi_0}(s)$ and $\Delta^S_{G_1}$, respectively.

Let $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$ be cusp forms. We consider the matrix coefficients

$$\begin{aligned}
\Phi_{\varphi_{1,v},\varphi_{1,v}}(g_1) =& \langle \pi_{1,v}(g_1)\varphi_{1,v}, \varphi_{1,v} \rangle_v, \quad g_1 \in G_1(k_v), \\
\Phi_{\varphi_{0,v},\varphi_{0,v}}(g_0) =& \langle \pi_{0,v}(g_0)\varphi_{0,v}, \varphi_{0,v} \rangle_v, \quad g_0 \in G_0(k_v)
\end{aligned}$$

and put

$$I(\varphi_{1,v}, \varphi_{0,v}) = \int_{G_0(k_v)} \Phi_{\varphi_{1,v},\varphi_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v},\varphi_{0,v}}(g_{0,v})}\, dg_{0,v}.$$

It can be proved that this integral is convergent.

Then we conjecture that there exists an integer β such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0\rangle|^2}{\langle \varphi_1, \varphi_1\rangle \langle \varphi_0, \varphi_0\rangle} = 2^\beta C_0 \Delta_{G_1}^S \mathcal{P}_{\pi_1,\pi_0}^S(1/2) \prod_{v \in S} \frac{I(\varphi_{1,v}, \varphi_{0,v})}{\langle \varphi_{1,v}, \varphi_{1,v}\rangle_v \, \langle \varphi_{0,v}, \varphi_{0,v}\rangle_v}.$$

The factor 2^β should be $(\sharp|\mathcal{S}_1| \cdot \sharp|\mathcal{S}_0|)^{-1}$, where $\mathcal{S}_i$ is the S-group of π_i, which plays an important roles in the theory of endoscopy.

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Some results on harmonic analysis on reductive p-adic symmetric spaces

Patrick Delorme

Let F be a p-adic field of characteristic zero, $G = \underline{G}(F)$ the group of F-points of a reductive algebraic group $\underline{G}$, defined over F, σ a rational involution of $\underline{G}$ defined over F. We denote by H the fixed point group in G of σ. We present results towards the decomposition of $L^2(G/H)$ into irreducible representations (Plancherel formula). The study of the corresponding problem for the real case is due to van den Ban-Schlichtkrull and D. [BSD]. Before, Harish-Chandra solved the real and the p-adic case ([HC], [W]) when $G = G_1 \times G_1$ and $\sigma(x, y) = (y, x)$. There are also results for spherical functions in special cases of p-adic symmetric spaces ([H],[HS], [O]).

We return to the general p-adic case.To obtain the Plancherel formula for G/H, it is natural to study the triples $(\pi, V, \xi)$where π is a smooth complex representation of G in V and ξ is an element of the space V^{*H} of H-fixed linear form on V. Also it is important to study of the generalized coefficients $c_{\xi,v}$ which are functions on G/H defined by :

$$c_{\xi,v}(gH) = < \pi^*(gH)\xi, v >, g \in G, v \in V$$

where π^* is the congragredient representation of π (not the smooth contragredient representation). Following the real case, one considers special parabolic subgroup of G: P is a σ parabolic subgroup iff P and $\sigma(P)$ are opposed parabolic subgroups. In such a case $P \cap \sigma(P)$ is the σ-stable Levi subgroup of P. Also a split torus A of G will be said σ-split if is contained in $\{g \in G | \sigma(g) = g^{-1}\}$. Helminck with S.P.

Wang and G. Helminck important properties of these notions [HH]. The behavior of the triples above under Jacquet functors for σ-parabolic subgroups has been studied by Lagier [L] on one hand and Kato-Takano [KT] on the other hand, for admissible modules. I extended recently the result to smooth modules using the generalized Jacquet Lemma of J. Bernstein:

Theorem 1 : Let P be a σ-parabolic subgroup of G. Denote by A a maximal σ-split torus of the cente of M. For $\varepsilon > 0$, denote by $A^-(\varepsilon)$ the set of elements of A on which the absolute value of the roots of A in the unipotent radical of P is less than ε. Let V be a smooth module a. Let j the projection of V on its Jacquet module, $(\pi_P, j(V))$, for P. Then there exists a map, denoted by j_P^* or simply j^*, , $j^* : V^{*H} \to (j(V))^{*M\cap H}$ such that, for $\xi \in V^{*H}$, $j^*(\xi) \in (j(V))^{*M\cap H}$ is characterized as follows:

For all $v \in V$, there exists $\varepsilon_v > 0$ such that :

$$< \pi^*(a)(\xi), v >=< \pi_P(a) j^*(\xi), j(v) >, a \in A^-(\varepsilon_v)$$

The following theorem is the analogue of the Cartan decomposition for G/H.

Theorem 2[BO], [DS] Let P_0 be a minimal σ- split parabolic subgroup of G and A_0 a maximal σ-split torus of the center of the σ-stable Levi subgroup of P_0. Then there exists a (finite) set of representative of the (H, P_0) open double cosets, $\mathcal{W}$ and compact subset of G such that :

$$G = \cup_{y \in \mathcal{W}} \Omega A_0^-(1) y^{-1} H$$

Concerning this theorem, [BO] is more general than [DS], but when G is split [DS] gives a more precise result.

Among other, the two previous theorems allow to prove that the generalized coefficients above are bounded when V is unitary [L]. The proof uses techniques of [C] for the group case. Also Kato and Takano [KT] studied the so called cuspidal ξ, i.e such that $j_P^*(\xi)$ is zero for all σ parabolic subgroup of G distinct from G .

Concerning induction from a σ parabolic subgroup P, in [BD] we construct families of H-fixed linear forms which depend of an $M \cap H$ linear form on the inducing representation δ and rationality of a unramified character, χ, of M which is anti invariant by σ. We denote them by $\xi(P, \delta, \eta, \chi)$.

By studying the asymptotic behaviour of generalized coeeficients for such ξ (which are called Eisenstein integrals), Lagier [L] was able to prove .

Theorem 3

For $\varepsilon > 0$ small enough the following union is disjoint:

$$\cup_{y \in \mathcal{W}} \Omega A_0^-(\varepsilon) y^{-1} H$$

Together with Theorem 1, I proved the following finiteness results.

Theorem 4

(i) Let K an open compact subgroup of G. Up to twisting by an antiinvariant unramified character of G, there exists finetely many irrecudible representations of G with a non zero H-cuspidal linear form.

(ii) If V is a smooth G-module of finite length, V^{*H} is finite dimensional.

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Automorphic Representations invariant under an automorphism.

Birgit Speh

(joint work with Dan Barbasch)

Abstract: Automorphic Representations invariant under an automorphism.

This is joint work with Dan Barbasch

Let $\mathbf{G}$ be semisimple simply connected algebraic group defined over $\mathbf{Q}$ and $\mathbf{A}$ the adeles of $\mathbf{Q}$. Suppose that τ is an automorphism of $\mathbf{G}$ defined over $\mathbf{Q}$. We show that there are representations in the cuspidal spectrum of $L^2(\mathbf{G}(\mathbf{A})/\mathbf{G}(\mathbf{A}))$ which are invariant under τ. We show in particular that there representations in the cuspidal spectrum of $L^2(\mathbf{G}(\mathbf{A})/\mathbf{G}(\mathbf{A}))$ with on trivial $(\mathfrak{g}, K_\infty)$–cohomology and so if the compact subgroup $K_{\mathbf{A}} = K_\infty K_{\mathbf{A}_f}$ is small enough the cuspidal cohomology of the locally symmetric space $K_{\mathbf{A}}\backslash\mathbf{G}(\mathbf{A})/\mathbf{G}(\mathbf{A})$ is nonzero. We prove these results using the twisted Arthur trace formula.

A combinatorial identity for root arrangements connected to Arthur's trace formula

Tobias Finis
(joint work with Erez Lapid, Werner Müller)

For any complete simplicial fan in a real vector space, there is a localization map from piecewise polynomial functions on the fan to global polynomial functions [B]. Composing this map with evaluation at zero, one obtains a linear form on piecewise polynomial functions of degree the dimension of the space, which can be expressed by a limit formula. For the case of the root hyperplane arrangements associated to pairs (G, M), G a reductive group and M a Levi subgroup of a parabolic, this construction is ubiquitous in Arthur's trace formula.

We study the localization map for root arrangements and special operator-valued piecewise power series obtained from collections of one-variable power series associated to the codimension one cones (walls) of the arrangement and subject to certain compatibility relations. This situation is abstracted from the case of global or normalized local intertwining operators, but we consider it in a purely formal algebraic-combinatorial setting. The main result is an expression for the value at zero of the localization purely in terms of first derivatives in linearly independent directions (and values at zero) of the one-variable functions entering into the construction. (Directly from the definition, we only get an expression in terms of derivatives of order up to the rank.) A special feature of the formula is its dependence (for relative rank greater than two) on non-canonical choices. An important special case are exponentials of piecewise linear functions. In this case one obtains a formula for the volume of a polytope in terms of products of its side lengths. In this case there is also a direct proof based on tessellations and the theory of mixed volumes. However, this argument does not extend to the general non-commutative case. Instead, we give a more elaborate inductive proof based on the consideration of the space of universal relations between products of derivatives. The crucial problem is to show that the dual module is generated in degrees less than the rank.

The combinatorial formula can be applied to rewrite the spectral side of Arthur's trace formula as an expression indexed by the discrete spectra of the Levi subgroups, which is absolutely convergent with respect to the trace norm. This result was previously known only for the groups $\mathrm{GL}(n)$ by the work of Müller and Speh [MS]. Even in this case, our new proof seems more natural, since it avoids the use of finer arithmetic information (bounds toward the Ramanujan conjecture).

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Formal degrees and adjoint gamma factors

Atsushi Ichino
(joint work with Kaoru Hiraga and Tamotsu Ikeda)

Harish-Chandra established the Plancherel formula for reductive groups over local fields of characteristic zero. Formal degrees and Plancherel measures are key ingredients in the Plancherel formula and are important objects in harmonic analysis. In 1976, Langlands [2] conjectured a formula which relates Plancherel measure and local L and ε-factors. His conjecture is significant in the sense that it relates objects in harmonic analysis and those in arithmetic. Shahidi [3] made a remarkable progress toward Langlands' conjecture. We give a natural extension of Langlands' conjecture to formal degrees in terms of the adjoint γ-factor

$$\gamma(s, \pi, \mathrm{Ad}, \psi) = \epsilon(s, \pi, \mathrm{Ad}, \psi) \cdot \frac{L(1-s, \check{\pi}, \mathrm{Ad})}{L(s, \pi, \mathrm{Ad})},$$

where Ad is the adjoint representation of the L-group of G on its Lie algebra. If π is an irreducible finite dimensional representation of a compact Lie group G of rank l, then our conjecture asserts that

$$\frac{\dim \pi}{\mathrm{vol}(G)} = \frac{1}{2^l} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|$$

and is compatible with the Weyl dimension formula. Moreover, if π is a stable discrete series representation of U(3) over a p-adic field, we show that the formal degree of π is equal to

$$\frac{1}{2} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|.$$

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Rankin-Selberg type spectral identities and Gelfand pairs

Andre Reznikov
(joint work with Joseph Bernstein)

We use the uniqueness of various invariant functionals on irreducible unitary representations of $PGL(2, \mathbb{R})$ in order to deduce the classical Rankin-Selberg identity for the sum of Fourier coefficients of Maass cusp forms and its new anisotropic analog. We deduce from these formulas non-trivial bounds for the corresponding unipotent and spherical Fourier coefficients of Maass forms. As an application we obtain a subconvexity bound for certain L-functions. Our main tool is the notion of a Gelfand pair from the representation theory.

We study periods of automorphic functions. We present a new method which allows one to obtain non-trivial spectral identities for weighted sums of certain periods of automorphic functions. These identities are modelled on the classical identity of R. Rankin and A. Selberg. We recall that the Rankin-Selberg identity relates the weighted sum of Fourier coefficients of a cusp form ϕ to the weighted integral of the inner product of ϕ^2 with the Eisenstein series.

We show how to deduce the classical Rankin-Selberg identity and similar new identities from the uniqueness principle in representation theory (also known under the following names: the multiplicity one property, Gelfand pair). The uniqueness principle is a powerful tool in representation theory; it plays an important role in the theory of automorphic functions.

We associate a non-trivial spectral identity to certain *pairs* of different *triples* of Gelfand subgroups. Namely, we associate a spectral identity (see the formula (1) below) with two triples $\mathcal{F} \subset \mathcal{H}_1 \subset \mathcal{G}$ and $\mathcal{F} \subset \mathcal{H}_2 \subset \mathcal{G}$ of subgroups in a group $\mathcal{G}$ such that pairs $(\mathcal{G}, \mathcal{H}_i)$ and $(\mathcal{H}_i, \mathcal{F})$ for $i = 1,\ 2$, are strong Gelfand pairs having the same subgroup $\mathcal{F}$ in the intersection (for the notion of Gelfand pair that we use, see Section 0.0.3). We call such a collection $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F})$ a *strong Gelfand formation.*

0.0.1. *Periods and Gelfand pairs.* We briefly review some well-known notions and constructions from representation theory and the theory of automorphic functions needed to formulate our identities.

Taking periods is the classical technique to study automorphic functions. It goes back, at least, to Hecke and Maass. In the classical language it means the following. Let ϕ be an automorphic function on Y and let ψ be an automorphic function on a cycle $C \subset Y$, where the cycle C is equipped with a measure dc. Then one can consider the period defined via the integral

$$\int_C \phi(c)\psi(c)\ dc\ .$$

It is well-known that in the modern language of automorphic representations this construction leads to the following setup.

0.0.2. *Automorphic representations, periods and representation theory.* Let $\mathcal{G}$ be a real reductive Lie group, let $\Gamma_{\mathcal{G}} \subset \mathcal{G}$ be a lattice and let $X_{\mathcal{G}} = \Gamma_{\mathcal{G}} \setminus \mathcal{G}$ be the corresponding automorphic space. Let $\mathcal{H} \subset \mathcal{G}$ be a subgroup (not necessarily reductive, e.g., a unipotent subgroup of $SL_2(\mathbb{R})$) and let $X_{\mathcal{H}} \subset X_{\mathcal{G}}$ be a closed orbit of $\mathcal{H}$ (e.g., $X_{\mathcal{H}} = \mathcal{H} \cap \Gamma_{\mathcal{G}} \setminus \mathcal{H}$). We fix a $\mathcal{H}$-invariant measure $\mu_{\mathcal{H}}$ on $X_{\mathcal{H}}$ and consider the unitary representation $L^2(X_{\mathcal{H}}, \mu_{\mathcal{H}})$ of $\mathcal{H}$. We denote by $r_{\mathcal{H}} = r_{X_{\mathcal{H}}} : C^\infty(X_{\mathcal{G}}) \to C^\infty(X_{\mathcal{H}})$ the corresponding restriction map.

Let (π, ν_π) be an automorphic representation of $\mathcal{G}$ on $X_{\mathcal{G}}$ and let (σ, ν_σ) an automorphic representation of $\mathcal{H}$ on $X_{\mathcal{H}}$. We consider the $\mathcal{H}$-*equivariant* map

$T^{aut}_{\pi,\sigma} = T^{aut}_{X_{\mathcal{H}},\pi,\sigma} = \nu^*_\sigma \circ r_{\mathcal{H}} \circ \nu_\pi : V \to W$ defined via the composition

$$T^{aut}_{\pi,\sigma} : V \xrightarrow{\nu_\pi} C^\infty(X_{\mathcal{G}}) \xrightarrow{r_{\mathcal{H}}} C^\infty(X_{\mathcal{H}}) \xrightarrow{\nu^*_\sigma} W \ .$$

We call this map the automorphic period map (or simply the period) associated to the collection (π, ν_π), (σ, ν_σ), $X_{\mathcal{G}}$, $X_{\mathcal{H}}$ and the choice of corresponding measures. This is the representation theoretic substitute for the classical period. Clearly $T^{aut}_{\pi,\sigma} \in \mathrm{Hom}_{\mathcal{H}}(V, W)$. We denote the vector space $\mathrm{Hom}_{\mathcal{H}}(V, W)$ by $\mathcal{P}(V, W)$ and call it the vector space of *periods* between π and σ.

0.0.3. *Gelfand pairs and periods.* In many cases interesting periods are associated to some special subgroup (or representations). Many of these examples come from what is called the multiplicity one representations or Gelfand pairs. In what follows we will use the notion of Gelfand pairs for *real Lie groups.*

A pair $(\mathcal{G}, \mathcal{H})$ of a real Lie group $\mathcal{G}$ and a real Lie subgroup $\mathcal{H} \subset \mathcal{G}$ is called a *strong Gelfand pair* if for any pair of irreducible representations (π, V) of $\mathcal{G}$ and (σ, W) of $\mathcal{H}$, the dimension of the space $\mathrm{Hom}_{\mathcal{H}}(V, W)$ of $\mathcal{H}$-equivariant maps (i.e., the space of periods $\mathcal{P}(V, W)$) for the spaces of *smooth* vectors is at most *one*. One of the important observations in the theory of automorphic functions is that for Gelfand pairs, automorphic periods $T^{aut}_{\pi,\sigma}$ lead to certain interesting numbers or functions (e.g., Fourier coefficients, L-functions). This is based on the fact that in many cases one can construct another vector in the *one-dimensional* vector space $\mathcal{P}(V, W)$ of periods.

Namely, usually representations π and σ could be constructed explicitly in some *model* spaces of sections of various vector bundles over appropriate homogeneous manifolds (e.g., for $SL_2(\mathbb{R})$, in the spaces of homogeneous functions on the plane $\mathbb{R}^2 \setminus 0$). These models usually exist for all representations and not only for the automorphic ones. If $\dim \mathcal{P}(V, W) = 1$, using these models one can construct an explicit non-zero map $T^{mod}_{\pi,\sigma} \in \mathcal{P}(V, W)$ by means of the corresponding kernel (i.e., construct this map as an integral operator with an explicit kernel). We call such a map a *model* period.

When $\dim \mathcal{P}(V, W) = 1$, such a choice of a non-zero model period $T^{mod}_{\pi,\sigma}$ gives rise to the automorphic coefficient of proportionality $a_{\pi,\sigma} = a_{\nu_\pi,\nu_\sigma} \in \mathbb{C}$ such that

$$T^{aut}_{\pi,\sigma} = a_{\pi,\sigma} \cdot T^{mod}_{\pi,\sigma} \ .$$

We explain now how sometimes one can obtain spectral identities involving certain coefficients $a_{\pi,\sigma}$.

0.0.4. **Rankin-Selberg type spectral identities.** Our main observation is that for a given pair of groups $\mathcal{F} \subset \mathcal{G}$ there might be different intermediate subgroups $\mathcal{H}$ as above leading to *different* spectral decompositions of the same functional $I_{\mathcal{F}}$ and hence to identities between the automorphic coefficients.

Let $\mathcal{G}$ be a real Lie group and $\mathcal{F} \subset \mathcal{H}_i \subset \mathcal{G}$, $i = 1, 2$, be a collection of subgroups such that each embedding is a strong Gelfand pair (i.e., pairs $(\mathcal{G}, \mathcal{H}_i)$ and $(\mathcal{H}_i, \mathcal{F})$ are strong Gelfand pairs). We call such a collection of subgroups a *strong Gelfand formation.*

Let $\Gamma \subset \mathcal{G}$ be a lattice and $X_{\mathcal{G}} = \Gamma \setminus \mathcal{G}$ the corresponding automorphic space. Let $X_i = X_{\mathcal{H}_i} \subset X_{\mathcal{G}}$ and $X_{\mathcal{F}} \subset X_{\mathcal{G}}$ be closed orbits of $\mathcal{H}_i$ and $\mathcal{F}$, respectively, satisfying the obvious set of embeddings, assumed to be compatible with the corresponding action of groups. We endow each orbit (as well as $X_{\mathcal{G}}$) with a measure invariant under the corresponding subgroup (to explain the idea, we assume that all orbits are compact, and hence, these measures could be normalized to have mass one).

We fix a decompositions $L^2(X_1) = \oplus_i \sigma_i$ into a direct sum (in general into a direct integral) of automorphic representations $(\sigma_i, \nu_{\sigma_i}, W_i)$ of $\mathcal{H}_1$ and similarly $L^2(X_2) = \oplus_j \tau_j$ for automorphic representations $(\tau_j, \nu_{\tau_j}, U_j)$ of $\mathcal{H}_2$.

Let (π, ν_π) be an automorphic representation of $\mathcal{G}$ and $I_{\mathcal{F}} : V \to \mathbb{C}$ the period defined by the integration over the cycle $X_{\mathcal{F}}$. It is easy to see that two different triples $\mathcal{F} \subset \mathcal{H}_1 \subset \mathcal{G}$ and $\mathcal{F} \subset \mathcal{H}_2 \subset \mathcal{G}$ lead to two spectral decompositions of the period $I_{\mathcal{F}}$. Namely, let us choose the model periods $T^{mod}_{\pi,\sigma_i} \in \mathrm{Hom}_{\mathcal{H}_1}(V, W_i)$, $T^{mod}_{\sigma_i,\mathbb{C}} \in \mathrm{Hom}_{\mathcal{F}}(W_i, \mathbb{C})$ and similarly $T^{mod}_{\pi,\tau_j} \in \mathrm{Hom}_{\mathcal{H}_2}(V, U_j)$, $T^{mod}_{\tau_j,\mathbb{C}} \in \mathrm{Hom}_{\mathcal{F}}(U_j, \mathbb{C})$. This leads to the introduction of constants $T^{aut}_{\pi,\sigma_i} = a_{\pi,\sigma_i} T^{mod}_{\pi,\sigma_i}$, $T^{aut}_{\sigma_i,\mathbb{C}} = b_{\sigma_i,\mathbb{C}} T^{mod}_{\sigma_i,\mathbb{C}}$ and $T^{aut}_{\pi,\sigma_i} = c_{\pi,\tau_j} T^{mod}_{\pi,\tau_j}$, $T^{aut}_{\tau_j,\mathbb{C}} = d_{\tau_j,\mathbb{C}} T^{mod}_{\tau_j,\mathbb{C}}$ as in Section 0.0.3. We denote by $\gamma_{\pi,\sigma_i} = a_{\pi,\sigma_i} \cdot b_{\sigma_i,\mathbb{C}}$, $I^{mod}_{\pi,\sigma_i} = T^{mod}_{\sigma_i,\mathbb{C}} \circ T^{mod}_{\pi,\sigma_i} \in V^*$ and similarly by $\delta_{\pi,\tau_j} = c_{\pi,\tau_j} \cdot d_{\tau_j,\mathbb{C}}$, $I^{mod}_{\pi,\tau_j} = T^{mod}_{\tau_j,\mathbb{C}} \circ T^{mod}_{\pi,\tau_j} \in V^*$. The spectral decomposition for two triples of Gelfand pairs $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F})$ and the corresponding orbits implies the following identity

$$\sum_{\sigma_i} \gamma_{\pi,\sigma_i} \cdot I^{mod}_{\pi,\sigma_i} = I_{\mathcal{F}} = \sum_{\tau_j} \delta_{\pi,\tau_j} \cdot I^{mod}_{\pi,\tau_j} \; . \tag{1}$$

We call such an identity the *Rankin-Selberg type spectral identity* or the *period identity* associated with the Gelfand formation $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{F})$, the corresponding orbits and the automorphic representation π. Note that the summation on the left in (1) is over the set of irreducible representations of $\mathcal{H}_1$ occurring in $L^2(X_1)$ and the summation on the right is over the set of irreducible representations of $\mathcal{H}_2$ occurring in $L^2(X_2)$. Since groups $\mathcal{H}_1$ and $\mathcal{H}_2$ might be quite different, the identity (1) is non-trivial in general. Surprisingly, even if $\mathcal{H}_1$ and $\mathcal{H}_2$ are conjugate in $\mathcal{G}$, the resulting identity is non-trivial in general (e.g., for $\mathcal{G} = PGL_2(\mathbb{R})$ and two unipotent subgroups intersecting over $\mathcal{F} = e$, this gives the Voronoĭ type summation formula for Fourier coefficients of cusp forms or of Eisenstein series).

The above identity is the identity between functionals on V. It is easy to translate it into the identities for weighted sums of coefficients γ's and δ's. Let $v \in V$ be a vector. It will play the role of a test function. As we explained in Section 0.0.3, in order to construct model periods, we have to consider model realizations of all corresponding representations. In particular, we can view $v \in V$ as a function on some manifold (or a section of a vector bundle). The resulting functionals I^{mod}_{π,σ_i} and I^{mod}_{π,τ_j} could be viewed as integral transforms on the spaces of such functions. Hence, we obtain for any $v \in V$, the identity

$$\sum_{\sigma_i} \gamma_{\pi,\sigma_i} \cdot I^{mod}_{\pi,\sigma_i}(v) = \sum_{\tau_j} \delta_{\pi,\tau_j} \cdot I^{mod}_{\pi,\tau_j}(v) \tag{2}$$

for the weighted sums of products of automorphic periods $\gamma_{\pi,\sigma_i} = a_{\pi,\sigma_i} \cdot b_{\sigma_i,\mathbb{C}}$, and $\delta_{\pi,\tau_j} = c_{\pi,\tau_j} \cdot d_{\tau_j,\mathbb{C}}$. The main point of (2) is that the weights $I^{mod}_{\pi,\sigma_i}(v)$ and $I^{mod}_{\pi,\tau_j}(v)$ could be computed in some explicit models without any reference to the automorphic picture. We show in [1] that as a special case, these identities include the classical Rankin-Selberg identity.

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Cuspidal and non-cuspidal discrete series for reductive symmetric spaces

Henrik Schlichtkrull

(joint work with Nils B. Andersen, Mogens Flensted-Jensen)

1. Recall the following results of Harish-Chandra [5]. Let G be a connected semisimple real Lie group with finite center (or more generally, reductive of Harish-Chandra's class), and let $K \subset G$ be a maximal compact subgroup with corresponding Cartan involution θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the corresponding decomposition of the Lie algebra, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace.

Let $\mathcal{C}(G)$ denote the Schwartz space for G, which is dense in $L^2(G)$. By definition, a *cusp form* on G is a function $f \in \mathcal{C}(G)$ such that

$$\int_N f(xny)\, dn = 0$$

for all parabolic subgroups $P = MAN \subsetneq G$, and all $x, y \in G$ (the integral converges absolutely for all $f \in \mathcal{C}(G)$). Let $L^2_{\mathrm{ds}}(G)$ denote the sum of all the discrete series represention in $L^2(G)$. It is both left and right invariant, and the intersection $\mathcal{C}_{\mathrm{ds}}(G) = L^2_{\mathrm{ds}}(G) \cap \mathcal{C}(G)$ is a dense subspace.

Theorem 1 (Harish-Chandra)**.** *$\mathcal{C}_{\mathrm{ds}}(G)$ is exactly the space of cusp forms. It is non-zero if and only if G and K have equal rank.*

The Plancherel decomposition splits $L^2(G)$ into a finite sum of *series*, each of which is related to a particular cuspidal parabolic subgroup $P = MAN$ (that is, $\operatorname{rank} M = \operatorname{rank} M \cap K$). The splitting can be accomplished as follows.

Let $\mathfrak{h}_1, \dots, \mathfrak{h}_r$ be a complete (up to conjugation) set of θ-stable Cartan subalgebras in $\mathfrak{g}$, and let $\mathfrak{a}_i = \mathfrak{h}_i \cap \mathfrak{p}$ for $i = 1, \dots, r$. For each $i = 1, \dots, r$, let P_i be a parabolic subgroup with Langlands decomposition $M_iA_iN_i$ such that $A_i = \exp \mathfrak{a}_i$. We can arrange that $\mathfrak{a}_1 = \mathfrak{a}$, then P_1 is a minimal parabolic subgroup.

We now define $\mathcal{C}_i(G) \subset \mathcal{C}(G)$ as the set of functions $f \in \mathcal{C}(G)$ for which

- f is orthogonal to all $h \in \mathcal{C}(G)$ with $\int_{N_i} h(xny)\, dn = 0$ for all $x, y \in G$.
- $\int_N f(xny) = 0$ for all $x, y \in G$, for all cuspidal parabolic subgroups some conjugate of which is properly contained in P_i.

In particular, for $i = 1$ the second condition is vacuous and $\mathcal{C}_1(G)$, which is called the *most-continuous* series, is just the orthocomplement of space of functions annihilated by all integrals $\int_{N_1} g(xny)\,dn$. On the other hand, if $\operatorname{rank} G = \operatorname{rank} K$, then we can arrange that $\mathfrak{a}_r = \{0\}$ and $N_r = \{e\}$. Then for $i = r$ the first condition is vacuous, and $\mathcal{C}_r(G)$ is the space $\mathcal{C}_{\mathrm{ds}}(G)$ of cusp forms.

Theorem 2 (Harish-Chandra)**.** *The following is an orthogonal direct sum*

$$\mathcal{C}(G) = \oplus_{i=1}^{r} \mathcal{C}_r(G).$$

In Harish-Chandra's Plancherel decomposition, each piece $\mathcal{C}_i(G)$ (or its closure in $L^2(G)$) is further decomposed into generalized principal series representations induced from P_i.

2. Let now G/H be a semisimple symmetric space, that is, the homogeneous space of a subgroup $G_0^\sigma \subset H \subset G^\sigma$, where $\sigma\colon G \to G$ is an involution, G^σ the group of its fixed points, and G_0^σ the identity component of this group. The problem of obtaining the Plancherel decomposition for $L^2(G/H)$ has been solved (see [4],[7],[1],[3],[2]), and the outcome is similar to what was described above for $L^2(G)$. One can define a Schwartz space $\mathcal{C}(G/H)$, and again there is a finite decomposition

$$\mathcal{C}(G/H) = \oplus_i \mathcal{C}_i(G/H)$$

where each piece decomposes as a direct integral of representations induced from a particular parabolic subgroup. However, the pieces in this decomposition are defined representation theoretically. The main topic of the talk is:

Question: *Is there a description of the $\mathcal{C}_i(G/H)$ through integrals over subgroups N (or $N/N \cap H$), similar to that for G? In particular, can the discrete series be characterized through some reasonable definition of cusp forms?*

3. The parabolic subgroup P_i from which the representations in $\mathcal{C}_i(G/H)$ are induced, is a so-called $\sigma\theta$-stable parabolic subgroup, and it would be tempting to apply the unipotent radical N_i of that in a definition of cusp forms on G/H:

$$\int_{N_i} f(gnH)\,dn = 0 \quad (g \in G, P_i \neq G). \tag{1}$$

However, in the group case, where G is considered as a symmetric space for $G \times G$, these subgroups of $G \times G$ are of the form $P \times \theta(P)$, where $P \subset G$ is as in Harish-Chandra's case, and thus the integral (1) becomes a double integral over both N and $\theta(N)$. In general, this integral does not converge (see however [6] for the special case of L^1-discrete series for G/H).

4. The *minimal* $\sigma\theta$-stable parabolic subgroup $P_{\min}$ is obtained as follows. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition according to σ. We may assume that σ and θ commute, and can arrange that $\mathfrak{a}_q := \mathfrak{a} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$. The set Σ of non-zero weights of $\mathfrak{a}_q$ in $\mathfrak{g}$ is a root system, and $P_{\min} = MA_qN$ is determined from a choice Σ^+ of positive roots. Here $A_q = \exp \mathfrak{a}_q$, and MA_q is its centralizer. The *most-continuous series* for G/H is induced from this parabolic (see [1]).

The following was suggested by Flensted-Jensen (Oberwolfach Tagungsbericht 28/2000). We choose a system of positive roots for $\mathfrak{a}$ in $\mathfrak{g}$, such that Σ^+ consists of the non-zero restrictions to $\mathfrak{a}_q$. Furthermore we choose an ordering on $\mathfrak{a}_q$ compatible with Σ^+, and subsequently an ordering on $\mathfrak{a}_h = \mathfrak{a} \cap \mathfrak{h}$, compatible with the positive system. We now define the following subspaces of the Lie algebra $\mathfrak{n}$ of N:

$\mathfrak{n}_+ = \sum_\beta \mathfrak{g}^\beta$, where β is a root with $\beta|_{\mathfrak{a}_q} > 0$ and $\beta|_{\mathfrak{a}_h} > 0$.

$\mathfrak{n}_- = \sigma\theta(\mathfrak{n}^+) = \sum_\beta \mathfrak{g}^\beta$, where β is a root with $\beta|_{\mathfrak{a}_q} > 0$ and $\beta|_{\mathfrak{a}_h} < 0$.

$\mathfrak{n}_0 = \sum_\beta \mathfrak{g}^\beta$. where β is a root with $\beta|_{\mathfrak{a}_q} > 0$ and $\beta|_{\mathfrak{a}_h} = 0$.

Then $\mathfrak{n} = \mathfrak{n}_+ \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_-$, and $\mathfrak{n}^* := \mathfrak{n}_+ \oplus \mathfrak{n}_0$ is a subalgebra. Let $N^* = \exp(\mathfrak{n}^*)$. The suggestion is to replace N by N^* in (1) and consider integrals

$$\int_{N^*} f(gnH)\,dn \tag{2}$$

for a possible definition of *cusp forms* on G/H. A similar construction can be done for the other $\sigma\theta$-stable parabolic subgroups. It is easily seen that in the group case, the integrals (2) amount exactly to those of the original definition of Harish-Chandra.

5. In joint work with N. Andersen and M. Flensted-Jensen (in preparation), we consider the special case of a hyperbolic symmetric space

$$G/H = \mathrm{SO}_0(p,q)/\mathrm{SO}_0(p,q-1).$$

In this case, if $p+q \geq 5$ there exist Schwartz functions for which the integral $\int_N g(nH)\,dn$ does not converge, when N is the unipotent radical of $P_{\min}$. However, with the subgroup N^* defined above, the situation is different.

Theorem 3. *Let* $G/H = \mathrm{SO}_0(p,q)/\mathrm{SO}_0(p,q-1)$.

i) The integral $\int_{N^*} f(gnH)\,dn$ *converges absolutely for all* $f \in \mathcal{C}(G/H)$.

ii) Assume $p \geq q$. *The integral* (2) *vanishes for all* $g \in G$ *and every* $f \in \mathcal{C}(G/H)$ *which generates a discrete series representation.*

iii) Assume $p < q$. *There exists a non-empty and finite family of discrete series representations with the following property. Assume* $f \in \mathcal{C}(G/H)$ *generates a discrete series representation. The integral* (2) *vanishes for all* $g \in G$ *if and only if this representation does not belong to the family.*

The exceptional discrete series which occur are called *non-cuspidal.* In fact, in the case $p < q$ it is shown that there exists a function $f \in \mathcal{C}(G/H)$, which generates a discrete series and for which $f(gH) > 0$ for all $g \in G$. It follows that the integral $\int_N f(nH)\,dn$ will be non-zero for *all* subgroups $N \subset G$, for which it converges. This means that the answer to the question given above is negative.

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The Kashiwara-Vergne conjecture and Drinfeld's associators

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(joint work with Charles Torossian)

The Kashiwara-Vergne conjecture [7] is the following property of the Campbell-Hausdorff series $z(x,y) = \ln(e^x e^y)$:

Conjecture: There exists a pair of Lie series $(A(x,y), B(x,y))$ such that

$$z(x,y) = x + y + (1 - e^{-\mathrm{ad}_x})A(x,y) + (e^{\mathrm{ad}_y} - 1)B(x,y) \tag{1}$$

and for all finite dimensional Lie algebras one has

$$\mathrm{Tr}(\mathrm{ad}_x \partial_x A + \mathrm{ad}_y \partial_y B) = \mathrm{Tr}(f(\mathrm{ad}_x) + f(\mathrm{ad}_y) - f(\mathrm{ad}_z)), \tag{2}$$

where $\partial_x A(z) = dA(x+tz,y)/dt|_{t=0}$ and f is a formal power series in one variable.

Remark. The even part of the power series f is completely determined by equation (1),

$$f_{even}(s) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{B_{2n}s^{2n}}{(2n)!},$$

where B_{2n} are Bernoulli numbers.

The Kashiwara-Vergne conjecture implies the Duflo theorem [6] on the isomorphism between the center of the universal enveloping algebra and the ring of invariant polynomials (for finite dimensional Lie algebras over a field of characteristic zero). It also implies the ring isomorphism in cohomology $H(g, Ug) \cong H(g, Sg)$ (see [8]) and an analogue of the Duflo theorem for distributions [2, 3, 4]. The Kashiwara-Vergne conjecture was proved in [1] based on an earlier work [9].

Here we address the uniqueness issue for equations (1-2). As a main tool, we introduce a family of pro-unipotent groups KV_n for $n \geq 2$. The group KV_2 acts freely and transitively on the set of solutions of the Kashiwara-Vergne conjecture. We show that KV_2 contains a subgroup isomorphic to the Grothendieck-Teichmüller group GRT introduced in [5]. Furthermore, we show that the set of solutions of the Kashiwara-Vergne conjecture is a direct product of the a $\mathrm{Assoc}(\mathrm{KV}_3)$ and a line. Here $\mathrm{Assoc}(\mathrm{KV}_3)$ stands for the set of solutions of the Drinfeld's equations for associators taking values in the group KV_3. We show that the set of Drinfeld's associators Assoc_D is a subset of $\mathrm{Assoc}(\mathrm{KV}_3)$.

As a byproduct, we obtain a new proof of the Kashiwara-Vergne conjecture. Indeed, in [5] Drinfeld shows that Assoc_D is nonempty. Hence, the same is true for $\mathrm{Assoc}(\mathrm{KV}_3)$. And this implies that the set of solutions of equations (1-2) is also nonepmty.

The group $\mathrm{GRT} \subset \mathrm{KV}_2$ acts on the set of solutions of the Kashiwara-Vergne conjecture. This action descends to the space of odd power series f (see equation (2)). We show that this action is transitive. That is, the Kashiwara-Vergne conjecture admits solutions for all possible formal power series f with even part given by Bernoulli numbers.

We conjecture that $\mathrm{KV}_2 \cong \mathrm{GRT}$. This would imply $\mathrm{Assoc}(\mathrm{KV}_3) = \mathrm{Assoc}_D$. At the moment there is no evidence niether in favor no against this conjecture.

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On Lipsman's Conjecture

Taro Yoshino

1. Lipsman's Conjecture on Proper Actions

In this note, we discuss Lipsman's conjecture on proper actions, and give its counterexample. First, let us recall the statement of the conjecture.

Conjecture 1 (Lipsman [L95]). (A) *Suppose a simply connected nilpotent Lie group L acts on $\mathbb{R}^n$ affinely and unipotently. Then, the action is proper iff it is fixed point free.*

(B) *Suppose G is a simply connected nilpotent Lie group and L, H are its closed subgroups. Then the L-action on the homogeneous space G/H is proper iff it is fixed point free.*

Note that (A) is a special case of (B). In fact, if we denote by $N(n)$ the nilpotent Lie group consisting of $n \times n$ upper triangular matrices, then any affine and unipotent action on $\mathbb{R}^n$ factors through the natural action of $N(n+1)$ on $G/H = N(n+1)/N(n) \simeq \mathbb{R}^n$.

The point of Lipsman's conjecture is to give a useful criterion for proper actions in the sense of Palais. Suppose a Lie group L acts on a manifold M.

Definition 2. *The L-action on M is proper if the following map is proper.*

$$\Phi : L \times M \to L \times L, \quad (l, x) \mapsto (lx, x).$$

In other words, the inverse image of any compact set $S \subset M \times M$ is again compact. This is also equivalent to the following condition:

$$L_S := \{l \in L \ : \ l(S) \cap S \neq \emptyset\} \text{ is compact for any compact set } S \subset M.$$

Recall that the action of non-compact group may have wild behaviour. For instance, the quotient space is not always Hausdorff, though the quotient space by a compact group is always Hausdorff. Among actions of non-compact Lie groups, 'proper action' has the following nice properties:

Property 3. *Suppose a Lie group L acts on a manifold M properly.*

(1) *The quotient space $L\backslash M$ is Hausdorff.*
(2) *If L is discrete, the L-action on M is properly discontinuous.*
(3) *If L is discrete and torsion free, then the quotient map $\pi : M \to L\backslash M$ is covering.*

2. Criterion of Proper Actions for Reductive Case

Kobayashi gave a criterion for proper actions in [K89] in the reductive case. This is restated in [K92] as follows:

Fact 4 ([K89])**.** *Suppose G is a linear reductive Lie group, and L, H are its reductive subgroups. Then the L-action on the homogeneous space G/H is proper iff it satisfies (CI)-property.*

For a Lie group L acting on a manifold M, we recall from [K92]:

Definition 5. *The L-action is said to have (CI)-property if the isotropy group at any point $x \in M$ is compact. In other words, $L_{\{x\}}$ is compact for any $x \in M$.*

Remark that properness implies (CI)-property, because $S = \{x\}$ is compact. Since (CI)-property is much easier to check, Fact 4 gives us an easy way to check the properness of actions. And this criterion became a big breakthrough in the study of discontinuous group. In fact, since middle of 90's, progress in this area has been made by using techniques from different fields of mathematics by a number of people, including Benoist, Labourie, Zimmer, Mozes, Witte, Iozzi, Margulis, Oh, Shalom, etc.

3. Criterion of Proper Actions for Nilpotent Case

It should be so nice if we can find a criterion for proper actions also in non-reductive case. Lipsman's conjecture is one of such attempts. More precisely, he conjectured that a similar statement to Fact 4 also holds in nilpotent case. Note that (CI)-property is equivalent to fixed point freeness in the nilpotent case, because the only compact subgroup of L is trivial. In Conjecture 1, (A) has been proved to be true for lower dimensional cases, and (B) was proved to be true if G is k-step nilpotent for $k \leq 3$. We summarize these known results:

(A) for n-dimensional case	(B) for k-step nilpotent case
$n = 2$ [K92]	$k = 2$ [N01]
$n = 3$ [L95]	$k = 3$ [B05, Y07, Pü07]
$n = 4$ [Y04]	

4. Counterexample to Lipsman's Conjecture

All the known results so far have been affirmative ones. However, our aim in this note is to give a counterexample to Lipsman's conjecture, (A) for $n \geq 5$ and (B) for $k \geq 4$. We set

$$X(a,b) := \begin{pmatrix} 0 & 0 & 0 & 0 & -b & a \\ & 0 & a & 0 & 0 & b \\ & & 0 & a & 0 & 0 \\ & & & 0 & a & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} \in \mathfrak{n}(6),$$

and let L be the two-dimensional abelian subgroup of $N(6)$ whose Lie algebra is given by

$$\mathfrak{l} := \{X(a,b) \ : \ a, b \in \mathbb{R}\}.$$

Then, we claim the L-action on $\mathbb{R}^5 = N(6)/N(5)$ is *not* proper, but fixed point free. To see this action is not proper, define a compact subset S of $\mathbb{R}^5$ by

$$S := \left\{ {}^t(0,0,0,p,q) \ : \ p = \pm\sqrt{3}, \quad 0 \leq q \leq 2\sqrt{3} \right\} \subset \mathbb{R}^5,$$

then the subset L_S contains an unbounded set

$$\exp\left\{ X\left(a, -\frac{a^2}{2\sqrt{3}}\right) \in \mathfrak{l} \ : \ a \in \mathbb{R} \right\} \subset L_S.$$

Thus, the L-action is not proper. On the other hand, fixed point freeness follows from direct calculation. Thus, (A) in the conjecture fails if $n \geq 5$. Furthermore, essentially the same example gives a counterexample to (B) for k-step nilpotent case for $k \geq 4$.

We end this note with a remark on the topology of the quotient space $Q := L\backslash\mathbb{R}^n$. There are two special lines l_1 and l_2 in Q. The space Q is *locally Hausdorff* in the sense that $Q \setminus l_i$ ($i = 1$ or 2) is Hausdorff. But, Q itself is not Hausdorff because any two points $x_1 \in l_1$ and $x_2 \in l_2$ cannot be separated by open sets.

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Distribution vectors invariant under small nilpotent subgroups and applications

Martin Olbrich

Let G be a connected simple Lie group of real rank one, $P = MAN \subset G$ a minimal parabolic subgroup, and let $U \subset N$ be a connected subgroup. I discuss the structure of the space of U-invariant distribution vectors of principal series representations induced from P. In the classical case $U = N$ this space is finite-dimensional, and it is closely related to the intertwining operators for principal series representations. In contrast, for $U \neq N$, this space is infinite-dimensional, and it has not been studied before. U-invariant distribution vectors appear naturally in the study of constant terms along cusps of automorphic forms on $\Gamma\backslash G$, where $\Gamma \subset G$ is a geometrically finite discrete subgroup (typically of infinite co-volume). The case $U \neq N$ appears precisely if the corresponding cusp has infinite volume.

For generic induction parameter, the space of U-invariants can be identified with the space of global sections of a sheaf on an auxiliary manifold Z_U (which mimics $U\backslash G/P$). I conjecture that this sheaf is canonically isomorphic to the sheaf of distributional sections of a certain vector bundle over Z_U. I am able to verify this conjecture in a number of cases, in particular for $G = SO_0(1,n)$. I regard this conjecture as a further manifestation of the principle "Quantization commutes with Reduction". Indeed, principal series representations arise as quantizations of loxodromic adjoint orbits of G, whereas the quantization of the symplectic reduction of the orbit with respect to the U-action results in a section space over the manifold Z_U. Note that the action of U on the orbit is not proper.

In a second part of the talk, I draw some consequences of the above for a geometric construction of Eisenstein series for $\Gamma\backslash G$, Γ geometrically finite. The talk is based on the recent version of the joint paper with U. Bunke [3], but it takes a quite different point of view. In [3] the meromorphic continuation of the Eisenstein series for $\Gamma\backslash G$ is obtained (except for $G = F_4^{(-20)}$). Note that this result includes the case of so-called irrational cusps as well as the finite volume case (for which it provides a new method of meromorphic continuation of the Eisenstein series). For further information on geometry and analysis related to geometrically finite groups I refer to [1, 4, 5, 2].

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A comparison of Paley-Wiener spaces for real reductive Lie groups

Sofiane Souaifi

(joint work with Erik P. van den Ban)

In [1], J. Arthur describes the Paley-Wiener space PW_A for a real reductive Lie group G using the so-called Arthur-Campoli relations. In 2006, E. P. van den Ban and H. Schlichtkrull generalized the Paley-Wiener theorem to the case of reductive symmetric spaces (see [2]). Their description of the Paley-Wiener space is given in some sense in terms of Arthur-Campoli relations.

In 2005, P. Delorme ([3]) re-established the Paley-Wiener theorem for G using a totally different approach. In the spirit of Zelobenko, [5], Delorme uses induction on the length of K-types of admissible representations and a subquotient theorem on Harish-Chandra modules ([4]). The conditions given by Delorme to describe the Paley-Wiener space PW_D are in terms of intertwining relations.

Obviously, knowing the Paley-Wiener theorem on real reductive Lie groups, these two sets of conditions are equivalent. The main point of this talk will be to give a proof of this equivalence without using the Paley-Wiener theorem.

The main idea deals with some Hecke algebra $\mathcal{H}(G,K)$ and its representation theory. Let P be a minimal parabolic subgroup of G with $P = MAN$ its Langlands decomposition. Let PW_{pre} be the pre-Paley-Wiener space for G (a space of functions on $\hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ with endomorphism values, satisfying some growth conditions in the second variable). The Arthur-Campoli relations can be rewritten as follows. Any element of PW_{pre} is in PW_A if, for a certain family $(\pi_\delta, V_\delta)_\delta$

of $\mathcal{H}(G,K)$-modules, it induces pointwise an endomorphism annihilating the orthogonal $\pi_\delta(\mathcal{H}(G,K))^\perp$ of the image of $\mathcal{H}(G,K)$ in $\mathrm{End}(V_\delta)$. The intertwining conditions given by Delorme can be described in the following way: any element of PW_{pre} is in PW_D if, for the same family $(\pi_\delta, V_\delta)_\delta$, it induces pointwise an element of $\mathrm{End}(V_\delta)^\#$ (endomorphisms of V_δ preserving invariant subspaces of $V_\delta^{\oplus n}$ for any $n \in \mathbb{N}$). Via some double orthogonal arguments, $\pi_\delta(\mathcal{H}(G,K))^{\perp\perp} = \pi_\delta(\mathcal{H}(G,K))$, and, using a kind of double commutant theorem, $\mathrm{End}(V_\delta)^\# = \pi_\delta(\mathcal{H}(G,K))$. Hence the equality $PW_A = PW_D$ follows.

In conclusion, we see that the two sets of conditions due respectively to Arthur and Delorme can be written simply in terms of the Hecke algebra $\mathcal{H}(G,K)$.

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Schwartz functions on Nash manifolds and applications to representation theory

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(joint work with Dmitry Gourevitch and Eitan Sayag)

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Let us start with the following motivating example. Consider the circle S^1, let $N \subset S^1$ be the north pole and denote $U := S^1 \setminus N$. Note that U is diffeomorphic to $\mathbb{R}$ via the stereographic projection. Consider the space $\mathcal{D}(S^1)$ of distributions on S^1, that is the space of continuous linear functionals on the Fréchet space $C^\infty(S^1)$. Consider the subspace $\mathcal{D}_{S^1}(N) \subset \mathcal{D}(S^1)$ consisting of all distributions supported at N. Then the quotient $\mathcal{D}(S^1)/\mathcal{D}_{S^1}(N)$ will not be the space of distributions on U. However, it will be the space $\mathcal{S}^*(U)$ of Schwartz distributions on U, that is continuous functionals on the Fréchet space $\mathcal{S}(U)$ of Schwartz functions on U. In this case, $\mathcal{S}(U)$ can be identified with $\mathcal{S}(\mathbb{R})$ via the stereographic projection.

The space of Schwartz functions on $\mathbb{R}$ is defined to be the space of all infinitely differentiable functions that rapidly decay at infinity together with all their derivatives, i.e. $x^n f^{(k)}$ is bounded for any n, k.

In this talk we extend the notions of Schwartz functions and Schwartz distributions to a larger geometric realm.

As we can see, the definition is of algebraic nature. Hence it would not be reasonable to try to extend it to arbitrary smooth manifolds. However, it is reasonable to extend this notion to smooth algebraic varieties. Unfortunately, sometimes this is not enough. For example, a connected component of real algebraic variety is not always an algebraic variety. By this reason we extend this notion to smooth semi-algebraic manifolds. They are called **Nash manifolds**.

For any Nash manifold M, we will define the spaces $\mathcal{G}(M)$, $\mathcal{T}(M)$ and $\mathcal{S}(M)$ of **generalized Schwartz functions**[1], **tempered functions** and **Schwartz functions** on M. Informally, $\mathcal{T}(M)$ is the ring of functions that have no more than polynomial growth together with all their derivatives, $\mathcal{G}(M)$ is the space of generalized functions with no more than polynomial growth and $\mathcal{S}(M)$ is the space of functions that decay together with all their derivatives faster than any inverse power of a polynomial.

As in the classical case, in order to define generalized Schwartz functions, we have to define Schwartz functions first. Both $\mathcal{G}(M)$ and $\mathcal{S}(M)$ are modules over $\mathcal{T}(M)$.

The triple $\mathcal{S}(M)$, $\mathcal{T}(M)$, $\mathcal{G}(M)$ is analogous to $C_c^\infty(M)$, $C^\infty(M)$ and $C^{-\infty}(M)$ but it has additional nice properties as we will see later.

We will show that for $M = \mathbb{R}^n$, $\mathcal{S}(M)$ is the space of classical Schwartz functions and $\mathcal{G}(M)$ is the space of classical generalized Schwartz functions. For compact Nash manifold M, $\mathcal{S}(M) = \mathcal{T}(M) = C^\infty(M)$.

Main results.

Result 1. *Let M be a Nash manifold and $Z \subset M$ be a closed Nash submanifold. Then the restriction maps $\mathcal{T}(M) \to \mathcal{T}(Z)$ and $\mathcal{S}(M) \to \mathcal{S}(Z)$ are onto.*

Result 2. *Let M be a Nash manifold and $U \subset M$ be a semi-algebraic open subset. Then a Schwartz function on U is the same as a Schwartz function on M which vanishes with all its derivatives on $M \setminus U$.*

This theorem tells us that extension by zero $\mathcal{S}(U) \to \mathcal{S}(M)$ is a closed imbedding, and hence restriction morphism $\mathcal{G}(M) \to \mathcal{G}(U)$ is onto.
Classical generalized functions do not have this property. This was our main motivation for extending the definition of Schwartz functions.

Schwartz sections of Nash bundles. Similar notions will be defined for Nash bundles, i.e. smooth semi-algebraic bundles.

For any Nash bundle E over M we will define the spaces $\mathcal{G}(M, E)$, $\mathcal{T}(M, E)$ and $\mathcal{S}(M, E)$ of generalized Schwartz, tempered and Schwartz sections of E.

As in the classical case, a generalized Schwartz function is not exactly a functional on the space of Schwartz functions, but a functional on Schwartz densities, i.e. Schwartz sections of the bundle of densities.

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[1] Here we distinguish between the (similar) notions of a generalized function and a distribution. They can be identified by choosing a measure. Without fixing a measure, a smooth function defines a generalized function but not a distribution.

Therefore, we will define generalized Schwartz sections by $\mathcal{G}(M, E) = (\mathcal{S}(M, \widetilde{E}))^*$, where $\widetilde{E} = E^* \otimes D_M$ and D_M is the bundle of densities on M.

Let $Z \subset M$ be a closed Nash submanifold, and $U = M \setminus Z$. Result 2 tells us that the quotient space of $\mathcal{G}(M)$ by the subspace $\mathcal{G}(M)_Z$ of generalized Schwartz functions supported in Z is $\mathcal{G}(U)$. Hence it is useful to study the space $\mathcal{G}(M)_Z$. As in the classical case, $\mathcal{G}(M)_Z$ has a filtration by the degree of transversal derivatives of delta functions. The quotients of the filtration are generalized Schwartz sections over Z of symmetric powers of normal bundle to Z in M, after a twist.

This result can be extended to generalized Schwartz sections of arbitrary Nash bundles.

Restricted topology and sheaf properties. Similarly to algebraic geometry, the reasonable topology on Nash manifolds to consider is a topology in which open sets are open semi-algebraic sets. Unfortunately, it is not a topology in the usual sense of the word, it is only what is called **restricted topology**. This means that the union of an infinite number of open sets does not have to be open. The only open covers considered in the restricted topology are finite open covers.

The restriction of a generalized Schwartz function (respectively of a tempered function) to an open subset is again a generalized Schwartz (respectively a tempered function). This means that they form pre-sheaves. We will show that they are actually sheaves, which means that for any finite open cover $M = \bigcup_{i=1}^{n} U_i$, a function α on M is tempered if and only if $\alpha|_{U_i}$ is tempered for all i. It is of course not true for infinite covers. We denote the sheaf of generalized Schwartz functions by $\mathcal{G}_M$ and of the sheaf of tempered functions by $\mathcal{T}_M$. By result 2, $\mathcal{G}_M$ is a flabby sheaf.

Similarly, for any Nash bundle E over M we will define the sheaf $\mathcal{T}_M^E$ of tempered sections and the sheaf $\mathcal{G}_M^E$ of generalized Schwartz sections.

As we have mentioned before, Schwartz functions behave similarly to compactly supported smooth functions. In particular, they cannot be restricted to an open subset, but can be extended by zero from an open subset. This means that they do not form a sheaf, but an object dual to a sheaf, a so-called cosheaf. We denote the cosheaf of Schwartz functions by $\mathcal{S}_M$. We will prove that $\mathcal{S}_M$ is actually a cosheaf and not just pre-cosheaf by proving a Schwartz version of the partition of unity theorem. Similarly, for any Nash bundle E over M we will define the cosheaf $\mathcal{S}_M^E$ of Schwartz sections.

Possible applications. Schwartz functions are used in the representation theory of algebraic groups. Our definition coincides with Casselman's definition (cf. [Cas1]) for algebraic groups. Our paper allows to use Schwartz functions in more situations in the representation theory of algebraic groups, since an orbit of an algebraic action is a Nash manifold, but does not have to be an algebraic group or even an algebraic variety.

Generalized Schwartz sections can be used for "devisage". We mean the following. Let $U \subset M$ be an open (semi-algebraic) subset. Instead of dealing with generalized Schwartz sections of a bundle on M, we can deal with generalized Schwartz sections of its restriction to U and generalized Schwartz sections of some other bundles on $M \setminus U$.

For example if we are given an action of an algebraic group G on an algebraic variety M, and a G-equivariant bundle E over M, then devisage to orbits helps us to investigate the space of G-invariant generalized sections of E.

Summary. To sum up, for any Nash manifold M we define a sheaf $\mathcal{T}_M$ of algebras on M (in the restricted topology) consisting of tempered functions, a sheaf $\mathcal{G}_M$ of modules over $\mathcal{T}_M$ consisting of generalized Schwartz functions, and a cosheaf $\mathcal{S}_M$ of modules over $\mathcal{T}_M$ consisting of Schwartz functions.

Moreover, for any Nash bundle E over M we define sheaves $\mathcal{T}_M^E$ and $\mathcal{G}_M^E$ of modules over $\mathcal{T}_M$ consisting of tempered and generalized Schwartz sections of E respectively and a cosheaf $\mathcal{S}_M^E$ of modules over $\mathcal{T}_M$ consisting of Schwartz sections of E.

Let us list the main properties of these objects that we will prove in this paper:

1. *Compatibility: For open semi-algebraic subset* $U \subset M$, $\mathcal{S}_M^E|_U = \mathcal{S}_U^{E|_U}$, $\mathcal{T}_M^E|_U = \mathcal{T}_U^{E|_U}$, $\mathcal{G}_M^E|_U = \mathcal{G}_U^{E|_U}$.

2. $\mathcal{S}(\mathbb{R}^n)$ = *Classical Schwartz functions on* $\mathbb{R}^n$.

3. *For compact* M, $\mathcal{S}(M, E) = \mathcal{T}(M, E) = C^\infty(M, E)$.

4. $\mathcal{G}_M^E = (\mathcal{S}_M^{\widetilde{E}})^*$, *where* $\widetilde{E} = E^* \otimes D_M$ *and* D_M *is the bundle of densities on M.*

5. *Let* $Z \subset M$ *be a closed Nash submanifold. Then the restriction maps* $\mathcal{S}(M, E)$ *onto* $\mathcal{S}(Z, E|_Z)$ *and* $\mathcal{T}(M, E)$ *onto* $\mathcal{T}(Z, E|_Z)$.

6. *Let* $U \subset M$ *be a semi-algebraic open subset, then*

$$\mathcal{S}_M^E(U) \cong \{\phi \in \mathcal{S}(M, E)| \quad \phi \textit{ is 0 on } M \setminus U \textit{ with all derivatives}\}.$$

7. *Let* $Z \subset M$ *be a closed Nash submanifold. Consider* $\mathcal{G}(M, E)_Z = \{\xi \in \mathcal{G}(M, E) | \xi \textit{ is supported in } Z\}$. *It has a canonical filtration such that its factors are canonically isomorphic to* $\mathcal{G}(Z, E|_Z \otimes S^i(N_Z^M) \otimes D_M^*|_Z \otimes D_Z)$ *where* N_Z^M *is the normal bundle of* Z *in* M *and* S^i *means i-th symmetric power.*

Remarks.

Remark 1. *Harish-Chandra has defined a Schwartz space for every reductive Lie group. However, Harish-Chandra's Schwartz space does not coincide with the space of Schwartz functions that we define in this paper even for the algebraic group* $\mathbb{R}^\times$.

Remark 2. *There is a different approach to the concept of Schwartz functions. Namely, if M is embedded as an open subset in a compact manifold K then one can define the space of Schwartz functions on M to be the space of all smooth functions on K that vanish outside M together with all their derivatives. This approach is implemented in* [CHM], [KS], [Mor] *and* [Pre]. *In general, this definition depends on the embedding into K. Our results show that for Nash manifolds M and K it coincides with our definition and hence does not depend on the embedding.*

Remark 3. *After the completion of this project we found out that many of the properties of Schwartz functions on affine Nash manifolds have been obtained already in* [dCl].

An application to representation theory. Using the theory of Schwartz functions we showed the following theorem:

Theorem 1. *Let F be either the field of real or complex numbers and consider the standard imbedding $\mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{n+1}(F)$. We consider the two-sided action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathrm{GL}_{n+1}(F)$ defined by $(g_1, g_2)h := g_1 h g_2^{-1}$. Then any $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ invariant distribution on $\mathrm{GL}_{n+1}(F)$ is invariant with respect to transposition.*

This theorem has the following corollary in representation theory.

Theorem 2. . *Let (π, E) be an irreducible admissible continuous representation of $\mathrm{GL}_{n+1}(F)$ on a Hilbert space E. Then*

$$dim Hom_{\mathrm{GL}_n(F)}(E^\infty, \mathbb{C}) \leq 1 \tag{1}$$

Clearly, the last theorem implies in particular that (1) holds for unitary irreducible representations of $\mathrm{GL}_{n+1}(F)$. That is, the pair $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$ is a generalized Gelfand pair in the sense of [vD].

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Mixed Periods for $GL(n)$ local and global

Omer Offen

(joint work with Eitan Sayag)

Globally, we study mixed (symplectic-Whittaker) period integrals of automorphic forms in the discrete spectrum of $GL(n)$. Let F be a number field and fix a non trivial character ψ of F. For every integer r denote by U_r the subgroup

GL_r consisting of upper triangular unipotent elements and let ψ_r be the generic character of U_r defined by

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r}).$$

For a decomposition $n = r + 2k$ let $H_{r,2k}$ be the subgroup of GL_n consisting of elements of the form

$$\begin{pmatrix} u & X \\ 0 & h \end{pmatrix},\ u \in U_r,\ X \in M_{r\times 2k},\ h \in Sp(2k)$$

and let $\psi_{r,2k}$ be the character of $H_{r,2k}$ defined by

$$\psi_{r,2k}\begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

A discrete spectrum automorphic representation π of $GL_n(\mathbb{A}_F)$ is called $(H_{r,2k}, \psi_{r,2k})$-distinguished if there exists an automorphic form ϕ in the space of π such that

$$\int_{H_{r,2k}(F)\backslash H_{r,2k}(\mathbb{A}_F)} \phi(h)\psi_{r,2k}\ dh \neq 0.$$

We show that for every irreducible, discrete spectrum automorphic representation π of $GL_n(\mathbb{A}_F)$ there is a unique integer $k = \kappa(\pi)$ such that $0 \leq k \leq [\frac{n}{2}]$ and π is $(H_{n-2k,2k}, \psi_{n-2k,2k})$-distinguished. Furthermore, when the data is unramified we express the mixed period integral in terms of special values of the Rankin-Selberg L-function of π and its contragradiant.

The global period integral is factorizable, and the corresponding local representations have an analogous theory of mixed models. In fact, the uniqueness of $\kappa(\pi)$ is a consequence of a local multiplicity one result that we now explain. We consider the models

$$\mathcal{M}_{r,2k} = \mathrm{Ind}_{H_{r,2k}(F)}^{GL_n(F)}(\psi_{r,2k}).$$

They were introduced by Klyachko in [Kl84] when F is a finite field. Let

$$m_\pi = \dim(\mathrm{Hom}_{GL_n(F)}(\pi, \oplus_{k=1}^{[\frac{n}{2}]}\mathcal{M}_{r,2k})).$$

It is proved in [IS91] that when F is a finite field then $m_\pi = 1$ for any irredicible representation of $GL_n(F)$. Over a local non archimedean field the problem was first considered in [HR90]. Heomus and Rallis showed, in particular, that there exist irreducible, admissible representations π of $GL_n(F)$ such that $m_\pi = 0$. We show that over a local field, the sum of the Klyachko models is of multiplicity one, i.e. that for every irreducible admissible representation π of $GL_n(F)$ we have $m_\pi \leq 1$ [OS2]. Furthermore, using the global results on purely symplectic periods [Off06b], we show over a p-adic field that if in addition π is unitary then $m_\pi = 1$ [OS1]. This is a joint work with Eitan Sayag.

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Branching laws for some unitary representations of GL(4,ℝ)

Bent Ørsted
(joint work with Birgit Speh)

1. Introduction

In this talk we consider the restriction of a unitary irreducible representation of type $A_{\mathfrak{q}}(\lambda)$ of $GL(4,\mathbb{R})$ to reductive subgroups H which are the fixpoint sets of an involution. We obtain a formula for the restriction to the symplectic group and to $GL(2,\mathbb{C})$, with applications to automorphic representations.

Understanding a unitary representation π of a Lie groups G often involves understanding its restriction to suitable subgroups H. This is in physics referred to as breaking the symmetry, and often means exhibiting a nice basis of the representation space of π. It may be thought of as the dual notion to induction of representations from H to G. Here (and to appear in [10]) we shall study in a special case a generalization of the method applied in [7] and again in [1]; this is a method of Taylor expansion of sections of a vector bundle along directions normal to a submanifold. This works nicely when the original representation is a holomorphic discrete series for G, and the subgroup H also admits holomorphic discrete series and is embedded in a suitable way in G. The branching law is a discrete sum decomposition, even with finite multiplicities, so-called admissibility of the restriction to H; and the summands are themselves holomorphic discrete series representations for H. Since holomorphic discrete series representations are cohomologically induced representations in degree zero, it is natural to attempt a generalization to other unitary representations of similar type, namely cohomologically induced representations in higher degree. We shall focus on the line bundle case, i.e. the $A_{\mathfrak{q}}(\lambda)$ representations. In this case T. Kobayashi [5] obtained necessary and sufficient conditions that the restriction is discrete and that each representation appears with finite multiplicity, the so-called *admissibility* of the representation relatively to the subgroup H. Using explicit resolutions and filtrations associated with the imbedding of H in G, we analyze the derived functor modules and obtain an explicit decomposition into irreducible representations. It is perhaps not surprising, that with the appropriate conditions on the imbedding of the subgroup, the class of (in our case derived functor) modules is preserved in the restriction from H to G.

Here is the general setting that we consider: Let G be a semisimple or reductive linear connected Lie group with maximal compact subgroup K and Cartan involution θ. Suppose that σ is another involution so that $\sigma \cdot \theta = \theta \cdot \sigma$ and let H be the fixpoint set of σ in G. Suppose that $L = L_x$ is the centralizer of an elliptic element $x \in G \cap H$ and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, $\mathfrak{q}^H = \mathfrak{q} \cap \mathfrak{h}$ be the corresponding θ–stable parabolic subgroups. Here we use as usual gothic letters for complex Lie algebras and subspaces thereof; a subscript will denote the real form, e.g. $\mathfrak{g}_o$. We say that pairs of parabolic subalgebras $\mathfrak{q}, \mathfrak{q}^H$ which are constructed this way are well aligned. For a unitary character λ of of L we define following Vogan/Zuckerman the unitary representations $A_{\mathfrak{q}}(\lambda)$.

We consider the example of the group $G = SL(4, \mathbb{R})$. There are 2 G-conjugacy classes of skew symmetric matrices with representants $Q_1 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$ and $Q_2 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let H_1 respectively H_2 be the symplectic subgroups defined by these matrices, and H_1', H_2' the centralizer of Q_1, respectively Q_2. All these subgroups are fixpoint sets of involutions $\sigma_i,\ i = 1, 2$ and $\sigma_i',\ i = 1, 2$ respectively.

The matrix Q_2 with centralizer L has finite order, is contained in all subgroups H_i and hence defines a θ stable parabolic subalgebra $\mathfrak{q}$ and also θ–stable parabolic subalgebras $\mathfrak{q}_{\mathfrak{h}_1} = \mathfrak{q} \cap \mathfrak{h}_1$ of $\mathfrak{h}_1$, respectively $\mathfrak{q}_{\mathfrak{h}_2} = \mathfrak{q} \cap \mathfrak{h}_2$ of $\mathfrak{h}_2$. We recall the construction of the representations $A_{\mathfrak{q}}(V)$, V an irreducible $(\mathfrak{q}, L \cap K)$ module, typically a character . We follow conventions of the book by Knapp/Vogan [3] (where much more detail on these derived functor modules is to be found - this is our standard reference) and will always consider representations of L and not of the metaplectic cover of L as some other authors. We consider $U(\mathfrak{g})$ as right $U(\mathfrak{q})$ module and write $V^\sharp = V \otimes \wedge^{\text{top}}\mathfrak{u}$. Let $\mathfrak{p}_L$ be a $L \cap K$–invariant complement of $\mathfrak{l} \cap \mathfrak{k}$ in $\mathfrak{l}$. We write $r_G = \mathfrak{p}_L \oplus \mathfrak{u}$. Now we introduce the derived functor modules as on page 167 in Knapp/Vogan - recall that this formalizes Taylor expansions of certain differential forms: Consider the complex

$$\begin{aligned} 0 \to \operatorname{Hom}_{L\cap K}(U(\mathfrak{g}), \operatorname{Hom}(\wedge^0 r_G, V^\sharp))_K \to& \\ \to \quad & \operatorname{Hom}_{L\cap K}(U(\mathfrak{g}), \operatorname{Hom}(\wedge^1 r_G, V^\sharp))_K \to \\ \to \quad & \operatorname{Hom}_{L\cap K}(U(\mathfrak{g}), \operatorname{Hom}(\wedge^2 r_G, V^\sharp))_K \to \dots \end{aligned}$$

which defines our $A_{\mathfrak{q}}$ as cohomology in the appropriate degree. We here consider trivial infinitesimal character, and using this complex we may analyze the branching law; the following proposition demonstrates how different imbeddings of the same subgroup (symplectic res. general linear complex) gives radically different branching laws; in essence it is contained, as a qualitative result, in the criteria of T. Kobayashi, see [4].

Proposition 1.

(1) *The restriction of $A_{\mathfrak{q}}$ to H_1 and to H_1' is a direct sum of irreducible representation each appearing with finite multiplicity.*

(2) *The restriction of* $A_{\mathfrak{q}}$ *to* H_2 *and to* H_2' *is not admissible and has continuous spectrum.*

We can prove the following

Theorem 2. *The representation* $A_{\mathfrak{q}}$ *restricted to* H_1 *is the direct sum of representations of the same cohomology type, each occuring with multiplicity one, namely (sum over suitable characters)*

$$A_{\mathfrak{q}|H_1} = \oplus_{n_1=0}^{\infty} A_{\mathfrak{q}\cap\mathfrak{h}_1}(\mu_1^{n_1} \otimes \lambda_{H_1}).$$

For the restriction of $A_{\mathfrak{q}}$ to the group H_1' we have a similar result, i.e. admissibility and the explicit branching law.

We may use our results to give different constructions of some known automorphic representations of $Sp(2,\mathbb{R})$ and $GL(2,\mathbb{C})$. We first explain the ideas in a more general setting. Again we may consider restrictions, this time in the obvious way of restricting functions on locally symmetric spaces to locally symmetric subspaces.

Assume first that G is a semisimple matrix group and Γ an arithmetic subgroup, H as before a subgroup of G. Then $\Gamma_H = \Gamma \cap H$ is an arithmetic subgroup of H. Let $V_\pi \subset L^2(G/\Gamma)$ be an irreducible $(\mathfrak{g}, K)$-submodule of $L^2(G/\Gamma)$. If $f \in V_\pi$ then f is a C^∞- function and so we define f_H as the restriction of f to H/Γ_H.

Suppose that the irreducible unitary $(\mathfrak{g}, K)$- module π is a submodule of $L^2(G/\Gamma)$ and that its restriction to H is a direct sum of unitary irreducible representations.

Proposition 3. *Under the above assumptions the restriction map* $RES_H(\pi)$ *is nonzero and its image is contained in the automorphic functions on* H/Γ_H.

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Heckman-Opdam hypergeometric functions and their specializations

Toshio Oshima
(joint work with Nobukazu Shimeno)

§0 Introduction. Heckman-Opdam hypergeometric function is defined by the second order differential operator

$$L(k) := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{\alpha\in\Sigma^+} k_\alpha \coth\langle\alpha, x\rangle \cdot \partial_\alpha.$$

Here Σ^+ is the set of positive roots of a root system Σ, $\partial_\alpha\phi(x) = \frac{d}{dt}\phi(x+t\alpha)|_{t=0}$ and the complex numbers k_α satisfy $k_\alpha = k_\beta$ if $|\alpha| = |\beta|$.

For a generic $\lambda \in \mathbb{C}^n$ we have a unique local solution

$$\Phi(\lambda, k; x) = e^{\langle\lambda-\rho,x\rangle} + \cdots \quad \text{(a series expansion at } \langle\alpha, x\rangle \to 0 \quad (\alpha \in \Sigma^+))$$

of the differential equation

$$L(k)u = \bigl(\langle\lambda, \lambda\rangle - \langle\rho(k), \rho(k)\rangle\bigr)u$$

and define Heckman-Opdam hypergeometric function

$$F(\lambda, k; x) := \sum_{w\in W} c(w\lambda)\Phi(\lambda, k; x)$$

as a generalization of the zonal spherical function of a Riemannian symmetric space. Here $\rho = \rho(k) = \sum_{\alpha\in\Sigma^+} k_\alpha\alpha$, W is the Weyl group of Σ and $c(\lambda)$ is a generalization of Harish-Chandra's c-function given by

$$c(\lambda) := \frac{\tilde{c}(\lambda)}{\tilde{c}(\rho(k))}, \quad \tilde{c}(\lambda) := \prod_{\alpha\in\Sigma^+} \frac{\Gamma\bigl(\frac{\langle\lambda,\check{\alpha}\rangle + k_{\alpha/2}}{2}\bigr)}{\Gamma\bigl(\frac{\langle\lambda,\check{\alpha}\rangle + k_{\alpha/2} + 2k_\alpha}{2}\bigr)} \quad \text{and} \quad \check{\alpha} := \frac{2\alpha}{\langle\alpha, \alpha\rangle}.$$

Put $\delta(k)^{\frac{1}{2}} = \prod_{\alpha\in\Sigma^+}\bigl(\sinh\langle\alpha, x\rangle\bigr)^{k_\alpha}$. Then the Schrödinger operator

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \sum_{\alpha\in\Sigma^+} \frac{k_\alpha(k_\alpha + 2k_{2\alpha} - 1)\langle\alpha, \alpha\rangle}{\sinh^2\langle\alpha, x\rangle} = \delta(k)^{\frac{1}{2}} \circ (L(k) + \langle\rho(k), \rho(k)\rangle) \circ \delta(k)^{-\frac{1}{2}}$$

is completely integrable and hence $L(k)$ is in a commuting system of differential operators with n algebraically independent operators.

Then we have the following fundamental result (cf. [1]).

Theorem [Heckman, Opdam]. *When k_α are generic, the function $F(\lambda, k; x)$ has an analytic extension on $\mathbb{R}^n$ and defines a unique simultaneous eigenfunction* *of the commuting system of differential operators with the eigenvalue parametrized by λ so normalized that the eigenfunction takes the value* 1 *at the origin.*

Heckman-Opdam hypergeometric system of differential equations characterizing $F(\lambda, k; x)$, which will be denoted by (HO), is a multi-variable analogue of a "rigid local system" among completely integrable quantum systems and we study three types of specializations of the system and the function $F(\lambda, k; x)$ as follows.

§1 **Confluence**. We examine its confluent limits such as Toda finite lattice (cf. [2] for the limiting procedure and the limits). For example, if Σ is of type A_{n-1}, under the correspondence $x_j \mapsto x_j + jt$ with $t \to \infty$ we remark that

$$\sum_{1\le i<j\le n} \frac{Ce^{2t}}{\sinh^2\big((x_i+it)-(x_j+jt)\big)} = \sum_{1\le i<j\le n} \frac{4Ce^{2(1-j+i)t}e^{2(x_i-x_j)}}{(1-e^{2(x_i-x_j)}e^{-2(j-i)t})^2}$$
$$\to \sum_{i=1}^{n-1} 4Ce^{2(x_j-x_{i+1})},$$

which holomorphically depends on $s = e^{-t}$.

Theorem 1. i) *For $v \in \mathbb{R}^n \setminus \{0\}$ the commuting system* (HO) *holomorphically continued to a confluent commuting system* $(\mathrm{HO})_{\mathrm{conf}}$ *by $x \mapsto x + tv$ with $t \to \infty$ and suitable $k_\alpha = k_\alpha(t)$. When Σ is of type BC_n (resp. F_4 or G_2), there exist three (resp. two) kinds of irreducible confluent limits.*

ii) *A suitably normalized Heckman-Opdam hypergeometric function has a non-zero holomorphic limit $W(x)$ with its expansion at an infinite point corresponding to a Weyl chamber $\mathcal{C}$. The limit has the moderate growth property:*

$$\exists C > 0,\ \exists m > 0 \text{ such that } |W(x)| \le Ce^{m|x|}.$$

iii) *The dimension of the solutions of the holomorphic family of the commuting systems including* $(\mathrm{HO})_{\mathrm{conf}}$ *with the moderate growth property is always one.*

iv) *For example, in the case of Toda finite lattice the limit $W(x)$ satisfies*

$$\exists C > 0,\ \exists m > 0,\ \exists K > 0 \text{ such that } |W(x)| \le C\exp(m|x| - e^{K\mathrm{dist}(x,\mathcal{C})}).$$

§2 **Restriction.** Let Ψ denote the fundamental system of Σ^+. For a subset Ψ' of Ψ let $H_{\Psi'}$ be the intersection of the walls defined by the elements of Ψ'. For a local solution u of (HO) at a generic point of $H_{\Psi'}$ we examine the differential equations satisfied by $u|_{H_{\Psi'}}$. Note that if $\#\Psi' = \#\Psi - 1$, the differential equations are ordinary differential equations. For example we have the following.

Theorem 2. *When (Ψ, Ψ') is of type (A_n, A_{n-1}) (resp. (BC_n, BC_{n-1})), the ordinary differential equations coincide with those satisfied by hypergeometric family ${}_{n+1}F_n$ of order $n+1$ (resp. even family of order $2n$). These are rigid local systems classified by Deligne-Simpson problem (cf.* [5]*).*

This theorem reduces the Gauss summation formula for (HO) given by [4] to the connection formula of the solutions of the ordinary differential equations.

§3 **Real forms.** For a signature

$$\epsilon : \Sigma \to \{\pm 1\} \qquad (\epsilon(\alpha+\beta) = \epsilon(\alpha)\epsilon(\beta) \text{ for } \forall\alpha,\ \beta,\ \alpha+\beta \in \Sigma)$$

of the root system Σ introduced by [3] we put

$$L(k)_\epsilon := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{\substack{\alpha\in\Sigma^+ \\ \epsilon(\alpha)>0}} k_\alpha \coth\langle\alpha, x\rangle \cdot \partial_\alpha + \sum_{\substack{\alpha\in\Sigma^+ \\ \epsilon(\alpha)<0}} k_\alpha \tanh\langle\alpha, x\rangle \cdot \partial_\alpha.$$

Note that $L(k)_\epsilon$ is obtained from $L(k)$ by the coordinate transformation $x \mapsto x + \sqrt{-1}v_\epsilon$ with a suitable $v_\epsilon \in \mathbb{R}^n$. We denote by $(\mathrm{HO})_\epsilon$ the corresponding commuting system of differential equations. Let W_ϵ be the Weyl group generated by the reflections with respect to the roots α satisfying $\epsilon(\alpha) = 1$.

Theorem 3. i) *If k_α are generic, the dimension of the solutions of $(\mathrm{HO})_\epsilon$ is $\#W/W_\epsilon$ and the vector $F_\epsilon(\lambda, k; x)$ of the independent solutions can be*

$$\big(F_\epsilon(\lambda, k; vx)\big)_{v\in W_\epsilon\backslash W} \sim \sum_{w\in W} A^\epsilon_w(\lambda, k)c(w\lambda, k)(e^{\langle w\lambda-\rho,x\rangle} + \cdots).$$

Here $A^\epsilon_w(\lambda, k)$ are intertwining matrices of size $\#W/W_\epsilon$ which satisfy

$$A^\epsilon_{wv}(\lambda, k) = A^\epsilon_w(v\lambda, k)A^\epsilon_v(\lambda, k) \qquad (w,\, v \in W).$$

If s_α is a simple reflection with respect to $\alpha \in \Psi$, $A^\epsilon_{s_\alpha}(\lambda, k)$ is a suitable direct product of the following matrices and scalars

$$A(\lambda, k) := \begin{pmatrix} \frac{\sin \pi k}{\sin \pi(\lambda+k)} & \frac{\sin \pi\lambda}{\sin \pi(\lambda+k)} \\ \frac{\sin \pi\lambda}{\sin \pi(\lambda+k)} & \frac{\sin \pi k}{\sin \pi(\lambda+k)} \end{pmatrix}, \quad \frac{\cos \frac{1}{2}\pi(\lambda - k)}{\cos \frac{1}{2}\pi(\lambda + k)} \quad \text{and} \quad 1.$$

ii) *We have a functional equation of the spherical functions:*

$$F_\epsilon(\lambda, k; x) = F_\epsilon(w\lambda, k; x)A^\epsilon_w(\lambda, k).$$

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Asymptotics of spherical functions for large rank after Okunkov and Olshanski

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An Olshanski spherical pair (G, K) is the inductive limit of an increasing sequence of Gelfand pairs $\big(G(n), K(n)\big)$. A spherical function for such a pair (G, K) is a continuous function φ on G which is K-biinvariant and such that

$$\lim_{n\to\infty}\int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α_n is the normalized Haar measure on $K(n)$ ([Olshanski, 1990]). When $G(n)/K(n)$ is a compact symmetric space it has been shown by Okunkov and Olshanski that a spherical function of positive type for the pair (G, K) can be obtained as limit of a sequence $\varphi^{(n)}$, where $\varphi^{(n)}$ is a spherical function for the pair $\big(G(n), K(n)\big)$ ([3], [4]).

1. In this talk we present the proof by Okunkov and Olshanski in the case of $G(n)/K(n)$ being the unitary group $U(n)$:

$$G(n) = U(n) \times U(n), \quad K(n) = \{(u, u) \mid u \in U(n)\} \simeq U(n).$$

A function φ on G which is K-biinvariant can be seen as a central function on the inductive limit $U(\infty) = \cup_{n=1}^{\infty} U(n)$. Consider the power series

$$\Phi(z) = \sum_{m=0}^{\infty} c_m z^m,\ c_m \geq 0,\ \sum_{m=0}^{\infty} c_m = 1,\ |z| \leq 1.$$

Define the function φ on $U(\infty)$ by

$$\varphi(u) = \det \Phi(u).$$

This means that φ is central and, for $u = \mathrm{diag}(z_1, \ldots, z_n, 1, \ldots)$,

$$\varphi(u) = \Phi(z_1)\ldots\Phi(z_n).$$

Theorem (Voiculescu, 1976) *The function φ is of positive type if and only if*

$$\Phi(z) = e^{\gamma(z-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(z-1)}{1 - \alpha_k(z-1)},$$

with $\alpha_k \geq 0$, $0 \leq \beta_k \leq 1$, $\gamma \geq 0$, and

$$\sum_{k=1}^{\infty}(\alpha_k + \beta_k) < \infty.$$

Define Ω_0 as the set of triples $\omega = (\alpha, \beta, \gamma)$, $\alpha = (\alpha_k)$, $\beta = (\beta_k)$, submitted to above conditions, and write $\Phi(z) = \Phi(\omega; z)$. For a continuous function f on $\mathbb{R}$ define the function L_f on Ω_0 as:

$$L_f(\omega) = \sum_k \alpha_k f(\alpha_k) + \sum_k \beta_k f(-\beta_k) + \gamma f(0).$$

We consider on Ω_0 the initial topology with respect to the functions L_f. For z fixed, the function $\omega \mapsto \Phi(\omega; z)$ is continuous.

Theorem (Vershik-Kerov, 1982, Boyer, 1983) *The spherical functions of positive type for the pair* (G, K) *are the following ones:*

$$\varphi(u) = \varphi(\omega_+, \omega_-; u) = \det \Phi(\omega_+; u) \det \Phi(\omega_-; u^{-1}),$$

with $\omega_+, \omega_- \in \Omega_0$.

Hence the spherical dual for the Olshanski spherical pair (G, K) is parametrized by $\Omega = \Omega_+ \times \Omega_-$.

2. Consider the Gelfand pair:

$$G(n) = U(n) \times U(n), \; K(n) = \{(u, u) \mid u \in U(n)\}.$$

A spherical function for the pair $\big(G(n), U(n)\big)$ is the normalized character of an irreducible representation of $U(n)$:

$$\varphi(\lambda; u) = \frac{\chi_\lambda(u)}{\chi_\lambda(\mathbf{1})}, \quad (\lambda \in \widehat{U(n)}).$$

The spherical dual $\Omega(n) \simeq \widehat{U(n)}$ is parametrized by signatures:

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_1 \geq \ldots \geq \lambda_n, \; \lambda_i \in \mathbb{Z}.$$

The restriction to the subgroup of diagonal matrices of the character χ_λ is a Schur function. Define, for $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$,

$$A_\mu(z) = \det(z_j^{\mu_i}).$$

The Schur function s_λ is defined as

$$s_\lambda(z) = \frac{A_{\mu+\delta}(z)}{V(z)},$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$, $V(z) = A_\delta(z)$ is the Vandermonde determinant. The character χ_λ is the central function on $U(n)$ such that

$$\chi_\lambda\big(\mathrm{diag}(z_1, \ldots, z_n)\big) = s_\lambda(z) \quad (z \in \mathbb{T}^n).$$

3. Assume for simplicity that λ is a positive signature: $\lambda_i \geq 0$. One defines the Frobenius parameters a_i and b_j of λ as
$a_i = \lambda_i - i$ if $\lambda_i \geq i$, $a_i = 0$ otherwise,
$b_j = \lambda'_j - j + 1$ if $\lambda'_j \geq j - 1$, $b_j = 0$ otherwise,
where λ' is the transpose of λ. For every n we define a map

$$T_n : \Omega(n) \to \Omega_0, \quad \lambda \mapsto \omega = (\alpha, \beta, \gamma),$$

with

$$\alpha_k = \frac{a_k}{n}, \; \beta_k = \frac{b_k}{n}, \; \gamma = 0.$$

Theorem (Vershik-Kerov, 1982) *Let* $\lambda(n) = \big(\lambda_1(n), \ldots, \lambda_n(n)\big)$ *be a sequence of signatures with* $\lambda(n) \in \Omega(n)$. *Assume that*

$$\lim_{n\to\infty} T_n\big(\lambda(n)\big) = \omega$$

for the topology of Ω_0. *Then*

$$\lim_{n\to\infty} \varphi\big(\lambda(n); u\big) = \det \Phi(\omega; u)$$

uniformly on every subgroup $U(m)$.

This means that, for every m,

$$\lim_{n\to\infty} \varphi\big(\lambda(n); \mathrm{diag}(z_1, \ldots, z_m, 1, \ldots, 1)\big) = \Phi(\omega; z_1) \ldots \Phi(\omega; z_m)$$

uniformly on $\mathbb{T}^m$.

Such a sequence $\lambda(n)$ is said to be a Vershik-Kerov sequence.

4. Let us sketch the method of proof by Okunkov and Olshanski.

a) The binomial formula for the Schur functions is the following expansion ([Okunkov, A., Olshanski, G.,1998]:

Proposition 1

$$\frac{s_\lambda(1+z_1, \ldots, 1+z_n)}{s_\lambda(1, \ldots, 1)} = \sum_{\mu_1 \geq \cdots \geq \mu_n \geq 0} \frac{\delta!}{(\mu+\delta)!} s^*_\mu(\lambda) s_\mu(z).$$

where s^*_μ *is the so-called shifted Schur function:*

$$s^*_\mu(\lambda) = \frac{\det([\lambda_i + \delta_i]_{\mu_j+\delta_j})}{\det([\lambda_i + \delta_i]_{\delta_j})}.$$

The following notation has been used: for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$

$$\mu! = \mu_1! \ldots \mu_n!,$$

and, for $\alpha \in \mathbb{C}$,

$$[\alpha]_k = \alpha(\alpha-1)\ldots(\alpha-k+1).$$

A polynomial f is said to be shifted symmetric if

$$f(\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots) = f(\lambda_1, \ldots, \lambda_{i+1}-1, \lambda_i+1, \ldots).$$

The shifted Schur function s^*_μ is shifted symmetric. The set Λ^* of shifted symmetric functions is an algebra.

b) One introduces the algebra morphism

$$\Lambda \to C(\Omega_0), \quad f \mapsto \tilde{f},$$

from the algebra of symmetric functions into the algebra of continuous functions on Ω_0 for which

$$\tilde{p}_1(\omega) = \sum_k \alpha_k + \sum_k \beta_k + \gamma,$$

$$\tilde{p}_m(\omega) = \sum_k \alpha_k^m + (-1)^{m-1} \sum_k \beta_k^m, \text{ if } m \geq 2,$$

where $\omega = (\alpha, \beta, \gamma)$, and p_m is the Newton sum:

$$p_m(x_1, \dots, x_k, \dots) = \sum_k x_k^m.$$

One establishes:

Proposition 2

$$\prod_{j=1}^{m} \Phi(\omega; 1 + z_j) = \sum_{\mu_1 \geq \cdots \geq \mu_n \geq 0} \tilde{s}_\mu(\omega) s_\mu(z).$$

c) The next step is the following:

Proposition 3 *Assume that*

$$\lim_{n\to\infty} T_n\big(\lambda(n)\big) = \omega$$

for the topology of Ω_0*. Let* $f^* \in \Lambda^*$ *be a shifted symmetric polynomial of degree* N*, and let* $f \in \Lambda$ *be the homogeneous part of* f^* *of degree* N*. Then*

$$\lim_{n\to\infty} \frac{1}{n^N} f^*\big(\lambda(n)\big) = \tilde{f}(\omega).$$

In particular, for $f^* = s_\mu^*$, a shifted Schur function,

$$\lim_{n\to\infty} \frac{1}{n^{|\mu|}} s_\mu^*\big(\lambda(n)\big) = \tilde{s}_\mu(\omega),$$

where $|\mu| = \mu_1 + \cdots + \mu_n$.

From Propositions 1, 2, and 3, and noticing that

$$\frac{(\mu+\delta)!}{\delta!} \sim n^{|\mu|} \quad (n \to \infty),$$

it is possible to deduce, by using properties of functions of positive type, that

$$\lim_{n\to\infty} \frac{s_{\lambda(n)}(1+z_1, \dots, 1+z_m, 1, \dots, 1)}{s_{\lambda(n)}(1, \dots, 1)} = \prod_{j=1}^{m} \Phi(\omega; 1+z_j).$$

This finishes the proof of the theorem.

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